CHAPTER II

ALGEBRAIC DIFFERENTIAL MANIFOLDS

MANIFOLDS AND THEIR DECOMPOSITION

1. Let $\Sigma$ be any finite or infinite system of d.p. in $\mathfrak{F}_1\{y_1, \cdots, y_n\}$. Let there be given an extension $\mathfrak{F}_1$ of $\mathfrak{F}$.\(^1\) Suppose that there exists in $\mathfrak{F}_1$ a set of $n$ elements $\eta_1, \cdots, \eta_n$ which are such that when each $y_i$ is replaced by $\eta_i$ in the d.p. of $\Sigma$, those d.p. all reduce to zero. We shall call the set $\eta_1, \cdots, \eta_n$ a zero of $\Sigma$. Thus a zero of $\Sigma$ is a solution of the system of equations obtained by equating the d.p. in $\Sigma$ to zero.

If $\Sigma$ has zeros, the totality of its zeros, for all possible extensions $\mathfrak{F}_1$ of $\mathfrak{F}$, will be called the manifold of $\Sigma$, or of the system of equations obtained by equating the d.p. in $\Sigma$ to zero.\(^2\) A zero of $\Sigma$ will at times be called a point of the manifold of $\Sigma$. The manifold of any system will be called an algebraic differential manifold, or, more briefly, a manifold.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be respectively the manifolds of systems $\Sigma_1$ and $\Sigma_2$.\(^3\) If $\mathfrak{M}_1$ is contained in $\mathfrak{M}_2$, we shall say that $\Sigma_2$ holds $\Sigma_1$. Also, we shall say that $\Sigma_2$ vanishes over or holds $\mathfrak{M}_1$. If $\Sigma$ is a system with no zeros, every system will be said to hold $\Sigma$.

Let $\Sigma$ be an infinite system, and $\Phi$ a basis of $\Sigma$ (I, §12).\(^4\) Because $\Sigma$ contains $\Phi$, $\Phi$ holds $\Sigma$. Because every d.p. in $\Sigma$ has a power in $[\Phi]$, $\Sigma$ holds $\Phi$. Thus, if $\Sigma$ has zeros, $\Sigma$ has the same manifold as $\Phi$. If $\Sigma$ has no zeros, $\Phi$ has no zeros. Thus the manifold of any infinite system of d.p. is the manifold of some finite subset of the system.\(^5\)

If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are manifolds of systems $\Sigma_1$ and $\Sigma_2$, the intersection $\mathfrak{M}_1 \cap \mathfrak{M}_2$, if not vacuous, is the manifold of $\Sigma_1 + \Sigma_2$. The union $\mathfrak{M}_1 + \mathfrak{M}_2$ is the manifold of the system of all products $AB$ with $A$ in $\Sigma_1$ and $B$ in $\Sigma_2$.

2. A manifold $\mathfrak{M}$ will be said to be reducible if it is the union of two manifolds, not necessarily mutually exclusive, which are proper parts of $\mathfrak{M}$. If $\mathfrak{M}$ is not reducible, it will be called irreducible.

A manifold $\mathfrak{M}$ of a system $\Sigma$ is irreducible if, and only if, whenever a product $AB$ vanishes over $\mathfrak{M}$, at least one of $A$ and $B$ vanishes over $\mathfrak{M}$. Suppose first that $AB$ holds $\mathfrak{M}$ while neither $A$ nor $B$ does. The manifolds of $\Sigma + A$ and

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\(^1\) The $y$ need not be indeterminates with respect to $\mathfrak{F}$.

\(^2\) Unfortunately, the totality of extensions of $\mathfrak{F}$ is an illegitimate totality. At the present time, there is no process of closure for differential fields analogous to the algebraic closure method. One knows, however, that troubles of this sort are not fatal to a theory.

\(^3\) All d.p. have coefficients in $\mathfrak{F}$, even though extensions are used in connection with zeros.

\(^4\) Chapter I, §12. When no chapter number is given, the chapter is that in which one is reading.

\(^5\) We understand this statement to stand for the two sentences which precede it.
\( \Sigma + B \), whose union is \( \mathcal{M} \), will be proper parts of \( \mathcal{M} \) and \( \mathcal{M} \) will be reducible. Again, let \( \mathcal{M} \) be the union of smaller manifolds \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), manifolds respectively of systems \( \Sigma_1 \) and \( \Sigma_2 \). Let \( A_i \), \( i = 1, 2 \), be a d.p. of \( \Sigma_i \) which does not hold \( \mathcal{M} \). The product \( A_1A_2 \) holds \( \mathcal{M} \).

Let \( \mathcal{M} \) be the manifold of a system \( \Sigma \). The totality \( \Omega \) of those d.p. which vanish over \( \mathcal{M} \) is an ideal, and, indeed, a perfect ideal. We shall call \( \Omega \) the perfect ideal associated with \( \mathcal{M} \). It will be seen in §7 that \( \Omega \) is \( \{ \Sigma \} \). \( \mathcal{M} \) is irreducible if, and only if, \( \Omega \) is prime. When \( \Omega \) is prime, we call it the prime ideal associated with \( \mathcal{M} \).

3. We prove the following fundamental theorem.

**Theorem:** Every manifold is the union of a finite number of irreducible manifolds.

Let the theorem not hold for the manifold \( \mathcal{M} \) of some system \( \Sigma \). Then \( \mathcal{M} \) is not irreducible. Let \( AB \) hold \( \mathcal{M} \), while neither \( A \) nor \( B \) does. Then \( \mathcal{M} \) is the union of the manifolds of \( \Sigma + A \) and \( \Sigma + B \). At least one of the latter manifolds must fail to be the union of a finite number of irreducible manifolds. Let such failure occur for the manifold of \( \Sigma + A \), which system we represent by \( \Sigma_1 \). Continuing, we produce, with the help of the axiom of selection, an infinite sequence

\[
\Sigma, \Sigma_1, \ldots, \Sigma_p, \ldots,
\]

each \( \Sigma_p \) containing, while not holding, its predecessor. Let \( \Omega \) be the union of the systems (1) and let \( \Phi \) be a basis of \( \Omega \). Then \( \Phi \) is contained in some system of (1), say in \( \Sigma_q \). We see that \( \Phi \) is a basis for \( \Sigma_q \). By §1, \( \Phi \) and \( \Sigma_q \) have the same manifold. But the same argument shows that \( \Phi \) and \( \Sigma_{q+1} \) have the same manifold. This furnishes the contradiction that \( \Sigma_{q+1} \) holds \( \Sigma_q \). The theorem is proved.

Let a manifold \( \mathcal{M} \) have a representation

\[
\mathcal{M} = \mathcal{M}_1 + \cdots + \mathcal{M}_p
\]

as a union of irreducible manifolds \( \mathcal{M}_i \). If an \( \mathcal{M}_i \) contains an \( \mathcal{M}_j \) with \( j \neq i \), then \( \mathcal{M}_j \) may be suppressed in (2). We thus suppose that no \( \mathcal{M}_i \) contains any \( \mathcal{M}_j \) with \( j \neq i \).

Now let \( \Sigma \) be the perfect ideal associated with \( \mathcal{M} \) and let, for each \( i \), \( \Sigma_i \) be the prime ideal associated with \( \mathcal{M}_i \). Each \( \Sigma_i \) is a divisor of \( \Sigma \). If \( A \) is a d.p. common to all \( \Sigma_i \), \( A \) holds \( \mathcal{M} \) and is thus in \( \Sigma \). Then \( \Sigma \) is the intersection of the \( \Sigma_i \). If \( j \neq i \), \( \Sigma_j \) is not a divisor of \( \Sigma_i \); otherwise \( \mathcal{M}_i \) would contain \( \mathcal{M}_j \). Then the \( \Sigma_i \) are the essential prime divisors of \( \Sigma \).

If \( \mathcal{M}' \) is an irreducible manifold contained in \( \mathcal{M} \), the prime ideal associated with \( \mathcal{M}' \) is a divisor of some \( \Sigma_i \) (I, §17). Thus \( \mathcal{M}' \) is contained in some \( \mathcal{M}_i \).

An irreducible manifold contained in \( \mathcal{M} \) which is not part of a larger irreducible manifold contained in \( \mathcal{M} \) will be called an *essential irreducible component* of \( \mathcal{M} \).
or a component of $\mathcal{M}$. The only components of $\mathcal{M}$ are the $\mathcal{M}_i$ in (2). Every irreducible manifold contained in $\mathcal{M}$ is contained in some component of $\mathcal{M}$. Our discussion shows that in every representation of $\mathcal{M}$ as the union of a finite number of irreducible manifolds, every component of $\mathcal{M}$ appears; all other irreducible manifolds in the union are redundant.

We partially summarize what precedes. A manifold $\mathcal{M}$ has a finite number of components, and is the union of them. The essential prime divisors of the perfect ideal associated with $\mathcal{M}$ are the prime ideals associated with the components of $\mathcal{M}$.

A component of the manifold of a system $\Sigma$ will at times be called a component of $\Sigma$.

**Illustrations in Analysis**

4. To illustrate the decomposition of manifolds, we shall employ differential equations of classical analysis.

We use an open region $A$ in the plane of the complex variable $z$. Our field $\mathfrak{F}$ will be supposed to consist of functions meromorphic throughout $A$.

Given a system $\Sigma$, we consider zeros of it obtained as follows. Let $B$ be any open region contained in $A$ and let $y_1(x), \ldots, y_n(x)$, analytic in $B$, annul every d.p. of $\Sigma$ in $B$. We shall call the entity composed of the $y_i(x)$ and $B$ an analytic zero, or a zero, of $\Sigma$. Two sets $y_i(x)$ which are identical from the standpoint of analytic continuation will give different zeros if they are not associated with the same open region. For instance, if we use an open region $B_1$ interior to $B$, and use, throughout $B_1$, the $y_i(x)$ as defined for $B$, we get a different zero of $\Sigma$.

The totality of analytic zeros of $\Sigma$ will be called the restricted manifold of $\Sigma$. At this point in our work, we have no need to consider other types of zeros of $\Sigma$ and it will turn out finally that the consideration of the restricted manifold produces a complete theory of the system $\Sigma$.

The case in which $\mathfrak{F}$ consists of meromorphic functions, and in which one uses restricted manifolds, will be called the analytic case. All definitions in §§1–3 following that of manifold retain their meaning and all proofs retain their validity, in the analytic case. Thus, in the analytic case, a system $\Sigma_1$ holds a system $\Sigma_2$ if every analytic zero of $\Sigma_1$ is a zero of $\Sigma_2$. The theorem of §3

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6 No misunderstanding can arise, since the only subsets of manifolds which we employ are essential irreducible components.

7 It is futile to seek greater generality through the use of functions analytic except for isolated singularities. If $f(x)$ has an isolated essential singularity for $x = a$ and if $c$ is a rational value assumed by $f(x)$ in every neighborhood of $a$, the reciprocal of $f(x) - c$ has a pole in every neighborhood of $a$.

8 No confusion with the term zero of the theory of functions will arise.

9 Given an analytic zero, we have to go through the formality of constructing an extension of $\mathfrak{F}$ in which its $y(x)$ are contained. This is done by forming all rational combinations of the $y(x)$ and their derivatives, with coefficients in $\mathfrak{F}$. If such a combination coincides in $B$ with a function $f(x)$ in $\mathfrak{F}$, we consider the combination to be identical with $f(x)$, and thus to be in $\mathfrak{F}$.

10 In §11, it will be seen that, in this case, every zero of $\Sigma_1$ is a zero of $\Sigma_2$. Thus the word hold will be established as a word of a single meaning.
becomes: Every restricted manifold is the union of a finite number of irreducible restricted manifolds. By a component of the restricted manifold \( M \) of a system \( \Sigma \), we mean an irreducible restricted manifold \( M' \) contained in \( M \), which is not part of a larger irreducible restricted manifold contained in \( M \). We may call \( M' \) a restricted component, or an analytic component of \( \Sigma \). It will be seen in §11 that the perfect ideal associated with the restricted manifold of \( \Sigma \) is identical with the perfect ideal associated with the complete abstract manifold. The essential prime divisors of this perfect ideal furnish both the analytic components of \( \Sigma \) and the full components discussed in §3.

In our present work under the analytic case, the term manifold will be understood to mean restricted manifold.

We consider some examples. \( \mathcal{F} \) will be any field of meromorphic functions.

Example 1. Let \( \Sigma \) consist of the single d.p. \( A = y_1^2 - 4y \) in \( \mathcal{F}\{y\} \). We call attention to the fact that \( A \), as a polynomial in \( y \) and \( y_1 \), cannot be factored in any field. The manifold of \( \Sigma \) consists of the functions \( y = (x + c)^2 \) with \( c \) constant, and of the function \( y = 0 \). The derivative of \( A \) is \( 2y_1 (y_2 - 2) \). Now \( y_2 - 2 \) vanishes for every \( (x + c)^2 \) but not for \( y = 0 \). Again, \( y_1 \) vanishes for \( y = 0 \), but for no \( (x + c)^2 \). Thus \( M \) is reducible and is the union of \( M_1 \), composed of the functions \( (x + c)^2 \), and of \( M_2 \), composed of \( y = 0 \). \( M_1 \) is the manifold of the system \( A, y_2 - 2 \) and \( M_2 \) is the manifold of \( A, y_1 \). It is obvious that \( M_2 \) is irreducible. As to \( M_1 \), let it be held by \( BC \). When \( y \) is replaced by \( (x + c)^2 \), \( B \) and \( C \) become polynomials in \( c \) with coefficients meromorphic in \( A \). If the product of two such polynomials vanishes identically in \( x \) and \( c \), one of the polynomials does. Thus one of \( B \) and \( C \) holds \( M_1 \) and \( M_2 \) is irreducible.

Example 2. Let \( \Sigma \) be the d.p. \( A = y_2^2 - y \) in \( \mathcal{F}\{y\} \). Differentiating \( A \) successively, we have, over \( M \),

\[
2y_2y_3 - y_1 = 0,
\]

(3)

\[
2y_2y_4 + 2y_3^2 - y_2 = 0,
\]

(4)

\[
2y_2y_5 + 6y_3y_4 - y_3 = 0.
\]

Multiplying (4) by \( 2y_5 \) and substituting into the result the expression for \( y_5^2 \) found from (3), we have, over \( M \),

\[
y_2 (4y_3y_5 - 12y_4^2 + 8y_4 - 1) = 0.
\]

Thus \( M \) is reducible. It is composed of \( M_1 \) and \( M_2 \), the respective manifolds of \( A, y_2 \); \( A, 4y_3y_5 - 12y_4^2 + 8y_4 - 1 \).

As \( M_1 \) consists of \( y = 0 \), it is irreducible. We shall see later that \( M_2 \), which is the general solution of \( A \), is irreducible.

Example 3. The manifold of \( y_1 (y_1 - y) \) decomposes into the two irreducible

\( ^{11} \) We shall not encumber our discussions with references to the areas in which the functions in a zero are analytic.
manifolds given by $y = c$ and $y = ce^y$. These two manifolds have $y = 0$ in common.

**Example 4.** Let $\Sigma$ consist of $A = y^2y_2 - y$. We find with a single differentiation that $\mathfrak{M}$ is reducible and is made up of the manifolds of

$$A, y_1; \quad A, y_1y_2 + 2y_2^2 - 1.$$  

We call attention to the fact that $A$ cannot be factored and is of the first degree in $y_2$.

**Example 5.** Let $\Sigma$ be composed of $A = uy - u_1^2$ in $\mathfrak{F} \{ u, y \}$. Differentiating, we find over $\mathfrak{M}$,

$$u_1y + uy_1 - 2u_1u_2 = 0.$$  

Multiplying this equation by $y$ and using $A = 0$, we find

$$u_1 (y^2 + uy_1 - 2uy) = 0.$$  

Neither factor in (5) holds $\mathfrak{M}$, so that $\mathfrak{M}$ is reducible. We call attention to the fact that $A$ cannot be factored and is of zero order in $y$.

**Example 6.** In $\mathfrak{F} \{ y, z \}$, let $\Sigma$ be

$$y - xy_1 + \frac{y_2z_1}{4}, \quad z - xz_1 + \frac{y_1z_1}{4}.$$  

We are dealing with a pair of Clairaut equations. $\mathfrak{M}$ consists of two irreducible manifolds which are, to speak geometrically, the two-parameter family of lines

$$y = ax - \frac{ab}{4}, \quad z = bx - \frac{ab}{4},$$  

and their one-parameter family of envelopes

$$y = (x + c)^3, \quad z = (x - c)^3.$$  

The above examples might lead one to conjecture that the manifold of any finite system can be decomposed into irreducible manifolds by differentiations and eliminations. We shall see in Chapter V that this is actually so.

**Prime Ideals and Regular Zeros**

5. We return to the use of an abstract field. We shall call $\mathfrak{F} \{ y_1, \cdots, y_n \}$ the unit ideal. The prime ideal consisting of the d.p. 0 will be called the zero ideal. A prime ideal distinct from the unit ideal and the zero ideal will be said to be nontrivial.

Let $\Sigma$ be a nontrivial prime ideal. Let

$$A_1, \cdots, A_r$$

be a characteristic set of $\Sigma$. The separatant and initial of $A_i$ will be denoted by $S_i$ and $I_i$, respectively. As the $S$ and $I$ are reduced with respect to (6), they are not in $\Sigma$ (I, 5).
We prove that, for a d.p. $G$ to belong to $\Sigma$, it is necessary and sufficient that the remainder of $G$ with respect to (6) be zero. Let $G$ be in $\Sigma$. As the remainder, $R$, is in $\Sigma$ and is reduced with respect to (6), we have $R = 0$. Again, let $R = 0$. There is a relation
\[(7) \quad S_1^a \cdots I_r^c G \equiv 0, \quad (\Sigma).\]
As $\Sigma$ is prime and the $S$ and the $I$ are not in $\Sigma$, it must be that $G$ is in $\Sigma$.

A zero of the characteristic set (6) for which every $S_i$ and every $I_i$ is distinct from zero will be called a regular zero of (6).\(^{12}\) We shall prove that every regular zero of a characteristic set of $\Sigma$ is a zero of $\Sigma$. Let $\eta_1, \cdots, \eta_n$ be a regular zero of (6). Let $G$ be any d.p. in $\Sigma$. In (7), the $S$ and the $I$ are not annulled by the $\eta$. Then $G$ is annulled by the $\eta$. The $\eta$ thus constitute a zero of $\Sigma$.

**Generic zeros of a prime ideal**

6. Let $\Sigma$ be a prime ideal distinct from the unit ideal.

Let $A$ be any d.p., not necessarily contained in $\Sigma$. We form a class $\alpha$ of d.p., putting into $\alpha$ every d.p. $G$ such that $G \equiv A, (\Sigma)$. We call $\alpha$ a remainder class, modulo $\Sigma$. Thus $\mathfrak{F}\{ y_1, \cdots, y_n \}$ is composed of a set of remainder classes. As $\Sigma$ contains no element of $\mathfrak{F}$ except zero, two distinct elements of $\mathfrak{F}$ belong to distinct remainder classes; there are thus an infinite number of remainder classes.

Let $\alpha$ and $\beta$ be two remainder classes. All sums $A + B$ with $A$ in $\alpha$ and $B$ in $\beta$ belong to the same remainder class. We call this class $\alpha + \beta$. Actually, every d.p. in $\alpha + \beta$ is the sum of a d.p. in $\alpha$ and a d.p. in $\beta$. We define $\alpha \beta$ as the remainder class which contains all products $AB$ with $A$ in $\alpha$ and $B$ in $\beta$. Usually $\alpha \beta$ contains d.p. which are not products $AB$. The derivative $\alpha'$ of $\alpha$ is defined as the remainder class which contains the derivatives of the d.p. in $\alpha$.

The remainder class which contains the d.p. 0 is $\Sigma$. We call $\Sigma$ the zero class. As $\Sigma$ is prime, a relation $AB \equiv 0, (\Sigma)$, implies that either $A \equiv 0, (\Sigma)$ or $B \equiv 0, (\Sigma)$. Thus, if each of two remainder classes is distinct from the zero class, their product is distinct from the zero class.

We now consider pairs $(\alpha, \beta)$ of remainder classes in which $\beta$ is not the zero class. Two pairs, $(\alpha, \beta)$ and $(\gamma, \delta)$, will be called equivalent if $\alpha \delta = \beta \gamma$. As the equivalence relation is transitive, the totality of pairs of classes separates into sets of equivalent pairs. If $\mathfrak{A}$ is the set containing $(\alpha, \beta)$ and $\mathfrak{B}$ that containing $(\gamma, \delta)$, we define $\mathfrak{A} + \mathfrak{B}$ as the set containing $(\alpha \delta + \beta \gamma, \beta \delta)$, and $\mathfrak{A}\mathfrak{B}$ as the set containing $(\alpha \gamma, \beta \delta)$. The operations of subtraction and division are then uniquely determined. In particular, $\mathfrak{A}/\mathfrak{B}$ can and must be taken as the set containing $(\alpha \delta, \beta \gamma)$.\(^{13}\) The derivative of the set containing $(\alpha, \beta)$ is defined as the set containing $(\beta \alpha' - \alpha \beta', \beta')$. With these operations, the sets of pairs of remainder classes become a differential field, which we denote by $\mathfrak{F}_1$.

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\(^{12}\) It will be seen in §6 that regular zeros exist.

\(^{13}\) We attempt division only when $\gamma$ is not the zero class.
With an element \( a \) of \( \mathfrak{F} \), we associate the set in \( \mathfrak{F}_1 \) containing the pair \((\alpha, \beta)\) in which \( \alpha \) contains \( a \) and \( \beta \) contains \( 1 \). In this way we obtain a subset \( \mathfrak{F}' \) of \( \mathfrak{F}_1 \) which is isomorphic with \( \mathfrak{F} \). We replace each set of \( \mathfrak{F}' \) by the corresponding element of \( \mathfrak{F} \), and \( \mathfrak{F}_1 \) becomes an extension of \( \mathfrak{F} \).

We are going to find a zero of \( \Sigma \) in \( \mathfrak{F}_1 \). Let \( \omega \) be that one of the remainder classes above which contains unity, and for \( i = 1, \ldots, n \), let \( \alpha_i \) be the class which contains the d.p. \( y_i \). Let \( \eta_i \) be the set in \( \mathfrak{F}_1 \) which contains \((\alpha_i, \omega)\).

We shall show that \( \eta_1, \ldots, \eta_n \) is a zero of \( \Sigma \).

Let \( G \) be any d.p. in \( \Sigma \). The derivative of \( \eta_i \) is the set containing \((\alpha'_i, \omega)\), and \( \alpha'_i \) contains \( y_i \). It follows that when the \( \eta \) are substituted for the \( y \) in \( G \), we obtain a set containing \((\beta, \omega)\), where \( \beta \) is the remainder class containing \( G \), that is, the zero class. The set just described has \( 0 \) as its proxy in \( \mathfrak{F}_1 \). We see that \( \eta_1, \ldots, \eta_n \) is a zero of \( \Sigma \).

We see immediately, in a converse way, that if \( \eta_1, \ldots, \eta_n \) annuls a d.p. \( G \), \( G \) is contained in \( \Sigma \).

A zero of \( \Sigma \), naturally contained in some extension of \( \mathfrak{F} \), which is such that every d.p. over \( \mathfrak{F} \) which is annihilated by the zero is contained in \( \Sigma \), will be called a generic zero\(^{14}\) of \( \Sigma \), or a generic point of the manifold of \( \Sigma \). We know that every prime ideal distinct from the unit ideal has a generic zero.

If we take \( \Sigma \) as in §5, we see that a generic zero of \( \Sigma \) is a regular zero of (6).

**The theorem of zeros**

7. We prove the following theorem:

**Theorem:** If \( \Sigma \) is a perfect ideal distinct from the unit ideal, \( \Sigma \) has zeros and every differential polynomial which holds \( \Sigma \) is contained in \( \Sigma \).\(^{15}\)

Let \( \Sigma \) be the intersection of essential prime divisors \( \Sigma_i, i = 1, \ldots, p \). No \( \Sigma_i \) is the unit ideal. For each \( \Sigma_i \), we form a generic zero. Each of these \( p \) zeros is a zero of \( \Sigma \). Now let \( G \) be a d.p. which holds \( \Sigma \). As \( G \) is annihilated by each of the generic zeros, \( G \) is in each \( \Sigma_i \) and therefore in \( \Sigma \).

We see, as was stated in §2, that, given a manifold \( \mathfrak{M} \) of a system \( \Sigma \), the perfect ideal associated with \( \mathfrak{M} \) is \( \{ \Sigma \} \); it is the only perfect ideal whose manifold is \( \mathfrak{M} \).

Modifying slightly the theorem just proved, we obtain the

**Theorem of zeros:** Let

\[
F_1, \ldots, F_p
\]

be any finite system of differential polynomials and let \( G \) be any differential polynomial which holds that system. Some power of \( G \) is a linear combination of the \( F \) and of their derivatives of various orders, with differential polynomials for coefficients. In particular, if \( F_1, \ldots, F_p \) has no zeros, some linear combination of the \( F \) and of their derivatives of various orders equals unity.

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\(^{14}\) Raudenbush, 20.

\(^{15}\) A.D.E., Chapter VII, and Raudenbush, 21.
Let $\Sigma$ be the perfect ideal determined by the $F$. If $\Sigma$ is the unit ideal, unity is a linear combination as described above. Let $\Sigma$ be distinct from the unit ideal. Then $G$ is in $\Sigma$.

8. Let us reexamine the decomposition theorem of I, §19. Let $\Sigma$ be an ideal with a manifold $M$ which has a representation

$$M = M_1 + \cdots + M_p$$

where no two $M_i$ have a point in common. If $\Omega_i$ and $\Omega_j$, $i \neq j$, the system $\Omega_i + \Omega_j$ has no zeros. Then $\{ \Omega_i + \Omega_j \}$ is the unit ideal. This implies a relation $A + B = 1$ with $A$ in $\Omega_i$ and $B$ in $\Omega_j$. Thus $\Omega_i$ and $\Omega_j$ are relatively prime. It follows that $\Sigma$ has a unique representation as the product of ideals whose manifolds are the $M_i$.

**Example:** We consider, as in Example 1 of §4, the manifold $M$ of $A = y_1^2 - 4y$. At the present time, we use the full abstract manifold. $M$ is the union of $M_1$ and $M_2$, the respective manifolds of $A, y_2 - 2$ and $A, y_1$. As $y_2 - 2$ and $y_1$ have no zero in common, $M_1$ and $M_2$ have no point in common. As $y_1 (y_2 - 2)$ is in $[A]$, we have, by I, §11,

$$\{ A \} = \{ A, y_2 - 2 \} \cap \{ A, y_1 \}.$$  

Of course, $\{ A, y_1 \} = \{ y \} = [y]$. As $[A]$ contains $y_1(y_2 - 2)$, it contains $y_2(y_2 - 2)^2$. Thus $[A]$ contains $BC$ where

$$B = (y_2 - 2)^2, \quad C = 4y_2 - y_2^2.$$

We have $B + C = 4$. It follows from (8) and I, §19, that

$$[A] = [A, B] [A, C].$$

Let $\Sigma_1 = [A, C]$. As $\Sigma_1$ contains $y_2B$, it contains $y_2 (B + C)$ and therefore $y_2$. Then, as $\Sigma_1$ contains $y_1(y_2 - 2)$, it contains $y_1$ and hence $y$. Thus $\Sigma_1 = [y]$ and

$$[y_1^2 - 4y] = [y] [y_1^2 - 4y, (y_2 - 2)^2].$$

9. We shall now obtain a theorem of zeros for the analytic case.

With $\mathfrak{F}$ a field of meromorphic functions, we take $\Sigma$ as in §5. Let $G$ be any d.p. not in $\Sigma$. We are going to prove the existence of a regular zero of (6), composed of functions $y_1(x), \ldots, y_n(x)$, which is not a zero of $G$.

The remainder $R$ of $G$ with respect to (6) is not zero. Let

$$K = RS_1 \cdots S_r I_1 \cdots I_r$$

where the $S$ and $I$ are as in §5. We wish, for a short time, to consider $K$ and the $A_i$ not as d.p., but as ordinary polynomials in the $y_{ij}$. A letter $y_{ij}$ enters into our present work only if it appears effectively in some of the $r + 1$ polynomials. Let $\sigma$ be the system of polynomials $A_i$. By a zero of $\sigma$, we shall mean any set of functions $y_{ij}(x)$, analytic in some area contained in $A$, which annul every $A_i$. We do not ask that $y_{i, j+1}(x)$ be the derivative of $y_{ij}(x)$. 
No power of $K$ is linear in the $A_i$ with coefficients which are polynomials in the $y_{ij}$.
16 Otherwise $K$, considered as a d.p., would be contained in the prime ideal $\Sigma$. We shall now invoke Hilbert’s theorem of zeros for polynomials, which is proved in IV, §14. The system $\sigma$ has at least one zero, composed of functions $\tilde{y}_{ij}(x)$, which do not annul $K$. Let $a$ be a value of $x$ at which the $\tilde{y}_{ij}(x)$ and all coefficients in the $A_i$ and $G$ are analytic, and at which $K$, when the $\tilde{y}_{ij}(x)$ are substituted into it, has a value distinct from zero.

10. We return now to the consideration of $K$ and the $A_i$ as d.p. Let $p_j$ be the class of $A_j$, $j = 1, \ldots, r$, and $m_j$ the order of $A_j$ in $y_{pi}$. It may be that $r < n$ in (6), so that there are $y_i$ which are not among the $y_{pi}$. Every such $y_i$, we replace in the $A$ by a function $y_i(x)$ analytic at $a$, which is chosen with the sole restriction that if some $y_{ij}$ is a letter used in §9, the $j$th derivative of $y_i$ has at $a$ the value $\tilde{y}_{ij}(a)$ as in §9. It is a matter of forming convergent series of powers of $x - a$, with a finite number of coefficients assigned in advance. For these replacements, each $A_j$ goes over into an expression $B_j$ in $y_{pi}, \ldots, y_{pj}$ and their derivatives.

We consider the equation $B_1 = 0$ as an equation determining $y_{pni}$ as a function of $x, y_{pi}, \ldots, y_{pni} - 1$. We work at $x = a$. To every $y_{pni}$ which is a letter of §9, we assign the value $\tilde{y}_{pni}(a)$. There may be $y_{pni}$ with $i < m_i$ which do not appear in §9. To them we assign arbitrary numerical values. For the values assigned to $x$ and the $y_{pni}, B_1$ vanishes. Now $\partial B_1/\partial y_{pni}$ does not vanish for these values.\textsuperscript{17} We can thus solve the equation $B_1 = 0$ for $y_{pni}$, finding

\begin{equation}
\begin{aligned}
y_{pni} &= f_1(x, y_{pni}, \ldots, y_{pni} - 1) \\
\end{aligned}
\end{equation}

with $f$ analytic for the assigned values of its arguments and equal to $\tilde{y}_{pni}(a)$ for those values.

We now regard (9) as a differential equation of order $m_i$ for $y_{pi}$. For the initial conditions assigned as above at $x = a$, we obtain a solution $y_{pi}(x)$ analytic at $x = a$. The functions $y_i(x), \ldots, y_{pi}(x)$ annul $A_1$ but neither $S_1$ nor $I_1$.

We now substitute $y_{pi}(x)$ for $y_{pi}$ in $B_2$ and treat the equation $B_2 = 0$ as above. Continuing, we construct a regular zero of (6). This zero does not annul $K$ at $x = a$. Thus $R_i$ and also $G$, are not annihilated by the zero at $x = a$.\textsuperscript{18}

11. The theorems of §7 now go over to the analytic case. Thus, if $\Sigma$ is a field of meromorphic functions, and if $\Sigma$ has a perfect ideal distinct from the unit ideal, $\Sigma$ has a nonvacuous restricted manifold. Every differential polynomial which vanishes over the restricted manifold of $\Sigma$ is contained in $\Sigma$.

In the theorem of zeros, if $F_1, \ldots, F_p$ has no analytic zeros, unity is contained in the ideal of the $F_i$, so that $F_1, \ldots, F_p$ has no zeros of any type. If there is a

\textsuperscript{15} With coefficients in $\Sigma$.

\textsuperscript{17} The partial derivative is what $S_1$ becomes for the replacements made above in the $A$. We note that $K$ does not vanish for the $\tilde{y}_{ij}(a)$.

\textsuperscript{18} The work of §10 shows that a characteristic set of a prime ideal may be regarded as furnishing a system of differential equations, in a standard form, whose solutions more or less make up the manifold of the ideal.
restricted manifold, every d.p. which holds it is in the perfect ideal determined by the F and is thus annulled by all zeros of $F_1, \ldots, F_p$.

To sum up, given any system $\Sigma$ with $\mathfrak{S}$ as above, and with $\{ \Sigma \}$ distinct from the unit ideal, $\Sigma$ has a restricted manifold $\mathfrak{M}$ and an abstract manifold $\mathfrak{M}'$ which contains $\mathfrak{M}$. Both $\mathfrak{M}$ and $\mathfrak{M}'$ have $\{ \Sigma \}$ for associated perfect ideal. We shall find on this basis, in dealing with differential equations of analysis, that it suffices generally to work with restricted manifolds.

**General solutions**

12. We use $\mathfrak{S}\{ y_1, \ldots, y_n \}$ with $\mathfrak{S}$ any field. A d.p. of positive class will be said to be *algebraically irreducible* if it is not the product of two d.p. of positive class.

Let $F$ be of positive class $p$ and algebraically irreducible. We are going to study the representation of $\{ F \}$ as an intersection of prime ideals.\(^{19}\)

Denoting the separant of $F$ by $S$, we let $\Sigma_1$ be the totality of those d.p. $A$ which are such that

$$SA = 0, \quad \{ F \}.$$  

By §7, $A$ is in $\Sigma_1$ if $A$ vanishes for every zero of $F$ which does not annul $S$.

Clearly, the sum of two d.p. in $\Sigma_1$ is in $\Sigma_1$, as is also the product of a d.p. in $\Sigma_1$ by any d.p. From (10) it follows, by I, §10, that $SA'$, with $A'$ the derivative of $A$, is in $\{ F \}$. Then $A'$ is in $\Sigma_1$. Thus $\Sigma_1$ is an ideal.

We prove now that the ideal $\Sigma_1$ is prime. Let $AB$ be in $\Sigma_1$. Let $F$ be of order $m$ in $y_p$. The process of reduction used for forming remainders shows the existence of relations

$$S^A = R, \quad S^B = T, \quad [F],$$

with $R$ and $T$ of order at most $m$ in $y_p$. We shall prove that at least one of $R$ and $T$ is divisible by $F$. From (11) we have $SRT = S^{a+b+1}AB$, $[F]$. As the second member of this congruence is in $\{ F \}$, the first member is also. Let then

$$(SRT)^e = MF + M_1F' + \cdots + M_qF^{(q)},$$

superscripts indicating differentiation. We have

$$F^{(q)} = Sy_{p, m+q} + U$$

where $U$ is of order less than $m + q$ in $y_p$. We replace $y_{p, m+q}$ in $F^{(q)}$ and in the $M$ by $-U/S$. Clearing fractions, we find a relation

$$S^{d}(RT)^e = NF + N_1F' + \cdots + N_{q-1}F^{(q-1)}.$$  

Continuing, we find that some $S^e(RT)^e$ is divisible by $F$. As $F$ is algebraically irreducible, and not a factor of $S$, $F$ must be a factor of at least one of $R$ and $T$.

\(^{19}\) Even if $p < n$, $\{ \mathfrak{S} \}$ will contain d.p. of class as high as $n$. 

Suppose that \( R \) is divisible by \( F \). By (11), \( SA \) is in \( \{ F \} \) so that \( A \) is in \( \Sigma_1 \). Thus \( \Sigma_1 \) is prime.

13. We prove now that for a d.p. \( A \) to belong to \( \Sigma_1 \), it is necessary and sufficient that the remainder of \( A \) with respect to \( F \) be zero. In particular, if \( A \) is in \( \Sigma_1 \) and if \( A \) has the same order in \( y \), as \( F \), \( A \) is divisible by \( F \).

Let \( A \) belong to \( \Sigma_1 \). We have a relation

\[
S \circ A \equiv B, \quad [F],
\]

with \( B \) of order at most \( m \) in \( y \). Now \( SB \) is in \( \{ F \} \) so that, as in §12, \( B \) is divisible by \( F \). This means that the remainder of \( A \) is zero. Conversely, if the remainder is zero, we have (12) with \( B \) divisible by \( F \) so that \( A \) is in \( \Sigma_1 \).

We see, in particular, that \( \Sigma_1 \) does not contain \( S \).

14. We prove that

\[
\{ F \} = \Sigma_1 \cap \{ F, S \}.
\]

\( \{ F \} \) is contained in each ideal in the second member, so that it will suffice to show that the second member is in \( \{ F \} \). Let \( A \) be in \( \{ F, S \} \). For some \( a \),

\[
A^a = B + C
\]

with \( B \) in \( \{ F \} \) and \( C \) in \( \{ S \} \). Now, let \( A \) also belong to \( \Sigma_1 \). Then \( SA \) is in \( \{ F \} \) so that, by I, §10, the product of \( A \) by any derivative of \( S \) is in \( \{ F \} \). Then \( AC \) is in \( \{ F \} \) so that \( A^{a+1} \) is in \( \{ F \} \).

15. Let

\[
\{ F, S \} = \Lambda_1 \cap \cdots \cap \Lambda_q
\]

where the \( \Lambda \) are the essential prime divisors of \( \{ F, S \} \). Certain \( \Lambda \) may be divisors of \( \Sigma_1 \). Suppressing these, and using symbols \( \Sigma_i \) with \( i > 1 \) for the remaining \( \Lambda \), we have

\[
\{ F \} = \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_r.
\]

Thus, \( \Sigma_1 \) is an essential prime divisor of \( \{ F \} \) and, in the representation of \( \{ F \} \) as an intersection of essential prime divisors, there is precisely one prime ideal, namely \( \Sigma_1 \), which does not contain \( S \).

16. An interchange of the subscripts of the \( y \) may give \( F \) a new separant. Any such separant involves only derivatives present in \( F \) and is not divisible by \( F \). Hence, for the original ordering of the \( y \), such a separant has a remainder with respect to \( F \) which is not zero. Thus, in (14), \( \Sigma_1 \) contains no separant of \( F \), while \( \Sigma_2, \cdots, \Sigma_r \) contain every separant.\(^{20}\)

We shall call the manifold of \( \Sigma_1 \) the general solution of \( F \), or of the equation \( F = 0 \).

\(^{20}\) It is only in our present work that we use several separatals for a d.p., one for each indeterminate appearing effectively in the d.p. This matter will not cause confusion elsewhere.
Singular zeros and solutions

17. We take \( F \) as in \( \S 12 \). A zero of \( F \) will be called nonsingular if it fails to annul at least one separant of \( F \), and singular if it annihilates every separant. Correspondingly, we speak of nonsingular, and of singular, solutions of \( F = 0 \).

Every nonsingular zero of \( F \) is contained in the general solution of \( F \). The other components of \( F \) are made up of singular zeros.

If a d.p. \( G \) vanishes for all nonsingular zeros of \( F \), then \( G \) is in \( \Sigma_1 \). This is an immediate consequence of the fact that a generic zero of \( \Sigma_i \) is a nonsingular zero of \( F \). In the analytic case, we get a less trivial result. If \( G \) vanishes for all nonsingular analytic zeros of \( F \), \( G \) is contained in \( \Sigma_1 \). This follows from the fact that the product of \( G \) and the separatants holds the restricted manifold of \( F \), therefore the restricted manifold of \( \Sigma_i \). By the theorem of zeros, the product is in \( \Sigma_1 \), so that \( G \) is in \( \Sigma_1 \).

In the analytic case, we call the restricted manifold of \( \Sigma_i \) the restricted general solution of \( F \), and, as a rule, since misunderstandings do not occur, the general solution of \( F \).

The general solution may contain singular solutions of \( F = 0 \), as well as the nonsingular ones. From what precedes, we see that a singular solution belongs to the general solution if, and only if, every d.p. which vanishes for all nonsingular solutions vanishes also for the singular solution. In the analytic case, one uses here only the analytic nonsingular solutions.

18. As \( \Sigma_i \) contains no nonzero d.p. reduced with respect to \( F \), \( F \) is a characteristic set for \( \Sigma_i \). Let \( \Sigma \) be any nontrivial prime ideal (\( \S 5 \)) which has a characteristic set consisting of a single d.p. \( G \). We assume \( G \) to be algebraically irreducible since, if it is not, we can replace it by one of its factors. As \( \Sigma \) consists of those d.p. which have zero remainders with respect to \( G \), the manifold of \( \Sigma \) is the general solution of \( G \). The case in which the number \( n \) of indeterminates is unity is of special interest. For a single indeterminate \( y \), every irreducible manifold distinct from the manifold of the zero ideal is the general solution of a differential polynomial in \( y \).

For \( n > 1 \), this result does not hold. It will be seen, however, in Chapter III, that if \( G \) is any d.p. of positive class, every component of \( G \) is the general solution of some d.p.

19. We consider some examples in the analytic case. In Example 1 of \( \S 4 \), the component \( y = (x + c)^2 \) is composed of nonsingular zeros and is the general solution of \( y^2 - 4y \). In Example 2, \( y = 0 \) is the only singular zero, so that \( M_2 \) is the general solution.

In Example 5, a consideration of the two separatants shows that the singular zeros are those for which \( u = 0 \). We denote the general solution by \( M_1 \). The factor

\[ B = y^2 + u_1y_1 - 2uy \]

in (5) vanishes, for \( u = 0 \), only if \( y = 0 \). As \( u_1 \) in (5) is not divisible by \( A \), \( u_1 \) does not hold \( M_1 \). Thus \( B \) holds \( M_1 \), so that the only zero with \( u = 0 \) which
can belong to $\mathcal{M}_1$ is $u = 0, y = 0$. The zeros of $A$ with $u = 0$ constitute an irreducible manifold, the manifold $\mathcal{M}_2$ of the d.p. $u$. Thus $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. We can now see that $\mathcal{M}_1$ contains the singular zero $u = 0, y = 0$. Let $G$ be any d.p. in $\Sigma$ and $\bar{a}, \bar{y}$, with $\bar{a} \neq 0$, a zero of $A$. For every constant $c \neq 0$, $c\bar{a}, c\bar{y}$ annuls $A$ and is thus in $\mathcal{M}_1$. Thus $G$ vanishes for $c\bar{a}, c\bar{y}$ and hence for $u = 0, y = 0$. This puts $u = 0, y = 0$ in $\mathcal{M}_1$.

For another example of a general solution which contains a singular zero, we consider $A = y_1^2 - 4y^3$, whose manifold is $y = (x + c)^{-2}$ and $y = 0$. The only singular zero is $y = 0$. We see, letting $|c|$ increase, that a d.p. which vanishes for every $(x + c)^{-2}$ vanishes for $y = 0$. Thus $y = 0$ is in the general solution.

20. The above formulation of the concept of the general solution of an algebraic differential equation appears to be the first fully precise one which has ever been given. In the literature in general, the term “general solution” is used in a loose sense. For a differential equation of order $n$, an $n$-parameter family of solutions is called the “general solution.” Some authors are aware that singular solutions should sometimes be considered as belonging to the general solution, but no sharp criterion is given.

It is interesting, however, that a paper on singular solutions published by Lagrange\textsuperscript{ii} in 1774 shows him to have possessed a really good idea of the nature of a general solution. Dealing with an equation

\begin{equation}
V(x, y, \frac{dy}{dx}) = 0,
\end{equation}

he supposes determined for it a one-parameter family of solutions $y = f(x, a)$, which he calls the complete integral. He seeks conditions for a particular (in modern parlance, singular) solution $y(x)$ to be considered as belonging to the complete integral. He furnishes conditions under which $y(x)$ satisfies not only (15), but also “all equations of higher orders which can be derived from it.” The satisfaction of all such higher equations is given as the condition for $y(x)$ to belong to the complete integral. How the higher equations are to be determined is left to be guessed. One is apparently supposed to perform differentiations and eliminations, as in the examples treated by Lagrange. It is proper, however, to credit Lagrange with the possession of a heuristic version of the criterion for membership in the general solution given in §17 above, and to regard his work on singular solutions, like that of Laplace and of Poisson which will be considered in Chapter III, as precursive to the present theory.

**Parametric indeterminates**

21. Let $\Sigma$ be a nontrivial prime ideal in $\mathcal{S}[y_1, \ldots, y_n]$.

There may be some $y$, say $y_j$, such that no nonzero d.p. in $\Sigma$ involves only $y_j$; that is, every d.p. in which $y_j$ appears effectively also involves some $y_i$ with $i \neq j$. If there exist such $y_j$, let us pick one of them, arbitrarily, and call it $u_i$.

There may be a $y$ distinct from $u_i$ such that no nonzero d.p. in $\Sigma$ involves only $u_i$ and the new $y$. If there exist such $y$, we pick one of them and call it $u_2$.

\textsuperscript{ii} Lagrange, 15.
Continuing, we find a set \( u_1, \ldots, u_q \) \((q < n)\), such that no nonzero d.p. of \( \Sigma \) involves the \( u \) alone and such that, given any \( y_j \) not among the \( u \), there is a nonzero d.p. of \( \Sigma \) in \( y_j \) and the \( u \) alone.\(^{22}\)

Let the indeterminates distinct from the \( u \), taken in any order, be represented now by \( y_1, \ldots, y_p \) \((p + q = n)\).

We now list the indeterminates in the order

\[
u_1, \ldots, u_q; \quad y_1, \ldots, y_p.
\]

We shall speak generally as if \( u \) exist. It will be easy to see, in every case, what slight changes of language are necessary when they do not.

Of the nonzero d.p. in \( \Sigma \) involving only \( y_1 \) and the \( u \), let \( A_1 \) be one of least rank. There certainly exist d.p. of \( \Sigma \) of class \( q + 2 \) which are reduced with respect to \( A_1 \); for instance, any nonzero d.p. in \( y_2 \) and the \( u \) is of this type. Of such d.p., let \( A_2 \) be one of least rank.

Continuing, we build a characteristic set of \( \Sigma \),

\[
A_1, A_2, \ldots, A_p.
\]

We shall say that \( A_i \) introduces \( y_i \).

We shall call \( u_1, \ldots, u_q \) a parametric set of indeterminates for \( \Sigma \), or for the manifold of \( \Sigma \).

The Resolvent

22. The investigation which we now undertake will show that every irreducible manifold except that of \( [0] \) may be regarded as a birational\(^{23}\) transform of the general solution of some d.p.\(^{24}\)

Through §23, we shall work with a field \( \mathcal{F} \) which contains at least one nonconstant element.

We present first two lemmas of a special character.

A set of elements \( \eta_1, \ldots, \eta_s \) of \( \mathcal{F} \) will be called linearly dependent if there exists a relation

\[
c_1 \eta_1 + \cdots + c_s \eta_s = 0
\]

where the \( c \) are constant elements of \( \mathcal{F} \), not all zero.

We prove that for \( \eta_1, \ldots, \eta_s \) to be linearly dependent, it is necessary and sufficient that

\[
\begin{vmatrix}
\eta_1 & \cdots & \eta_s \\
\eta'_1 & \cdots & \eta'_s \\
\vdots & \cdots & \vdots \\
\eta_1^{(s-1)} & \cdots & \eta_s^{(s-1)}
\end{vmatrix} = 0,
\]

where superscripts indicate differentiation.

\(^{22}\) It will be seen in §32 that \( q \) does not depend on the particular manner in which the \( u \) are selected.

\(^{23}\) The birational transformations which we use will involve derivatives.

\(^{24}\) A.D.E., Chapter II, and Kolchin, 10. The treatment given here is taken over from Kolchin’s paper.
The proof is conducted as in analysis. For the necessity, we differentiate (18) \( s - 1 \) times. We secure a set of \( s \) homogeneous equations for the \( c \). The determinant must vanish, since there is a solution with some \( c \) distinct from zero. For the sufficiency proof, we proceed by induction. For \( s = 1 \), (19) is evidently sufficient. We treat the case of \( s = r \), supposing earlier cases to have been examined. By (19) the equations

\[
c_{0}\eta_{1}^{(0)} + \cdots + c_{r}\eta_{r}^{(0)}, \quad j = 0, \ldots, r - 1,
\]

are satisfied by elements \( c_{1}, \cdots, c_{r} \) of \( \mathcal{F} \), not all zero. We may evidently suppose that when the last row and last column are suppressed in (19), the resulting determinant is not zero. Then, in (20), \( c_{r} \neq 0 \). We may thus take \( c_{r} \) equal to unity. For \( j \leq r - 2 \), we differentiate (20) and then subtract the equation (20) corresponding to \( j + 1 \).

We find that

\[
c_{0}'\eta_{1}^{(j)} + \cdots + c_{r-1}'\eta_{r-1}^{(j)} = 0, \quad j = 0, \ldots, r - 2,
\]

where accents indicate differentiation. As the determinant of (21) is not zero, the \( c_{i} \) with \( i < r \) are constants. This completes the proof.

We prove now that if \( G \) is a nonzero d.p. in \( \mathcal{F} \{ u_{1}, \ldots, u_{q} \} \), there exist elements \( \mu_{1}, \ldots, \mu_{q} \) in \( \mathcal{F} \) such that \( G \) is not zero for \( u_{i} = \mu_{i}, i = 1, \ldots, q \).

It suffices to treat a d.p. \( G \) in a single indeterminate \( u \). Let \( \xi \) be a nonconstant element in \( \mathcal{F} \). Let \( r \) be any nonnegative integer. We shall prove that if \( G \) is a nonzero d.p. of order not exceeding \( r \), there is an element

\[
c_{0} + c_{1}\xi + c_{2}\xi^{2} + \cdots + c_{r}\xi^{r},
\]

where the \( c \) are constants in \( \mathcal{F} \), which does not annul \( G \). Let this be false, and let \( H \) be a nonzero d.p. of lowest rank which vanishes for every element (22). Let the order of \( H \) be \( s \). We know that \( s \leq r \). It is easy to see that \( s > 0 \).

When \( u \) is replaced in \( H \) by (22) and each \( u_{i} \) with \( 1 \leq i \leq s \) by

\[
c_{0}\xi^{(0)} + c_{1}(\xi^{(0)})' + \cdots + c_{r}(\xi^{(0)})^{r},
\]

the superscript \( (i) \) denoting \( i \) differentiations, \( H \), considered as a polynomial in the indeterminates \( c \), must vanish identically. Its partial derivatives with respect to the \( c \) are thus all zero. We have thus, from \( c_{0}, \cdots, c_{r} \),

\[
\frac{\partial H}{\partial u} = 0,
\]

\[
\frac{\partial H}{\partial u} \xi + \frac{\partial H}{\partial u_{1}} \xi' + \cdots + \frac{\partial H}{\partial u_{s}} \xi^{(s)} = 0,
\]

\[
\frac{\partial H}{\partial u} \xi^{2} + \frac{\partial H}{\partial u_{1}} (\xi^{2})' + \cdots + \frac{\partial H}{\partial u_{s}} (\xi^{2})^{(s)} = 0,
\]

\[
\frac{\partial H}{\partial u} (\xi^{3}) + \frac{\partial H}{\partial u_{1}} (\xi^{3})' + \cdots + \frac{\partial H}{\partial u_{s}} (\xi^{3})^{(s)} = 0.
\]

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In each \( \partial H/\partial u_i \), \( i = 0, \cdots, s \), the substitutions (22), (23) are supposed to be made. We regard equations (24) as equations for the \( \partial H/\partial u_i \). As \( \partial H/\partial u_s \) is of lower rank than \( H \), it does not vanish identically in the \( c \). The determinant of (24) is therefore zero. This means, by the preceding lemma, that there is a relation

\[
a_1 \xi' + a_2 (\xi')' + \cdots + a_s (\xi')' = 0
\]

where the \( a \) are constants in \( \mathcal{F} \), not all zero. Then

\[
a_1 \xi + a_2 \xi' + \cdots + a_s \xi' = a_0
\]

with \( a_0 \) a constant. Thus \( \xi \) satisfies an algebraic equation whose coefficients are in \( \mathcal{F} \) and are not all zero. Let an equation of this type of least degree be

\[
f(\xi) = 0.
\]

Then \( f'(\xi) \xi' = 0 \). As \( f'(\xi) \neq 0 \), we have \( \xi' = 0 \). We reach the contradiction that \( \xi \) is a constant and the lemma is proved.

23. Working in \( \mathcal{F} \{ u_1, \cdots, u_q; y_1, \cdots, y_p \} \), we28 consider a nontrivial prime ideal \( \Sigma \) for which the \( u \) are a parametric set (§21). We are going to show the existence in \( \mathcal{F} \) of elements

\[
(25) \quad \mu_1, \cdots, \mu_p
\]

and the existence of a nonzero d.p. \( G \), free of the \( y \), such that either

(a) there exist no two distinct zeros of \( \Sigma \), contained in a single extension of \( \mathcal{F} \),

\[
(26) \quad u_1, \cdots, u_q; \quad y_1, \cdots, y_p,
\]

\[
\bar{u}_1, \cdots, \bar{u}_q; \quad y_1', \cdots, y_p',
\]

with the same \( u \), which \( u \) do not annul \( G \), or

(b) such pairs of zeros exist and, for each pair,

\[
(27) \quad \mu_1 (y_1' - y_1'') + \cdots + \mu_p (y_p' - y_p'')
\]

is not zero.26

We consider the system \( \Sigma' \) obtained from \( \Sigma \) by replacing each \( y \), by a new indeterminate \( z_i \). Introducing \( p \) more indeterminates \( \lambda_1, \cdots, \lambda_p \), we consider the perfect ideal \( \Omega \) determined by \( \Sigma, \Sigma' \) and

\[
\lambda_1 (y_1 - z_1) + \cdots + \lambda_p (y_p - z_p).
\]

We have thus \( 3p + q \) indeterminates, the \( u, y, z, \lambda \), and we operate in \( \mathcal{F} \{ u; y; z; \lambda \} \).

Let \( \Lambda \) be any essential prime divisor of \( \Omega \). Suppose that not every \( y_i - z_i \),

\[28 \text{ We recall that } \mathcal{F} \text{ is supposed to contain nonconstant elements.}
\[26 \text{ If no } u \text{ exist, this is to mean that, if } \Sigma \text{ has a pair of distinct zeros in a single extension of } \mathcal{F}, \text{ (27) does not vanish for the pair. We take } G = 1 \text{ in this case.} \]
\( i = 1, \ldots, p, \) is in \( \Lambda. \) We shall prove that \( \Lambda \) contains a nonzero d.p. which involves no indeterminates other than the \( u \) and \( \lambda. \)

If \( \Lambda \) contains a d.p. in the \( u \) alone,\(^{27}\) we have our result. Suppose that \( \Lambda \) contains no such d.p.

Since \( \Lambda \) has all d.p. in \( \Sigma, \) \( \Lambda \) has, for \( j = 1, \ldots, p, \) a d.p. \( B_j \) in \( y_j \) and the \( u \) alone. Let \( B_j \) be taken so as to be of as low a rank as possible in \( y_j. \) Then \( S_j, \) the separant of \( B_j, \) is not in \( \Lambda. \)

Similarly let \( C_j, j = 1, \ldots, p, \) be a d.p. of \( \Lambda \) in \( z_j \) and the \( u, \) of as low a rank as possible in \( z_j. \) Letting \( z_j \) follow the \( u \) in \( C_j, \) we see that the separant \( S'_j \) of \( C_j \) is not in \( \Lambda. \)

To fix our ideas, suppose that \( y_1 - z_1 \) is not in \( \Lambda. \) Consider any generic zero of \( \Lambda. \) For it, we have

\[
(28) \quad \lambda_1 = - \frac{\lambda_2 (y_2 - z_2) + \cdots + \lambda_p (y_p - z_p)}{y_1 - z_1}.
\]

From (28) we find, for the \( j \)th derivative of \( \lambda_1 \) in the generic zero, an expression

\[
(29) \quad \lambda_{ij} = \rho_j (\lambda_2, \cdots, \lambda_p; y_1, \cdots, y_p; s_i, \cdots, s_p),
\]

in which \( \rho_j \) is rational in the \( \lambda, y, z \) and their derivatives, with coefficients in \( \mathfrak{f}. \) The denominator in each \( \rho_j \) is a power of \( y_1 - z_1. \)

Let \( B_i \) be of order \( r_i \) in \( y_i \) and \( C_i \) be of order \( s_i \) in \( z_i, i = 1, \cdots, p. \)

If a \( \rho_j \) involves derivatives of \( y_i \) of order higher than \( r_i, \) we can get rid of those derivatives by using their expressions in the derivatives of \( y_i \) of order \( r_i \) or less, found from \( B_i = 0. \) Similarly, we transform each \( \rho_j \) so as to be of order not exceeding \( s_i \) in \( z_i, i = 1, \cdots, p. \)

The new expression for each \( \rho_j, \) which will involve the \( u, \) will have a denominator which is a product of powers of \( y_1 - z_1, S_i, S'_i, i = 1, \cdots, p. \) Let \( g \) be the maximum of the integers \( r_i, s_i. \) Let

\[
h = 2p(g + 1) + 1.
\]

Let \( k \) be the total number of letters \( y_{ij}, z_{ij} \) which appear in the relations (29), transformed as indicated. Then \( h > k. \)

We consider the first \( h \) of the relations (29).\(^{28}\) (That is, we let \( j = 0, 1, \cdots, h - 1. \)) Let \( D, \) an appropriate product of powers of \( y_1 - z_1, \) the \( S_i, S'_i, \) be a common denominator for the second members of these relations. We write

\[
(30) \quad \lambda_{ij} = \frac{E_j}{D}, \quad j = 0, \cdots, h - 1.
\]

Let \( D \) and the \( E_j \) be written as polynomials in the \( k \) letters \( y_{ij}, z_{ij} \) present in them, with coefficients which are d.p. in \( \lambda_2, \cdots, \lambda_p \) and the \( u. \) Let \( m \) be the maximum of the degrees of these polynomials (total degrees in the \( y_{ij}, z_{ij} \)).

---

\(^{27}\) At times the term nonzero will be omitted. One will always know when it is being tacitly employed.

\(^{28}\) When \( j = 0, \) (29) is (28).
Let $\alpha$ represent a positive integer to be fixed later. The total number of distinct power products of degree $m\alpha$ or less, in $k$ letters, is

$$\frac{(m\alpha + k) \cdots (m\alpha + 1)}{k!}$$

(31)

Using (30), let us form expressions for all power products of the $\lambda_{ij}$ in (30) of degree $\alpha$ or less. Let each expression be written in the form

$$\frac{F}{D^\alpha}$$

(32)

Then $F$, as a polynomial in the $y_{ij}, z_{ij}$, will be of degree at most $m\alpha$.

The number of power products of the $h$ letters $\lambda_{ij}$ of degree $\alpha$ or less is

$$\frac{(\alpha + h) \cdots (\alpha + 1)}{k!}$$

(33)

Now (31) is a polynomial of degree $k$ in $\alpha$, whereas (33) is of degree $h$ in $\alpha$. As $h > k$ and as $m, h, k$ are fixed, (33) will exceed (31) if $\alpha$ is large. Let $\alpha$ be taken large enough for this to be realized.

If now the $F$ in (32) are considered as linear expressions in the power products in the $y_{ij}, z_{ij}$, we shall have more linear expressions than power products. Hence the linear expressions $F$ are linearly dependent. That is, some linear combination of the $F$, with coefficients which are d.p. in $\lambda_0, \ldots, \lambda_r$ and the $u$, not all zero, vanishes identically.

The same linear combination of the power products of the $\lambda_{ij}$ will vanish for the generic zero of $\Lambda$ for which (28) was written. Now this last linear combination is a d.p. $H$ in the $u$ and $\lambda$. $H$ is not identically zero, since the power products in the $\lambda_{ij}$ in $H$ are all distinct.

As $H$ vanishes for a generic zero of $\Lambda$, $H$ is in $\Lambda$.

Let $\Lambda_1, \ldots, \Lambda_r$ be the essential prime divisors of $\Omega$. Let $\Lambda_1, \ldots, \Lambda_r$ each not contain some $y_i - z_i$ and let $\Lambda_{s+1}, \ldots, \Lambda_r$ each contain every $y_i - z_i$. Let $H_i$ be a nonzero d.p. in $\Lambda_i$, $i = 1, \ldots, s$, involving only the $u$ and $\lambda$. Let $K = H_1 \cdots H_r$.

Using the second lemma of §22, we replace each $\lambda_i$ in $K$ by an element $\mu_i$ of $\Omega$, in such a way that $K$ reduces to a nonzero d.p. $G$ in the $u$. We shall show that $G$ and the $\mu$ serve as in the statement at the head of this section.

The zeros of $\Omega$ with $\lambda_j = \mu_j$, $j = 1, \ldots, p$, will be the zeros of the $\Lambda_i$ with $\lambda_i = \mu_i$. Now the zeros with $\lambda_j = \mu_j$ of $\Lambda_1, \ldots, \Lambda_r$ have $u$ which annul $G$. The zeros of $\Lambda_{s+1}, \ldots, \Lambda_r$, even with $\lambda_j = \mu_j$, have $y_i = z_i$, $i = 1, \ldots, p$.

Suppose now that $\Lambda_1, \ldots, \Lambda_r$ actually exist. Then there exist distinct pairs (26); the $y'$ can be taken as the $y$ in a zero of some $\Lambda_i$, $i \leq s$, and the $y''$ as the $z$.

For any such pair (26), (27) is zero only if the $u$, $y'$, $y''$ are in a zero, with

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*Perron, Lehrbuch der Algebra, vol. 1, p. 46.*

*We are not supposing here that the $\lambda$ are replaced by the $\mu$.***
\(\lambda_j = \mu_j, j = 1, \ldots, p,\) of some \(\Lambda_i\) with \(i \leq s.\) In that case, \(G\) vanishes for the \(u.\)

When every \(\Lambda_i\) contains every \(y_i - z_i,\) we take \(G = 1, \mu_1 = \cdots = \mu_p = 0.\)

We have thus produced the required \(G\) and \(\mu.\)

24. We shall now relinquish the condition that \(\mathcal{F}\) contain a nonconstant element. Let us assume that parametric indeterminates \(u\) exist. We are going to prove the existence of d.p. \(G, M_1, \cdots, M_p,\) in the \(u\) alone, with \(G \neq 0,\) such that, for two distinct zeros (26) of \(G\) does not vanish,

\[
M_1(y'_1 - y''_1) + \cdots + M_p(y'_p - y''_p)
\]

is not zero.

The discussion of §23 holds through the construction of \(K.\) We are going to prove the existence of d.p. \(M_1, \cdots, M_p,\) in the \(u\) alone, such that when \(\lambda_i\) is replaced by \(M_i\) in \(K,\) the resulting d.p. \(G\) is not identically zero.

Let \(K\) be arranged as a polynomial in the \(\lambda_{ij},\) with d.p. in \(\lambda_0, \cdots, \lambda_p\) and the \(u\) for coefficients. Let \(u_{1k}\) be a derivative of \(u_1\) of order greater than that of any derivative of \(u_1\) which may appear in the coefficients. If \(\lambda_1\) is replaced by \(u_{1k},\) \(K\) becomes a d.p. \(K_1\) in \(\lambda_0, \cdots, \lambda_p\) and the \(u\) which is not identically zero. Similarly, if we replace \(\lambda_3\) by a sufficiently high derivative of \(u_1\) in \(K_1,\) we obtain a nonzero d.p. \(K_2\) in \(\lambda_0, \cdots, \lambda_p\) and the \(u.\) Continuing these replacements, we obtain a nonzero d.p. \(G\) in the \(u\) alone.

Continuing as in §23, we see that the zeros of \(\Omega\) with \(\lambda_j = M_j, j = 1, \cdots, p,\) are the zeros of the \(\Lambda_i\) with \(\lambda_j = M_j.\) Now the zeros with \(\lambda_j = M_j\) of \(\Lambda_1, \cdots, \Lambda_s\) have \(u\) which annul \(G.\) The zeros of \(\Lambda_{s+1}, \cdots, \Lambda_s\) even with \(\lambda_j = M_j,\) have \(y_i = z_i\) for \(i = 1, \cdots, p.\) This proves our statement.

25. The results of §§23, 24 permit us to state that if either

(a) \(\mathcal{F}\) does not consist purely of constants, or

(b) there exist \(u,\)

there exists d.p. \(G, P, Q\) exist in \(\mathcal{F}\{ u_1, \cdots, u_2; y_1, \cdots, y_p \},\) with \(G\) and \(P\) not in \(\Sigma\) and \(G\) free of the \(y,\) such that, for two distinct zeros of \(\Sigma\) in a single extension of \(\mathcal{F},\) with the same \(u,\) the zeros annulling neither \(G\) nor \(P,\) the expression \(Q/P\) has two distinct values. For instance, if (a) holds, we can take \(P = 1\) and \(Q = \mu_1 y_1 + \cdots + \mu_p y_p.\)

The ideas will be more complete, and even simpler, if we use general d.p. \(P.\) The following is a nontrivial case in which \(P\) is of positive class. Let \(\mathcal{F}\) be the totality of rational functions of \(x.\) We take \(\Sigma\) as \(\{ y_1, y_2 \} \) in \(\mathcal{F}\{ y_1, y_2 \}.\) The zeros are \(y_1 = c, y_2 = d\) with \(c\) and \(d\) constant, but otherwise unrestricted. We take \(G = 1.\) If

\[
P = y_1 + xy_2, \quad Q = y_1^2 + x^2,
\]

\(\footnote{The following example shows that \(\Sigma\) may have many zeros with given \(u\) and that a \(G\) may exist such that, for \(G \neq 0,\) there is only one zero for given \(u.\) Let \(\Sigma\) be the perfect ideal generated by \(u_0 y_1 - u_2 \) in \(\mathcal{F}\{ u_0, u_2, y_1 \} \) \(\Sigma\) is prime, since the separant for \(u_2\) is unity. The set \(u_0, u_2\) is parametric. Let \(G = u_0.\) If \(u_0 = u_2 = 0, y_1\) may be taken arbitrarily, but, for given \(u_0, u_2 \) with \(G \neq 0,\) there is only one \(y_1.\)}\]
the expression $Q/P$ assumes distinct values for distinct zeros of $\Sigma$ with $P \neq 0$.\footnote{As usual, we compare only zeros contained in the same extension.}

In certain cases in which $F$ consists purely of constants and in which no $u$ exist, there may exist no pair $P, Q$ as described above. For instance, let $F$ be the totality of complex numbers. Let $\Sigma$ be as in the preceding example. For every zero, the $y_i$ are zero for $j > 0$. We therefore lose no generality in seeking a $P$ and $Q$ of order zero in $y_1$ and $y_2$. For any such $P$ and $Q$, $Q/P$ will yield the same result for infinitely many distinct pairs of constants $y_1, y_2$.

In developing the theory of a prime ideal $\Sigma$ for the case in which $F$ has only constants and in which there are no $u$, two courses are open to us. If we adjoin an element $z$ to $F$, as in I, §29, $\Sigma$ will generate, for the enlarged field, a prime ideal whose theory may be expected to be equivalent to that of $\Sigma$; in the analytic case, the ideals have the same restricted manifold. Again, by I, §27, we can introduce a new indeterminate $u$ and $\Sigma$ will generate a prime ideal in $F\{ u; y_1, \ldots, y_n \}$. After either type of adjunction, the theory which follows will apply.

26. From this point on, through §30, we work with a nontrivial prime ideal $\Sigma$. We assume that either

(a) $F$ does not consist purely of constants, or

(b) parametric indeterminates exist.

We take a triad $G, P, Q$ as in §25. Introducing a new indeterminate, $w$, we let $\Omega$ represent the ideal $\{ \Sigma, Pw - Q \}$ in $F\{ u; y; w \}$. Let $\Omega$ be the totality of those d.p. $G$ in $F\{ u; y; w \}$ which have the property that

$$PG = 0,$$

(\Lambda).

We see immediately that $\Omega$ is an ideal. We shall prove that $\Omega$ is prime.

Let $B$ and $C$ be such that $BC$ is in $\Omega$. For $s$ appropriate, $P^sB$ minus a linear combination of $Pw - Q$ and its derivatives is a d.p. $R$ free of $w$. We obtain similarly, from a $P^sC$, a d.p. $S$ free of $w$. As $RS$ is in $\Omega$, $PRS$ is in $\Lambda$. A generic zero of $\Sigma$ does not annul $P$, and thus furnishes a zero of $\Lambda$. Thus a generic zero of $\Sigma$ annuls $RS$, so that one of $R$ and $S$ is in $\Sigma$. If $R$ is in $\Sigma$, $P^sB$ is in $\Lambda$. Then $B$ is in $\Omega$, so that $\Omega$ is prime.

We notice that those d.p. of $\Omega$ which are free of $w$ are precisely the d.p. of $\Sigma$. In particular, $\Omega$ contains no d.p. in the $u$ alone.

We are going to show that $\Omega$ contains a d.p. in $w$ and the $u$ alone.

Let $B_i, i = 1, \ldots, p$, be a d.p. of $\Sigma$ involving only $y, u_1, \ldots, u_n$, of minimum rank in $y_i$. Let $S_i$ be the separant of $B_i$. Consider any generic zero of $\Omega$. For it, we have

$$w = \frac{Q}{P}.$$

For the $j$th derivative of $w$, we have an expression

$$w_j = \frac{Q_j}{P_j + 1}.$$

\footnote{As usual, we compare only zeros contained in the same extension.}
Using the relations \( B_i = 0 \), we free each \( Q_j \) from those derivatives of each \( y_i \) which are of order higher than the maximum of the orders of \( Q, P \) and \( B_i \) in \( y_i \). Each \( w_j \) will then be expressed as a quotient of two d.p., the denominator being a product of powers of \( P, S_1, \ldots, S_p \). If we use a sufficient number of the relations (35), as just transformed, we will have more \( w_i \) than there are \( y_{ij} \) in the second members. Using the process of elimination employed in §23, we obtain a d.p. \( K \) in \( w; u_1, \ldots, u_q \) which vanishes for a generic zero of \( \Omega \) and is therefore in \( \Omega \).

27. We now list the indeterminates in the order

\[
\begin{align*}
&u_1, \ldots, u_q; w; y_1, \ldots, y_p
\end{align*}
\]

and take a characteristic set of \( \Omega \),

\[
A, A_1, \ldots, A_p.
\]

Here \( w, y_1, \ldots, y_p \) are introduced in succession (§21). The separants for (36) will be represented by \( S, S_1, \ldots, S_p \) and the initials by \( I, I_1, \ldots, I_p \).

If \( A \) is not algebraically irreducible, we can replace it by one of its irreducible factors. We assume therefore that \( A \) is algebraically irreducible.

We are going to prove that \( A_1, \ldots, A_p \) are of order 0 in \( y_1, \ldots, y_p \), respectively and, indeed, that \( A_i \) is of the first degree in \( y_i \). Thus, since, for \( i > j \), \( A_i \) is of lower degree in \( y_j \) than \( A_j \), each equation \( A_i = 0 \) expresses \( y_i \) rationally in terms of \( w; u_1, \ldots, u_q \) and their derivatives.

The determination of the manifold of \( \Sigma \) will in this way be made to depend on the determination of the general solution of \( A = 0 \) (§16), which equation will be called a resolvent of the prime ideal \( \Sigma \), or of the system of equations obtained by equating the d.p. in \( \Sigma \) to zero.

28. Let us suppose that our claim with respect to the \( A_i \) is false and let \( A_k \) be the \( A_i \) of highest subscript for which it breaks down. Thus the \( A_i \) with \( i > k \), if they exist, are of zero order in \( y_{k+1}, \ldots, y_p \) respectively and are linear in those letters. On the other hand, either \( A_k \) is of positive order in \( y_k \), or \( A_k \) is of zero order in \( y_k \) and is not linear in \( y_k \). We shall force a contradiction.

Let \( P_1 \) be the remainder with respect to (36) of \( P \) of §26 and let \( U \) be the remainder with respect to (36) of

\[
P_1 S_k I_i I_{k+1} \cdots I_p.
\]

In \( \mathfrak{F} \{ u_1, \ldots, u_q; w; y_1, \ldots, y_k \} \), let

\[
\Xi = (A, A_1, \ldots, A_k, U).
\]

\( U \) is not zero and is reduced with respect to (36).\(^{22} \) Of all nonzero d.p. in \( \Xi \) which are reduced with respect to (36), let \( B \) be one of a least degree in \( y_{kr} \), where \( r \) is the order of \( A_k \) in \( y_k \). We say that \( B \) is free of \( y_{kr} \).

\(^{22} \) The fact that \( A_{k+1}, \ldots, A_p \) involve \( y_{k+1}, \ldots, y_p \), which do not figure in \( \Xi \), need give no concern. Note that \( U \) is free of those indeterminates.
Suppose that this is not so. Let $C$ be the initial of $B$, that is, the coefficient of the highest power of $y_{kr}$ in $B$. For $m$ appropriate,

$$C^mA_k = DB + E$$

with $D$ of lower degree than $A_k$ in $y_{kr}$, and $E$, if not zero, of lower degree than $B$ in $y_{kr}$. $E$ is in $\mathcal{E}$.

We shall prove that $E$ is in $\Omega$. This is certainly true if $E = 0$. Suppose that $E$ is not zero. Let $F$ be the remainder of $E$ with respect to $A, A_1, \ldots, A_{k-1}$. Then $F$ is in $\mathcal{E}$ and is reduced with respect to (36). If $F$ were not zero, it would be, like $E$, of lower degree than $B$ in $y_{kr}$. Thus $F = 0$ and $E$ is in $\Omega$.

Thus $DB$ is in $\Omega$. $B$ is not, since it is reduced with respect to (36). Then $D$ is in $\Omega$. With $t$ the degree of $D$ in $y_{kr}$, let

$$D = G_0 + G_1y_{kr} + \cdots + G_t y_{kr}.$$ 

As $E$, if not zero, is of lower degree than $A_k$ in $y_{kr}$, the initial of $DB$ is identical with that of $C^mA_k$. Now $C$, reduced with respect to (36), is not in $\Omega$. Thus $G_i$ is not in $\Omega$. It is easy to see that there exist integers $a, a_1, \ldots, a_{k-1}$ such that

$$1 \cdot I_1^{a_1} \cdot I_2^{a_2} \cdots I_{k-1}^{a_{k-1}} G_i \equiv G_i' \pmod{\Omega}, \quad i = 1, \ldots, t,$$

where each $G_i'$ is reduced with respect to $A, A_1, \ldots, A_{k-1}$. We see that $G_i' \neq 0$. Then

$$G_0' + \cdots + G_t' y_{kr}$$

is a nonzero d.p. in $\Omega$ which is reduced with respect to (36). This contradiction proves that $B$ is free of $y_{kr}$.

29. Now let $\Omega'$ be the totality of those d.p. in $\Omega$ which are free of $y_{k_1}, \ldots, y_p$. We see immediately that $\Omega'$ is a prime ideal with $A, \ldots, A_{k-1}$ as a characteristic set.

Let

$$u_i = \tau_i, i = 1, \cdots, q; w = \xi; y_i = \eta_i, i = 1, \cdots, p,$$

be a generic zero of $\Omega$, contained in an extension $\mathfrak{F}_1$ of $\mathfrak{F}$. Then

$$\tau_1, \cdots, \tau_q; \xi; \eta_1, \cdots, \eta_{k-1}$$

is a generic zero of $\Omega'$. We replace $u_1, \cdots, u_q, w, y_1, \cdots, y_{k-1}$ in $A_k$ by the quantities (38). We secure a d.p. $H_k$ in $\mathfrak{F}_1 \setminus \{ y_k \}$.

We examine $H_k$. Let $A_k$ be arranged as a polynomial in the $y_{ki}, i = 0, \cdots, r$, with nonzero coefficients. The coefficients are not in $\Omega$ and hence do not vanish for (38). Thus $H_k$ has the same degree in $y_{kr}$ that $A_k$ has.

Let $H_k$ be expressed as a product of irreducible factors over $\mathfrak{F}_1$ and let $K$ be an irreducible factor which is of order $r$ in $y_k$. Let $\xi_k$ be a generic point (§6) in the general solution of $K$.

---

34. For $k = 1$, we take the remainder of $E$ with respect to $A$. 

For (38), \( B \) becomes a d.p. \( L \) in \( \mathfrak{F}_1 \{ y_k \} \). As \( B \) is not in \( \Omega \), \( L \) is not identically zero. \( L \) is of order less than \( r \) in \( y_k \) if \( r > 0 \) and is an element of \( \mathfrak{F}_1 \) if \( r = 0 \). Thus \( L \) cannot vanish for \( y_k = \xi_k \) (§13). Then \( B \) does not vanish for

\[ (39) \]
\[ \tau_1, \cdots, \tau_p; \xi; \eta_1, \cdots, \eta_{k-1}; \xi_k. \]

As \( A, A_1, \cdots, A_k \) vanish for (39), \( U \) does not. Then (39) annuls none of

\[ P_1, S_k, I_k, \cdots, I_p. \]

The failure of \( I_{k+1}, \cdots, I_p \) to vanish for (39) shows that, when (39) is substituted into an \( A_j \) with \( j > k \), the equation \( A_j = 0 \) determines \( y_j \) as a quantity \( \xi_j \) in the extension of \( \mathfrak{F}_1 \) which contains \( \xi_k \). The quantities

\[ (40) \]
\[ \tau_1, \cdots, \tau_q; \xi; \eta_1, \cdots, \eta_{k-1}; \xi_k, \cdots, \xi_p \]

are seen to constitute a regular zero of (36) which does not annul PG. Thus (40) is a zero of \( \Omega \) and

\[ (41) \]
\[ \tau_1, \cdots, \tau_q; \eta_1, \cdots, \eta_{k-1}; \xi_k, \cdots, \xi_p \]

is a zero of \( \Sigma \) which does not annul PG.

Suppose now that \( r > 0 \). We cannot have \( \eta_k = \xi_k \). Otherwise \( y_k - \eta_k \) would be a d.p. in \( \mathfrak{F}_1 \{ y_k \} \) of lower rank than \( K \) which is annulled by \( \xi_k \). In (41) and in the generic zero of \( \Sigma \)

\[ \tau_1, \cdots, \tau_q; \xi_1, \cdots, \eta_p, \]

we have two zeros of \( \Sigma \) which do not annul PG and which yield the same value \( \xi \) for \( w \). This contradicts the nature of \( G, P, Q \).

We have thus proved that \( A_k \) is of order zero in \( y_k \).

30. The denial made at the beginning of §28 now becomes a claim that \( A_k \) is not linear in \( y_k \). We use the material of §29. As \( \xi_k \) must equal\(^{48} \) \( \eta_k \), \( y_k - \eta_k \) must be divisible by \( K \). This means, if \( H_k \) is of degree \( t \) in \( y_k \), that

\[ (42) \]
\[ H_k = \alpha (y_k - \eta_k)^t \]

where \( \alpha \) is the coefficient of \( y_k^t \) in \( H_k \). Let \( \beta \) be the coefficient of \( y_k^{t-1} \) in \( H_k \).

By (42),

\[ (43) \]
\[ t \alpha \eta_k + \beta = 0. \]

In \( \alpha \) and \( \beta \), we reverse the substitution made to convert \( A_k \) into \( H_k \). Also, in the first member of (43), we replace \( \eta_k \) by \( y_k \). We obtain a d.p.

\[ My_k + N \]

which is in \( \Omega \), since, by (43), it is annulled by (37). As \( \alpha \neq 0 \), \( M \) is not zero.

\(^{48} G \), which is not in \( \Omega \), cannot vanish for the \( r \).

\(^{48} \) Note that the theory of the general solution applies to d.p. which do not involve proper derivatives.
Furthermore, $M$, which is the product by $t$ of a coefficient of $A_k$, is reduced with respect to $A, \ldots, A_k-1$; so is $N$.

We have a final contradiction of the assumption of falsity made in $\S 28$.

Thus, every $A_i$ is linear in $y_i$ and, in the manifold of $\Sigma$, each $y_i$ has an expression rational in $w; u_1, \ldots, u_a$ and their derivatives, with coefficients in $F$.

31. We say that if

$$\bar{a}_1, \ldots, \bar{a}_q; \bar{w}; \bar{y}_1, \ldots, \bar{y}_p$$

is a zero of $\Omega$, then $\bar{a}_1, \ldots, \bar{a}_q; \bar{w}$ belongs to the general solution of $A$.$^{37}$ Let $K$ be any d.p. in $w$ and the $u$ belonging to the prime ideal whose manifold is the general solution of $A$. As the remainder of $K$ with respect to $A$ is zero, $K$ is in $\Omega$ and therefore vanishes for $\bar{a}_1, \ldots, \bar{a}_q; \bar{w}$.

32. The introduction of the resolvent accomplishes the following:

(a) It reduces the study of an irreducible manifold $M$ to the study of the general solution $M'$ of some d.p. The correspondence between $M$ and $M'$ may be described as birational. Of course, in the expression for $w$ in terms of the $y$, and in those of the $y$ in terms of $w$, derivatives may appear. For zeros in $M$ with $P = 0$, there may be no corresponding $w$, and for other zeros in $M$ the initial of some $A_i$ may vanish. For restricted manifolds, we shall gain information on these special zeros in Chapter VI.

(b) It extends into the theory of differential equations a property of systems of algebraic functions of several variables. It is well known that, given a finite system of algebraic functions, we can find a single algebraic function in terms of which, and of the variables, the functions in the system can be expressed rationally.

(c) It furnishes an instrument useful in the treatment of various problems.

**Dimension of an irreducible manifold**

33. Let $\Sigma$ be a nontrivial prime ideal in $F\{y_1, \ldots, y_n\}$ with $F$ any field.

We propose to show that, if parametric indeterminates exist, their number, $q$, does not depend on the manner in which they are selected; in other words, two sets of parametric indeterminates contain the same number of indeterminates.

Let us suppose that a set $u_1, \ldots, u_q$ has been selected, and that one has, in addition, $y_1, \ldots, y_p$. It will suffice to show that, given any $q + 1$ indeterminates among the $u$ and $y$

$$z_1, \ldots, z_q + 1,$$

there exists a d.p. in $\Sigma$ which involves only the $z$.

We form a resolvent for $\Sigma$. As $u$ exist, this is possible. Let us consider a generic zero of $\Omega$. The $z$ in that zero have expressions rational in $w$, the $u$ and their derivatives. If a $z_i$ happens to be a $u$, say $u_i$, the expression for $z_i$ is simply $u_i$. We write

$^{37}$ Here we consider $A$ as a d.p. in $w$ and the $u$ alone.
\[ z_i = \rho_i(w; u_1, \cdots, u_q), \quad i = 1, \cdots, q + 1. \]

On differentiating (45) repeatedly, we get expressions for the \( z_{ij} \) which are rational in \( w \) and \( u_{ij} \). Making use of the relation \( A = 0 \), we transform these expressions so as not to contain derivatives of \( w \) of order higher than \( r \), where \( r \) is the order of \( A \) in \( w \).

Since there are \( q + 1 \) of the \( z \) and only \( q \) of the \( u \), it follows that if we differentiate (45) often enough (and then transform), the \( z_{ij} \) will become more numerous than the \( u_{ij} \) and \( w, w_1, \cdots, w_r \).

It follows as in §23 that there exists a nonzero d.p. in the \( z \) which vanishes for a generic zero of \( \Sigma \). Such a d.p. is in \( \Sigma \).

We shall call the number \( q \) the dimension of \( \Sigma \), or of the manifold of \( \Sigma \). To a nontrivial prime ideal without parametric indeterminates, we attribute the dimension 0. The dimension of \( [0] \) will be defined as \( n \).

From §18, it follows that, for \( \mathcal{S} \{ y_1, \cdots, y_n \} \), every manifold of dimension \( n - 1 \) is the general solution of a differential polynomial.

**Order of the Resolvent**

34. We work with a nontrivial prime ideal \( \Sigma \) of dimension \( q \) in \( \mathcal{S} \{ u_1, \cdots, u_q; y_1, \cdots, y_r \} \), the \( u \) being parametric for \( \Sigma \).\(^{38}\) We suppose that triads \( G, P, Q \), and therefore resolvents, exist. Let

\[ A_1, \cdots, A_p \]

be a characteristic set for \( \Sigma \), the separant and initial of \( A_i \) being \( S_i \) and \( I_i \) respectively. We denote the order of \( A_i \) in \( y_i \) by \( r_i \). Let

\[ h = r_1 + \cdots + r_p. \]

We shall prove that every resolvent of \( \Sigma \) is of order \( h \) in \( w \).\(^{39}\)

We begin by proving that \( \Omega \) contains a d.p. in \( w; u_1, \cdots, u_q \) whose order in \( w \) does not exceed \( h \).

Consider a generic zero of \( \Omega \),

\[ \tilde{u}_1, \cdots, \tilde{u}_q; \tilde{w}; \tilde{y}_1, \cdots, \tilde{y}_r. \]

For it, we have

\[ w = \frac{Q}{P}. \]

We shall show the existence of d.p. \( R \) and \( T \), each of order not exceeding \( r_i \) in \( y_i \), \( i = 1, \cdots, p \), such that, for (47), \( T \) is not zero and

\[ w = \frac{R}{T}. \]

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\(^{38}\) When \( q = 0 \), there are no \( u \).

\(^{39}\) For a brief proof, based on the theory of algebraic fields, see Kolchin, 13.
Let $Q_1$ and $P_1$ be the remainders of $Q$ and $P$ respectively relative to (46). Let $Q_1$ be obtained by subtracting a linear combination of the $A_i$ and their derivatives from

$$S_1^r \cdots I_p^s Q$$

and let $P_1$ be obtained similarly from

$$S_1^{r_1} \cdots I_p^{s_1} P.$$ 

Then, for (47), we have

$$w = \frac{Q_1 S_1^{r_1} \cdots I_p^{s_1}}{P_1 S_1^{r_1} \cdots I_p^{s_1}}.$$ 

For $R$ and $T$ in (49), we take the numerator and denominator in (50).

We find from (49), for the $j$th derivative of $w_j$, an expression

$$w_j = \frac{B_j}{T_j + 1}.$$ 

If $U_j$ is the remainder of $B_j$ with respect to (46), we can write (51)

$$w_j = \frac{U_j}{W_j}$$

where $W_j$ is a product of powers of $T$, $S_1$, $\cdots$, $I_p$.

Consider (49) and the first $h$ of the relations (52). Let $D$ be a common denominator for the second members in these $h + 1$ relations. We write

$$w_j = \frac{E_j}{D},$$

$j = 0, \cdots, h$.

Let $D$, the $E$, and the $A$ in (46) be written as polynomials in the $y_{i,j}$ with coefficients which are d.p. in the $u$. Let $m$ be the maximum of the degrees of these polynomials.

For convenience, we represent the $r_i$th derivative of $y_i$ by $z_i$. Let $A_i$ be of degree $v_i$ in $z_i$.

Let $\alpha$ be a positive integer, to be fixed later. In (53), let us form all power products in the $w_j$ of degree $\alpha$ or less. Let the expression for each power product be written in the form

$$\frac{F}{D^\alpha}$$

Then each $F$ is a polynomial in the $y_{i,j}$ of degree not exceeding $m\alpha$.

Let each expression (54) be written

$$\frac{FI_p^{m\alpha}}{D^\alpha I_p^{\alpha}}.$$ 

Consider a particular $F$, and let it be written as a polynomial in $z_p$. Suppose
that its degree $d$ in $z_p$ is not less than $m$. Then, as $A_p = 0$ for (47), we have, letting

$$M = A_p - I_p z_p^d,$$

the relation

$$I_p z_p^d = -M z_p^{d_{-v_p}}.$$  (56)

If

$$F = J_0 + J_1 z_p + \cdots + J_d z_p^d,$$

with the $J$ free of $z_p$, we may write the numerator in (55) in the form

$$(J_0 I_p + \cdots + J_d I_p z_p^d) I_p^{m_{\alpha - 1}}.$$  (57)

Since $I_p$ is of degree less than $m$ in the $y_{ij}$, each term in the parentheses in (57) is of degree less than $m(\alpha + 1)$.

We replace $J_d I_p z_p^d$ by $-J_d M z_p^{d_{-v_p}}$ in (57). As $J_d$ is of degree not exceeding $m\alpha - d$ in the $y_{ij}$ and as $M$ is of degree at most $m$, then $J_d M z_p^{d_{-v_p}}$ is of degree less than $m(\alpha + 1)$ in the $y_{ij}$. Thus (55) goes over into

$$\frac{F_1 I_p^{m_{\alpha - 1}}}{D^a I_p^{m_{\alpha}}},$$

where $F_1$ is of degree less than $m(\alpha + 1)$ in the $y_{ij}$ and of degree less than $d$ in $z_p$. If the degree of $F_1$ in $z_p$ is not less than $m$, we repeat the above operation. After $t \leq m\alpha$ operations, we get an expression

$$\frac{H I_p^{m_{\alpha-t}}}{D^a I_p^{m_{\alpha}}},$$

with $H$ of degree less than $m$ in $z_p$ and of degree less than $m(\alpha + t)$ in the $y_{ij}$. The numerator in (58) is of degree in the $y_{ij}$ less than

$$m(\alpha + t) + m(m\alpha - t) \leq 2m^2\alpha.$$

Thus, if we let $D_1 = D I_p^a$, we can write each power product in the $w_j$ of degree $\alpha$ or less in the form

$$\frac{K}{D_1^t}$$

where $K$ is of degree less than $2m^2\alpha$ in the $y_{ij}$ and of degree less than $m$ in $z_p$.

We now write each expression (59) in the form

$$\frac{K I_p^{2m\alpha}}{D_1^{2t_{-2}^{-1}}},$$

and employ, with respect to $z_{p-1}$, the procedure used above. We find for each expression (60) an equivalent expression

$$\frac{L}{D_2^t}.$$
with $D_2 = D_1 P_{m-1}^{2m}$ and with $L$ of degree less than $4m^3\alpha$ in the $y_{ij}$ and of degree less than $m$ in $z_p$ and $z_{p-1}$. Continuing, we find, for each power product of the $w_j$, an expression

$$W \quad \frac{\text{D}_2^p}{\text{D}_1^p}$$

where $W$ is of degree less than $2^p m^p + 1\alpha$ in the $y_{ij}$ and of degree less than $m$ in $z_i$, $i = 1, \ldots, p$. Let $c$ represent $2^p m^p + 1$.

The number of power products in $z_i, \ldots, z_p$ of degree less than $m$ in each letter is $m^p$. Thus, as the $y_{ij}$ with $j < r_i$ are $h$ in number, the number of power products of the $y_{ij}$ of degree $c\alpha$ or less, and of degree less than $m$ in each $z_i$, is not more than

$$m^p \frac{(c\alpha + h) \cdots (c\alpha + 1)}{h!}.$$  

(63)

The number of power products of degree $\alpha$ or less in the $h + 1$ letters $w_j$ is

$$\frac{(\alpha + h + 1) \cdots (\alpha + 1)}{(h + 1)!}.$$  

(64)

As (64) is of degree $h + 1$ in $\alpha$ and (63) is only of degree $h$, (64) will exceed (63) for $\alpha$ large. This, we know from §23, implies the existence of a nonzero d.p. of $\Omega$ in $w$ and the $u$ alone, of order not exceeding $h$ in $w$.

This shows that the order in $w$ of the resolvent does not exceed $h$. Suppose that the order of $A$ in $w$ is $k < h$. For (47), we have relations

$$y_i = \frac{C_i}{D_i}$$

(65)

where the $C$ and $D$ are d.p. in $w$ and the $u_i$ of order not exceeding $k$ in $w$. We obtain from (65) expressions for the $y_{ij}$, $j = 0, \ldots, r_i - 1$, which are rational in the $w_j$ and $u_{ij}$ with powers of the $D$ for denominators. Using the relation $A = 0$, we depress the orders in $w$ of the numerators until they do not exceed $k$.

The transformed expressions will have denominators which are power products of $S$ and the $D$.

By an elimination, we obtain a nonzero d.p. $W$ in the $u$ and $y$ which belongs to $\Omega$, hence to $\Sigma$. This $W$, which is of order less than $r_i$ in each $y_{ij}$, is reduced with respect to (46). This is impossible.

We have thus proved that the order in $w$ of every resolvent is $h$.

35. Let $\Sigma$ be as in §34, except that we waive the condition that resolvents exist.

If we consider any $h + 1$ of the $y_{ij}$, the elimination process of §34 shows that $\Sigma$ contains a nonzero d.p. which, in addition to those $y_{ij}$, involves only the $u$ and their derivatives. Thus, if $\mathfrak{M}$ is the manifold of $\Sigma$, there exist $h$ of the $y_{ij}$ such that no algebraic relation among those $y_{ij}$ and any set of $u_{ij}$ holds throughout

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40 The statement which follows is an informal one, whose meaning is clear.
$\mathcal{M}$, whereas, given any $h + 1$ of the $y_{ij}$, an algebraic relation holds throughout $\mathcal{M}$ for those $y_{ij}$ and certain $u_{ij}$.

The quantity $h$ will be called the order of $\Sigma$ (or of $\mathcal{M}$) relative to $u_1, \cdots, u_q$. When $q = 0$, we call $h$ the order of $\Sigma$.

The two numbers $q$ and $h$ measure the extentiveness of $\mathcal{M}$. In the analytic case, we may think of $q$ as the number of arbitrary functions which figure in $\mathcal{M}$, and of $h$ as the number of arbitrary constants at one's disposal when the arbitrary functions are selected. This can be seen from §10.

The relative order depends, as one would expect, on the choice of the $u$. For instance, the manifold of $y_{11} - y_2$ is irreducible. If we let $u_1 = y_2$ we have $h = 1$. If $u_1 = y_1$, $h = 0$.

If $\mathcal{F}$ consists purely of constants and if $\mathcal{F}_1$ is secured from $\mathcal{F}$ by the adjunction of an element $x$ of derivative unity, the prime ideal $\Sigma_1$ of d.p. over $\mathcal{F}_1$ which $\Sigma$ generates has the same parametric sets and the same relative orders as $\Sigma$. This is because a characteristic set of $\Sigma$ is also one of $\Sigma_1$.\(^{41}\)

**Embedded Manifolds\(^{42}\)**

36. **Theorem:** Let $\Sigma$ and $\Sigma'$ be nontrivial prime ideals, with $\Sigma'$ a proper divisor of $\Sigma$, of the respective dimensions $q$ and $q'$. Then $q \geq q'$. If $q = q'$, every parametric set $u_1, \cdots, u_q$ for $\Sigma'$ is such a set for $\Sigma$ and the order of $\Sigma'$ relative to $u_1, \cdots, u_q$ is less than that of $\Sigma$.\(^{43}\)

To show that $q \geq q'$, we observe that $\Sigma$, which is contained in $\Sigma'$, can have no d.p. in the $u_1, \cdots, u_q'$ of a parametric set for $\Sigma'$. Thus we can build a parametric set for $\Sigma$ starting with $u_1, \cdots, u_q'$.

Suppose now that $q = q'$. By the final remark of §35, we may suppose, even if $q = 0$, that resolvents exist for $\Sigma$ and $\Sigma'$.

We can build resolvents simultaneously for $\Sigma$ and $\Sigma'$, using a single relation

$$w = \mu_1 y_1 + \cdots + \mu_p y_p.$$  

The $\mu$ and $G$ which serve for $\Sigma$ will serve also for $\Sigma'$, because the manifold of $\Sigma'$ is part of that of $\Sigma$. For $\Sigma$ we obtain an $\Omega$, and for $\Sigma'$ an $\Omega'$ which is a proper divisor of $\Omega$. Let

$$A, A_1, \cdots, A_p; A', A'_1, \cdots, A'_p$$

be characteristic sets of $\Omega$ and $\Omega'$ respectively. As $A$ is in $\Omega'$, $A'$ is not of higher order in $w$ than $A$. Suppose that $A'$ is of the same order in $w$ as $A$. By §13, $A'$ is divisible by $A'$. The algebraic irreducibility of $A$ and $A'$ implies that $A = cA'$ with $c$ in $\mathcal{F}$. This implies that $A, A_1, \cdots, A_p$ is a characteristic set for $\Omega'$ as well as for $\Omega$. Now a prime ideal is the totality of those d.p. which have zero remainders with respect to one of its characteristic sets. Thus $\Omega'$

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\(^{41}\) A d.p. in $\Sigma$, which is a polynomial in $z$ has coefficients in $\Sigma$.  \(I, \S 29).\)

\(^{42}\) Gourin, 5.

\(^{43}\) If $q = 0$, $\Sigma'$ is of lower order than $\Sigma$.\)
and \( \Omega \) are identical. This contradiction shows that \( A' \) is of lower order in \( w \) than \( A \). The theorem is proved.

When \( q = q' \), not every parametric set for \( \Sigma \) need be such a set for \( \Sigma' \). Let \( \Sigma = \{ y_1, y_2, y_3 \} \). Either \( y_1 \) or \( y_2 \) is a parametric set. If \( \Sigma' = \{ y_1 \} \), \( y_2 \) is parametric and \( y_1 \) is not.

**Prime ideals and field extensions**

37. Let \( \Sigma \) be a nontrivial prime ideal. Let \( \mathfrak{T}_1 \) be an extension of \( \mathfrak{T} \) and \( \Sigma' \) the ideal of d.p. over \( \mathfrak{T}_1 \) which \( \Sigma \) generates. We are going to show that \( \Sigma' \) is perfect and we shall discuss the essential prime divisors of \( \Sigma' \).

Let us suppose first that \( \Sigma \) is of dimension \( q > 0 \), with a parametric set \( u_1, \cdots, u_q \). We build a resolvent for \( \Sigma \), using a d.p.

\[
(66) \quad w = \mu_1 y_1 - \cdots - \mu_p y_p.
\]

Let

\[
(67) \quad A, A_1, \cdots, A_p
\]

be a characteristic set of \( \Omega \), with \( A = 0 \), of order \( r \) in \( w \), a resolvent for \( \Sigma \).

Suppose now that the irreducible factors of \( A \) over \( \mathfrak{T}_1 \) are \( B_1, \cdots, B_s \). Then each \( B_i \) is of order \( r \) in \( w \). Otherwise, the coefficients of the powers of \( w \), in \( A \), having a common factor over \( \mathfrak{T}_1 \), would have one over \( \mathfrak{T} \) and \( A \) would not be algebraically irreducible.

We consider some \( B_j \). Let its general solution have a generic point

\[
(68) \quad \tau_1, \cdots, \tau_q, \xi.
\]

We now examine any \( A_i \) in (67), denoting its initial by \( I_i \). It cannot be that \( I_i \) vanishes for (68). Otherwise \( I_i \) being of order not greater than \( r \) in \( w \), would be divisible by \( B_j \). Thus \( A_i \) and \( I_i \) would have a common factor over \( \mathfrak{T}_1 \), hence one over \( \mathfrak{T} \). This is impossible because \( I_i \) is of lower rank than \( A \). Thus the equation \( A_i = 0 \), when \( w \) and the \( u \) are as in (68), determines \( y_i \) as a quantity \( \eta_i \) in the extension of \( \mathfrak{T}_1 \) which contains \( \mathfrak{T} \).

We consider the quantities

\[
(69) \quad \tau_1, \cdots, \tau_q; \xi; \eta_1, \cdots, \eta_p.
\]

The totality \( \Omega_j \) of d.p. over \( \mathfrak{T}_1 \) which vanish for (69) is easily seen to be a prime ideal. We shall prove that \( \Omega_j \) contains \( \Omega \).

To take care of a point which arises later, let us start with any d.p. \( G \) over \( \mathfrak{T}_1 \). Let \( H \) be the remainder of \( G \) with respect to

\[
A_1, \cdots, A_p.
\]

For some \( a \), if \( S \) is the separant of \( A \),

\[
S^a H = K, \quad [A],
\]

where \( K \) is of order not higher than \( r \) in \( w \).
Suppose now that $G$ is in $\Omega$. Then $K$ is divisible by $A$ and hence by $B_j$. Thus $K$ vanishes for (68). Now $S$ does not vanish for (68); if it did, $S$ would be divisible by $B_j$ and would have a factor over $\mathfrak{F}$ in common with $A$. Then $G$ vanishes for (69) and is in $\Omega_j$. Thus $\Omega_j$ contains $\Omega$.

Let $\Omega'$ be the ideal of d.p. over $\mathfrak{F}_1$ which $\Omega$ generates. Because a d.p. in $\Omega$ goes over into one in $\Sigma$ when $w$ is replaced by $\mu_1 y_1 + \cdots + \mu_p y_p$, those d.p. of $\Omega'$ which are free of $w$ constitute $\Sigma'$. Each $\Omega_j$ contains $\Omega'$. Let $G$ be any nonzero d.p. which is contained in each $\Omega_j$. Let $K$ be found from $G$ as above. Then $K$ is divisible by each $B_j$, and hence by $A$. Thus $S_j H$ is in $\Omega'$. Then some $J G$, with $J = S_j P_1^{a_j} \cdots P_p^{a_j}$, is in $\Omega'$. Let $G$ be written, as in I, §28, in the form

\begin{equation}
\gamma_1 C_1 + \cdots + \gamma_m C_m
\end{equation}

with the $C_i$ d.p. over $\mathfrak{F}$ and the $\gamma$ linearly independent with respect to $\mathfrak{F}$. When we multiply by $J$ in (70), we get a d.p. in $\Omega'$. Hence each $J C_i$ is in $\Omega$. Then each $C_i$ is in $\Omega$ and $G$ is in $\Omega'$. Thus $\Omega'$ is the intersection of the $\Omega_j$.

On this basis, if $\Sigma_j$ is the prime ideal consisting of those d.p. in $\Omega_j$ which are free of $w$, $\Sigma'$ is the intersection of $\Sigma_1, \cdots, \Sigma_p$. Thus $\Sigma'$ is perfect.

No $\Omega_j$ contains any $\Omega_i$ with $i \neq j$; if it did, $B_j$ would be divisible by $B_j$. Now $\Omega_j$ is the ideal of d.p. over $\mathfrak{F}_1$ generated by $\Sigma_j$ and the d.p. in (66). Thus none of the $\Sigma_j$ contains any other, and the $\Sigma_j$ are the essential prime divisors of $\Sigma'$.

Consider some $\Sigma_j$. If it contained a d.p. $G$ in the $u$ alone, $G$ would vanish for the $r$ in (68). Thus each $\Sigma_j$ is of dimension $q$, with the same parametric sets as $\Sigma$. One can see now that $B_j = 0$ is a resolvent for $\Sigma_j$. Thus the order of $\Sigma_j$ relative to any parametric set equals that of $\Sigma$.

Suppose now that $q = 0$. We adjoin a new indeterminate $u$. $\Sigma$ generates a prime ideal $\Lambda$ of d.p. in $u$ and the $y$ (I, §27). $\Lambda$ is of dimension unity, with $u$ as a parametric set and with an order relative to $u$ equal to the order of $\Sigma$.44

Let $\Lambda'$ be the ideal of d.p. over $\mathfrak{F}_1$ generated by $\Lambda$. Then $\Sigma'$ consists of those d.p. in $\Lambda'$ which are free of $u$. Let the essential prime divisors of $\Lambda'$ be $\Lambda_1, \cdots, \Lambda_s$. If $\Sigma_j$ is the prime ideal composed of those d.p. in $\Lambda_j$ which are free of $u$, $\Lambda_j$ contains the prime ideal $\Sigma_j$ in $\mathfrak{F}_1\{u; y_1, \cdots, y_n\}$ generated by $\Sigma_j$. As $\Sigma_j$ is a divisor of $\Sigma'$, $\Sigma_j$ is a divisor of $\Lambda'$. This means that $\Lambda_j = \Sigma_j$. Then no $\Sigma_j$ contains any $\Sigma_i$ with $i \neq j$, and the $\Sigma_j$ are the essential prime divisors of $\Sigma'$.

As the order of any $\Sigma_j$ equals that of $\Lambda_j$ relative to $u$, each $\Sigma_j$ has the same order as $\Sigma$.

We summarize. Let $\Sigma$ be a nontrivial prime ideal of dimension $q$, and $\Sigma'$ the ideal of d.p. over $\mathfrak{F}_1$, an extension of $\mathfrak{F}$, generated by $\Sigma$. Then $\Sigma'$ is perfect and each of its essential prime divisors $\Sigma_j$, $j = 1, \cdots, s$, is of dimension $q$. If $q > 0$, every parametric set for $\Sigma$ is such a set for every $\Sigma_j$ and the orders of the $\Sigma_j$ relative to such a set all equal that of $\Sigma$. If $q = 0$, every $\Sigma_j$ has the same order as $\Sigma$.45

44 A characteristic set of $\Sigma$ is one for $\Lambda$.
45 A.D.E., Chapter VI, and Koelein, 13.
ADJUNCTIONS TO FIELDS

38. Let $\mathcal{F}$ be a field and $\mathcal{F}_1$ an extension of $\mathcal{F}$. Let $\sigma$ be any set of elements of $\mathcal{F}_1$. There exist fields which are contained in $\mathcal{F}_1$ and contain $\mathcal{F}$ and $\sigma$. The intersection of all such fields is a field which will be denoted by $\mathcal{F} < \sigma >$ and will be called the field obtained by the adjunction of $\sigma$ to $\mathcal{F}$. $\mathcal{F} < \sigma >$ consists of all rational combinations of elements of $\sigma$, and of derivatives of such elements, with coefficients in $\mathcal{F}$.

A quantity $\eta$ lying in an extension of $\mathcal{F}$ will be said to be differential with respect to $\mathcal{F}$ if $\eta$ annihilates a nonzero d.p. in one indeterminate over $\mathcal{F}$.

Theorem: Let $\mathcal{F}$ contain a nonconstant element. Let $\eta_1, \cdots, \eta_n$ be elements lying in an extension of $\mathcal{F}$, each differential with respect to $\mathcal{F}$. The field $\mathcal{F} < \eta_1, \cdots, \eta_n >$ contains an element $\xi$ such that

$$\mathcal{F} < \eta_1, \cdots, \eta_n > = \mathcal{F} < \xi >.
$$

Let $\Sigma$ be the set of those d.p. in $\mathcal{F} \{ y_1, \cdots, y_n \}$ which vanish for $y_i = \eta_i$, $i = 1, \cdots, n$. Then $\Sigma$ is a prime ideal of dimension zero. We form a resolvent for $\Sigma$, using a d.p. as in (66) with $p = n$. Let $\xi = \Sigma \mu \eta_i$. Consider the initial $I_i$ of some $A_i$ in (67). If $I_i$ vanished for $\xi$ and the $\eta_i, I_i$ would go over into a d.p. in $\Sigma$ when $w$ is replaced by the sum of the $\mu \eta_i$; thus $I_i$ would be in $\Omega$. It follows that each $\eta_i$ is contained in $\mathcal{F} < \xi >$. This proves the theorem.

ANALOGUE OF LÜROTH'S THEOREM

39. Let $\mathcal{F}$ be any field and $u$ an indeterminate. The totality of rational combinations of the $u_i$, with coefficients in $\mathcal{F}$, is a field which, by §38, it is proper to call $\mathcal{F} < u >$.

We prove the following theorem.

Theorem: Let $\mathcal{F}'$ be any extension of $\mathcal{F}$ which is contained in $\mathcal{F} < u >$. Then $\mathcal{F}'$ contains an element $v$ such that $\mathcal{F} < v > = \mathcal{F}'$.

This theorem is analogous to a well known theorem on algebraic fields which is equivalent to Lüroth's theorem on the parametrization of unicursal curves.

40. Every element of $\mathcal{F} < u >$ can be written in various ways in the form $P/R$ with $P$ and $R$ in $\mathcal{F} \{ u \}$. We shall write $P(u)$ for a d.p. $F$ in $u$, irrespective of the number of derivatives of $u$ which appear in $F$.

Lemma: Let $P$, $Q$, $R$ be in $\mathcal{F} \{ u \}$, with $R$ not zero. Let the relation

$$(71) \quad \frac{P(\eta)}{R(\eta)} = \frac{P(\tau)}{R(\tau)},$$

where $\eta$ and $\tau$ lie in the same extension of $\mathcal{F}$ and do not annul $R$, imply the relation

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46 Kolchin, 12.
47 A.D.E., Chapter VIII, and Kolchin, 12.
48 van der Waerden, Modern Algebra, vol. 1, p. 126.
\[ \frac{Q(\eta)}{R(\eta)} = \frac{Q(\tau)}{R(\tau)}. \]

Then \( Q(u)/R(u) \) is contained in the field obtained by adjoining \( P(u)/R(u) \) to \( \mathfrak{F} \).

If \( Q(u)/R(u) \) is an element of \( \mathfrak{F} \), the conclusion holds in a trivial way. If \( P(u)/R(u) \) is an element of \( \mathfrak{F} \), (71) holds when \( \eta \) and \( \tau \) are indeterminates. Then (72) holds when \( \eta \) and \( \tau \) are indeterminates and \( Q/R \) is in \( \mathfrak{F} \). In what follows, we assume that neither \( P/R \) nor \( Q/R \) is in \( \mathfrak{F} \).

Let \( F \) be a d.p. in \( \mathfrak{F}[u, y, z] \). For the substitution \( y = P(u)/R(u), \ z = Q(u)/R(u) \), \( F \) becomes an element of \( \mathfrak{F}<u> \). There exist d.p. in \( \mathfrak{F}[u, y, z] \), for instance, 0, which vanish for the indicated substitution. The totality \( \Sigma \) of such d.p. is a prime ideal.

We show first that \( \Sigma \) contains no d.p. in \( y \) alone. Let such a d.p. \( G \) exist. Let \( P/R \) be written \( P_1/R_1 \) with \( P_1 \) and \( R_1 \) relatively prime as polynomials in the \( u \). We consider the algebraically irreducible d.p. \( K = P_1(u) - yR_1(u) \). Let \( u = \tau, y = \eta \) be a generic point in the general solution of \( K \). \( R \) is not annulled by \( \tau \). Hence \( G(\eta) = 0 \). This contradicts the fact that no nonzero d.p. in \( y \) alone holds the general solution of \( K \), and our statement is proved. Similarly, \( \Sigma \) contains no nonzero d.p. in \( z \) alone.

A generic zero of \( \Sigma \) satisfies the relations \( y = P(u)/R(u), \ z = Q(u)/R(u) \). With an elimination, we find that \( \Sigma \) contains a d.p. in \( y \) and \( z \) alone.

For the order \( y, z, u \), let \( Z, U \) be a characteristic set of \( \Sigma \) with \( Z \) algebraically irreducible. Here \( y \) is parametric, \( Z \) introduces \( z \) and \( U \) introduces \( u \). We claim that \( Z \) is of order zero in \( z \) and linear in \( z \). The justification of this claim will amount to the proof of our lemma.

We denote the order of \( Z \) in \( z \) by \( r \). Let \( y = \eta, z = \zeta, u = \tau \) be a generic zero of \( \Sigma \). In \( Z, \) we replace \( y \) by \( \eta \), securing a d.p. \( Z_1 \) in \( \mathfrak{F}<\eta> \{ z \} \), of order \( r \) in \( z \). Let \( Z_1 \) be factored in \( \mathfrak{F}<\eta, \zeta, \tau> \), and let \( Z_2 \) be one of those irreducible factors of \( Z_1 \) which are of order \( r \) in \( z \). Let \( \zeta' \) be a generic point in the general solution of \( Z_2 \).

We shall show that \( \eta, \zeta' \), which annuls \( Z \), is a generic point in the general solution of \( Z \). It will suffice to show that \( \eta, \zeta' \) annuls no d.p. \( B \) which is reduced with respect to \( Z \). On the one hand, this will show that \( \eta, \zeta' \) does not annul the separatant of \( Z \), and is therefore in the general solution. On the other hand, it will prove that a d.p. whose remainder with respect to \( Z \) is not zero cannot vanish for \( \eta, \zeta' \). We shall show thus that the only d.p. which vanish for \( \eta, \zeta' \) are those which hold the general solution of \( Z \).

Let \( \eta, \zeta' \) annul a \( B \) as above. By \$28\$, some linear combination \( C \) of \( B \) and \( Z \) is reduced with respect to \( Z \) and free of \( z \). As \( \eta, \zeta' \) cannot annul \( C \), it cannot annul \( B \).

Thus, if we substitute \( \eta, \zeta' \) into \( U \), we obtain a d.p. \( U_1 \) in \( u \) whose order in \( u \) is the same as that of \( U \). Let \( s \) be that common order. We factor \( U_1 \) in \( \mathfrak{F}<\eta, \zeta, \tau, \zeta'> \). Let \( U_2 \) be one of those irreducible factors of \( U_1 \), which are of order \( s \) in \( u \) and let \( \tau' \) be a generic point in the general solution of \( U_2 \).
We say that \( \eta, \xi', \tau' \) is a generic zero of \( \Sigma \). For this, it suffices to show that \( \eta, \xi', \tau' \) annuls no \( C \) which is reduced with respect to \( Z, U \). Given such a \( C \), some linear combination of \( Z, U \), and \( C \) is, by \( \$28 \), reduced with respect to \( Z, U \) and free of \( u_4 \). The proof is now easily completed.

We see now that \( \xi' = \xi \). Otherwise \( \tau \) and \( \tau' \), which do not annul \( R \), would produce the same \( P/R \) and two distinct \( Q/R \).

We find as in \( \$29, 30 \) that \( r = 0 \) and that \( Z \) is linear in \( z \).

41. There exist d.p. in \( \mathfrak{F}' \{ y \} \) which vanish for \( y = u \). For instance, if \( P(u)/R(u) \) is an element of \( \mathfrak{F}' \), we can use

\[
(73) \quad P(y) - \frac{P(u)}{R(u)} R(y).
\]

The totality \( \Sigma \) of such d.p. is a prime ideal. Clearly \( y = u \) is a generic zero of \( \Sigma \). We shall prove that the manifold of \( \Sigma \) is the general solution of a d.p. of the type (73).

We know that the manifold of \( \Sigma \) is the general solution of some d.p. \( B \) (\( \$18 \)).

We suppose each coefficient in \( B \) to be written as the ratio of two d.p. in \( \mathfrak{F}' \{ u \} \). Multiplying \( B \) by a suitable element of \( \mathfrak{F}' \{ u \} \), we obtain a d.p. \( C \) in \( \mathfrak{F} \{ u, y \} \) which is not divisible by any d.p. in \( \mathfrak{F}' \{ u \} \) actually involving one or more \( u_i \). If \( C \) is arranged as a polynomial in the \( y_i \), the ratio of any two of its coefficients will be in \( \mathfrak{F}' \).

42. There must be a pair of coefficients, \( P(u) \) and \( R(u) \), in \( C \), whose ratio is not an element of \( \mathfrak{F}' \). Otherwise, we could secure from \( C \) a d.p. in \( \mathfrak{F}' \{ y \} \) vanishing for \( y = u \). Let

\[
(74) \quad D = R(u)P(y) - P(u)R(y).
\]

We are going to show that \( D \) is the product of \( C \) by an element of \( \mathfrak{F}' \).

Let \( E = D/R(u) \). We consider \( E \) as a d.p. in \( \mathfrak{F}' \{ y \} \). Then \( E \) vanishes for \( y = u \) and so is in \( \Sigma \). Hence, if \( S_i \) is the separant of \( B_i \), there is a relation

\[
(75) \quad S_i E \equiv 0, \quad [B].
\]

From (75), if we represent the separant of \( C \) for the order \( u, y \) by \( S \), we secure a relation

\[
(76) \quad FS^n D \equiv 0, \quad [C],
\]

with \( F \) in \( \mathfrak{F} \{ u \} \). Let

\[
(77) \quad C = G_1 \cdots G_p
\]

be a resolution of \( C \) into factors which are algebraically irreducible in \( \mathfrak{F} \). Each \( G \) involves \( y \). Let \( r \) be the order of \( C \) in \( y \). We say that each \( G \) is of order \( r \) in \( y \).

Suppose that \( G_1 \) is of order \( s < r \) in \( y \). Then, when \( C \) is arranged as a polynomial in \( y \), each coefficient is divisible by \( G_1 \). Let \( B \) above be arranged as a polynomial in \( y_r \). Let \( H_1, \cdots, H_1 \) be the coefficients in \( B \). The \( H \), considered
as polynomials in \( y_1, \ldots, y_{r-1} \), are relatively prime. Hence, there is a relation

\[ M_1H_1 + \cdots + M_sH_s = N \]

where \( N \) and the \( M_i \) are polynomials in \( y_1, \ldots, y_{r-1} \) and where \( N \) is distinct from zero and free of \( y_r \). We can obtain from (78) a relation which shows that the coefficients of the powers of \( y_r \) in \( C \) are not divisible by a d.p. of order \( s \) in \( y \).

No two of the \( G \) in (77) have a ratio which is an element of \( \mathfrak{T} \). Otherwise \( S \) would have a factor in common with \( C \). As above, we see that this is impossible for the reason that \( S_1 \) has no factor in common with \( B \). By §13, no \( G_i \) holds the general solution of a \( G_j \) with \( j \neq i \).

We wish to show that \( D \) holds the general solution of each \( G \). This will follow from (76) if we can show that \( S \) holds no such general solution. For this, we observe first that \( S \) is of order not more than \( r \) in \( u \). As \( S \) has no factor in common with \( C \), \( S \) is not divisible by any \( G \).

Let \( s \) be the order of \( C \) in \( u \). By (74) the order \( s' \) of \( D \) in \( u \) does not exceed \( s \). Let \( G_1, \ldots, G_m \) be those \( G \) which are of order \( s \) in \( u \). As \( s' \leq s \) and as \( D \) holds the general solutions of \( G_1, \ldots, G_m \), it must be that \( s' = s \) and that \( D \) is divisible by each \( G_i \) with \( i \leq m \). Then let

\[ D = KG_1 \cdots G_m. \]

The degree of \( G_1 \cdots G_m \) in \( u_s \) is that of \( C \). As, by (74), \( D \) has a degree in \( u_s \) which does not exceed that of \( C \), \( K \) is of order less than \( s \) in \( u \).

Let \( G_{m+1}, \ldots, G_m \) be those \( G \) which are of order \( s - 1 \) in \( u \). Their general solutions are held by \( D \) but by no \( G_i \) with \( i \leq m \). Thus \( K \) holds the general solutions and is divisible by \( G_{m+1} \cdots G_m \).

If \( C \) and \( D \) are arranged as power products in the \( u_i \) and if such power products are ordered as in I, §22, the highest product in \( D \) will not be higher than that in \( C \). It follows from (77) and (79) that

\[ K = LG_{m+1} \cdots G_m' \]

with \( L \) of order less than \( s - 1 \) in \( u \). Continuing, we find \( D \) to be the product of \( C \) by a d.p. \( M(y) \) in \( \mathfrak{T} y \). \( M \) has to be an element of \( \mathfrak{T} \). Otherwise, \( D \) by its symmetry, would be divisible by \( M(u) \) and \( M(u) \) would be a factor of \( C \).

43. Let \( T \) be the d.p. in \( \mathfrak{T} y \) obtained by dividing \( D \) by \( R(u) \). Then \( T \) is of type (73) and is the product of \( B \) by an element of \( \mathfrak{T} \).

Let \( v = P(u)/R(u) \). We are going to prove that \( \mathfrak{T}' = \mathfrak{T} <v> \).

Let \( U(u)/V(u) \) be any element of \( \mathfrak{T}' \). We shall show that \( U(u)/V(u) \) is in \( \mathfrak{T} <v> \). Let

\[ \text{Perron, \textit{Lehrbuch der Algebra}, vol. 1, p. 204.} \]
\[ \text{The remainder of \( D \) with respect to } G_i \text{ for the order } y, u \text{ is zero.} \]
\[ \text{It is easy now to prove that the } G \text{ are all of order } r \text{ in } u. \]
\[ W = U(y) - \frac{U(u)}{V(u)} V(y). \]

Then \( W \) is in \( \Sigma \). If \( S_1 \) is the separant of \( T \), there is a relation
\[ S_1^2 W = 0, \quad [T]. \]
This gives
\[ X S^2 [V(u)U(y) - U(u)V(y)] = 0, \quad [D], \]
with \( X \) in \( \mathfrak{S} \{ u \} \) and \( S \) the separant of \( D \). Let \( u = \tau, y = \eta \) be a zero of \( D \).
We suppose that \( X(\tau)R(\tau) \neq 0 \). Let \( \tau, \eta \) annul \( S \). Then, if
\[ Y = P'(y)R(y) - R'(y)P(y), \]
where \( P' = \partial P/\partial y, \) and \( R' = \partial R/\partial y, \) \( \eta \) annuls \( Y \). We shall show that \( Y(y) \) is not zero. As \( C \) is not divisible by a d.p. in \( u \) alone, \( D \) is not. Hence \( P \) and \( R \)
are relatively prime. If, for instance, \( R' \) is not zero, \( R' \) is not divisible by \( R \).
Thus \( Y(y) \) is not zero. If \( Y(\eta) \neq 0, \tau, \eta \) annuls \( V(u)U(y) - U(u)V(y) \).

We now write \( P(u)/R(u) \) and \( U(u)/V(u) \) with a common denominator \( RVXY \). Applying the lemma of \( \S 40 \), we find that \( U/V \) is rational in \( P/R \) and its derivatives. This proves the theorem of \( \S 39 \).

44. Let \( w \) be any element of \( \mathfrak{S}' \) such that \( \mathfrak{S} < w > = \mathfrak{S}' \). We seek a relation
between \( w \) and the \( v \) found above. The totality of those d.p. in \( \mathfrak{S} \{ y, z \} \) which
vanish for \( y = v, z = w \) is a prime ideal \( \Sigma \) whose manifold is the general solution
of a d.p. \( F \) (\( \S 33 \)). As \( w \) has an expression rational in \( v \) and its derivatives,
\( \Sigma \) contains a d.p. which is of order zero in \( z \) and linear in \( z \). \( F \) must be such a
d.p. Similarly, \( F \) is of zero order in \( y \) and linear in \( y \). This means that
\[ w = (\alpha v + \beta)/(\gamma v + \delta) \]
where \( \alpha, \beta, \gamma, \delta \) are elements of \( \mathfrak{S} \).

45. From the theorem of \( \S 39 \), it follows that if \( v \) and \( w \) are elements of \( \mathfrak{S} < u > \),
that is, quotients of two d.p. in \( u \), there exists a quotient \( t \) of two d.p. in \( u \) such that
\( v \) and \( w \) are rational in \( t \) and its derivatives while \( t \) is rational in \( v, w \) and
their derivatives. This result parallels Luroth's theorem on unicursal curves.