CHAPTER III
STRUCTURE OF DIFFERENTIAL POLYNOMIALS

I. Manifold of a Differential Polynomial

THEOREM ON DIMENSION OF COMPONENTS

1. If $F$ is an algebraically irreducible d.p. in $\mathcal{F}\{y_1, \ldots, y_n\}$, the dimension of the general solution of $F$ is $n - 1$. One might inquire as to the dimensions of the other components of $F$ (II, §3). This question is answered by the following theorem:

**Theorem:** Let $F$ be a differential polynomial\(^1\) of positive class in $\mathcal{F}\{y_1, \ldots, y_n\}$. Every component of $F$ is of dimension $n - 1$.

From II, §33, it follows that every component of $F$ is the general solution of a differential polynomial.

2. Let the essential prime divisors of $\{F\}$ be $\Sigma_1, \ldots, \Sigma_s$. We have to show that every $\Sigma_i$ is of dimension $n - 1$. Consider some $\Sigma_j$ and let $\eta_1, \ldots, \eta_n$ be a generic zero of $\Sigma_j$. We shall show that $\eta_1, \ldots, \eta_n$ is a zero of some $\Sigma_i$ of dimension $n - 1$. Any such $\Sigma_i$ must be contained in $\Sigma_j$ and must therefore be identical with $\Sigma_j$. This will prove that $\Sigma_j$ is of dimension $n - 1$.

3. We use new indeterminates $z_1, \ldots, z_n$. In $F$, we replace each $y_i$ by $z_i + \eta_i$. Then $F$ goes over into a d.p. $K$ in $\mathcal{F}_0\{z_1, \ldots, z_n\}$ where $\mathcal{F}_0$ is $\mathcal{F}<\eta_1, \ldots, \eta_n>$. $K$ vanishes when each $z_i$ is replaced by 0.

4. Now let $W$ be the sum of the terms of lowest degree in $K$ considered as a polynomial in the $z_{ij}$. Let $V$ be a factor of $W$, algebraically irreducible in $\mathcal{F}_0$. Changing subscripts if necessary, we shall assume that $V$ involves $z_1$ effectively. Let $\xi_1, \ldots, \xi_n$ be a generic point in the general solution of $V$ and let $\mathcal{F}_1$ represent $\mathcal{F}_0<\xi_1, \ldots, \xi_n>$.

**Arbitrary constants**

5. We shall explain now what is to be meant by the term *arbitrary constant*. At each stage of our work we operate in a definite field; thus far we have met $\mathcal{F}$, $\mathcal{F}_0$, $\mathcal{F}_1$. A field having been given, we understand by an arbitrary constant with respect to the field, a quantity $c$ which can be adjoined to the field,\(^2\) which is transcendental with respect to the field (I, §29), and whose derivative is zero.

**The polygon process**

6. We are going to show that $K$ has a zero

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\(^1\) Algebraic irreducibility is not necessary.

\(^2\) That is, $c$ lies in an extension of the field.
\begin{equation}
\begin{aligned}
z_i &= \xi_i c, & i &= 2, \cdots, n, \\
z_1 &= \xi_1 c + \phi_1 c^\rho_1 + \cdots + \phi_k c^\rho_k + \cdots.
\end{aligned}
\end{equation}

Here the \( \phi \) are elements of a field \( \mathfrak{S}' \) which contains \( \mathfrak{S}_1 \) while \( c \) is an arbitrary constant with respect to \( \mathfrak{S}' \). The \( \rho \) are rational numbers with a common denominator; they exceed unity and increase with their subscripts.\(^4\)

7. It may be that \( K \) vanishes for \( z_i = \xi_i c, \ i = 1, \cdots, n \), where \( c \) is an arbitrary constant with respect to \( \mathfrak{S}_1 \). In that case, the \( \xi_i c \) are suitable expressions \((1)\) with \( \mathfrak{S}' = \mathfrak{S}_1 \).\(^4\) In what follows, we assume that such vanishing does not occur.

We put in \( K \)
\begin{equation}
\begin{aligned}
z_i &= \xi_i c, & i &= 2, \cdots, n; \\
z_1 &= \xi_1 c + u_1.
\end{aligned}
\end{equation}

Then \( K \) goes over into an expression \( K' \) which is a polynomial in \( c \) and the \( u_1 \). We may write
\begin{equation}
K' = a'(c) + \sum_{i=1}^{p} b_i'(c) U_i'.
\end{equation}

Here \( a'(c) \) and the \( b_i'(c) \) are polynomials in \( c \) with coefficients in \( \mathfrak{S}_1 \), \( p \) is a positive integer and the \( U_i' \) are power products, of positive degree, in the \( u_1 \). We know that \( a'(c) \) is not zero. We understand that no \( b_i'(c) \) is zero.

8. Let \( \sigma' \) be the least exponent of \( c \) in \( a' \) and \( \sigma'_i \) the least exponent of \( c \) in \( b_i' \).

Let \( d_i \) be the total degree of \( U_i' \). Finally, let
\begin{equation}
\rho_2 = \text{Max} \frac{\sigma'_{i} - \sigma'_i}{d_i}.
\end{equation}

We shall prove that \( \rho_2 > 1 \).

To begin with, if \( d \) is the degree of \( W \) of \( \S 4 \), \( \sigma' > d \) since \( W \) is annulled by the \( \xi \). Under (2), the constituent \( W \) of \( K \) contributes to \( K' \) terms which effectively involve one or more \( u_1 \).\(^5\) The total degree of any such term in \( c \) and the \( u_1 \) is \( d \). Thus, for at least one \( i \) in (3), we have \( \sigma'_i + d_i = d \). As \( \sigma' > d \), we have then \( \sigma'_{i} - \sigma'_i > d_i \). Thus \( \rho_2 > 1 \).

9. Let \( g' \) be the coefficient of \( c^{\sigma'} \) in \( a' \). Let \( h'_i \) denote the coefficient of \( c^{\sigma'_i} \), or denote zero, according as \( (\sigma'_{i} - \sigma'_i)/d_i \) equals \( \rho_2 \) or is less than \( \rho_2 \). Let
\begin{equation}
L'(u_1) = g' + \sum_{i=1}^{p} h'_i U_i'.
\end{equation}

\(^{4}\) The zero (1) will lie in an extension of \( \mathfrak{S}' \). How to use formal infinite series, and how the fractional powers of \( c \) are to be regarded, will be obvious.

\(^{4}\) One sees how to go through the formality of building an extension of \( \mathfrak{S}' \) which contains the \( z_i \) presented.

\(^{5}\) To see this, it suffices to show that \( W \) does not vanish identically for \( z_1 = \xi_1, \cdots, z_n = \xi_n \).

Let \( W \) be arranged as a polynomial in the \( z_1 \). The coefficients are d.p. in \( z_1, \cdots, z_n \) and thus cannot hold the general solution of \( V \), which d.p. involves \( z_1 \).
We consider \( L' \) as a d.p. in \( \mathfrak{F}_1 \{ u_1 \} \). Let \( \{ L' \} \) have \( \Omega_1, \ldots, \Omega_q \) for essential prime divisors. Each \( \Omega_i \) has a generic zero \( \psi_i \) in an extension (depending on \( i \)) of \( \mathfrak{F}_1 \). We select one of the \( \psi_i \) in the following manner.

Let \( L' \) be of effective degree \( f \) in the \( u_{1} \). Then certain partial derivatives of \( L' \), of order \( f \), with respect to the \( u_{1} \) are elements of \( \mathfrak{F}_1 \) distinct from zero. Of all positive integers \( r \) for which there is a \( \psi_i \) which does not annul every partial derivative of \( L' \) of order \( r \), let \( f_1 \) be the least. We choose a \( \psi_i \) which does not annul every partial derivative of order \( f_1 \) and designate it by \( \varphi_2 \). Let \( \mathfrak{F}_2 = \mathfrak{F}_1 < \varphi_2 > \).

10. From now on we understand that \( c \), used above, is an arbitrary constant with respect to \( \mathfrak{F}_2 \). It may be that \( \varphi_2 c^a \) causes \( K' \) to vanish when substituted for \( u_1 \). In that case we have suitable expressions (1) with

\[
z_1 = f_1 c + \varphi_2 c^a.
\]

Let us suppose that the vanishing does not occur.

We make in \( K' \) the substitution

\[
u_1 = \varphi_2 c^a + u_2.
\]

Then \( K' \) goes over into an expression \( K'' \) in \( c \) and \( u_2 \) which may be written

\[
K'' = a''(c) + \sum b''_i(c) U''_i.
\]

Here \( a'' \) and the \( b'' \) are sums in which each term is the product of a rational power of \( c \) and an element of \( \mathfrak{F}_2 \). We know that \( a'' \neq 0 \), and we assume that no \( b'' \) vanishes. The sums \( \sum \) in (3) and in (7) do not necessarily involve the same power products.

Let \( \sigma'' \) be the least exponent of \( c \) in \( a'' \); \( \sigma'_i \) the least exponent in \( b''_i \); \( d_i \) the degree of \( U''_i \). Let

\[
\rho_3 = \text{Max} \left( \frac{\sigma'' - \sigma'_i}{d_i} \right).
\]

We are going to prove that \( \rho_3 > \rho_2 \).

Using an indeterminate \( v \), we replace \( u_1 \) in \( K' \) by \( c^a v \). The \( i \)th term of \( \sum \) in (3) will produce a set of terms, each of the type \( \beta c^\varsigma T \), where \( T \) is a power product in the \( v_i \)\( \# \) \( \beta \) an element of \( \mathfrak{F}_1 \), and where

\[
q \geq \sigma_i + \rho_2 d_i.
\]

By (4), \( q \geq \sigma' \). We will have \( q = \sigma' \) only if \( \beta \) is an \( h' \) in (5). On this basis, we may write

\[
K'(c^a v) = c'^a L'(v) + c'^a M'(c, v),
\]

where, in regard to \( L', M', \tau', \) the following statements apply.

\( L' \) as in (5) with \( u_1 \) replaced by \( v \). \( M' \) is a polynomial in the \( v_i \) with co-

\( T \) is the same in all terms and is merely \( U'_i \) with \( u_1 \) replaced by \( v \).
coefficients which are sums of terms, each the product of a nonnegative rational power of \( c \) by an element of \( \mathfrak{F}_1 \). As to \( \tau' \), which we understand to be taken as large as possible, it is a rational number greater than \( \sigma' \).

We now put \( v = \varphi_2 + c^{\rho_2} u_2 \). Then (10) gives, by (6),

\[
K''(u_2) = c^{\sigma'}L'(\varphi_2 + c^{\rho_2}u_2) + c^{\sigma'}M'(c, \varphi_2 + c^{\rho_2}u_2).
\]

Let \( K' \) be of order \( r \) in \( u_1 \). Suppose that, for some set of nonnegative integers \( l_0, \ldots, l_r \),

\[
\frac{\partial^{b+\cdots+b} L'(u_4)}{\partial^{b} u_{10} \cdots \partial^{r} u_{1r}}
\]

does not vanish for \( u_1 = \varphi_2 \). This implies that at least one \( l \) is positive. Let \( Z = u_{20} \cdots u_{2r} \). We shall prove that \( Z \) is present in \( \Sigma \) in (7) and we shall determine the \( \sigma'' \) associated with that power product.

The coefficient of any \( u_{20} \cdots u_{2r} \) in the second member of (11), whether (12) vanishes for it or not, is the quotient by \( l_0! \cdots l_r! \) of

\[
c^{\sigma'' - \rho_2} L_{i_0}^{\lambda} \cdots L_{i_r}^{\lambda} (\varphi_2) + c^{\sigma'' - \rho_2} M_{i_0}^{\lambda} \cdots M_{i_r}^{\lambda} (c, \varphi_2)
\]

where \( \lambda = l_0 + \cdots + l_r \); \( L_{i_0}^{\lambda} \cdots L_{i_r}^{\lambda} \) is (12) with \( u_1 \) replaced by \( \varphi_2 \); \( M_{i_0}^{\lambda} \cdots M_{i_r}^{\lambda} \) is obtained by the same differentiation and substitution from \( M'(c, u_1) \).

The assumption that (12) does not vanish for \( u_1 = \varphi_2 \) implies that \( Z \) is present in (7). The associated \( \sigma'' \) is given by

\[
\sigma'' = \sigma' - \rho_2 \lambda = \sigma' - \rho_2 d_i.
\]

On the other hand, if \( \varphi_2 \) annihilates (12) and if \( Z \) is present in (7), we have

\[
\sigma'' > \sigma' - \rho_2 d_i.
\]

We can now study \( \rho_2 \) in (8). We have, for every \( i \),

\[
\frac{\sigma'' - \sigma_i''}{d_i} = \frac{\sigma'' - \sigma'}{d_i} + \frac{\sigma' - \sigma_i''}{d_i}.
\]

By (14) and (15) we have

\[
\frac{\sigma' - \sigma_i''}{d_i} = \rho_2
\]

or

\[
\frac{\sigma' - \sigma_i''}{d_i} < \rho_2,
\]

according as (12) with suitable \( l \) does not vanish or does vanish. From (17) and (18) we see that \( (\sigma' - \sigma_i'')/d_i \) is a maximum, namely \( \rho_2 \), for those \( i \) for which (12) does not vanish. Such \( i \) exist, as was seen in connection with the stipulation made in regard to \( \varphi_2 \) in §9.

From (13) with every \( l \) zero, we see now that \( \sigma'' > \sigma' \). Turning now to (16), we see that there are \( i \) for which the first member of (16) exceeds \( \rho_2 \).
This proves that \( \rho_3 > \rho_2 \).

11. We now form for \( K'' \) a d.p. \( L'' \) analogous to \( (5) \) and obtain a zero \( \varphi_3 \) of \( L'' \) in the manner followed for \( \varphi_2 \). In this, we consider \( L'' \) as a d.p. in \( \mathfrak{F}_2 \{ u_2 \} \).

We continue this procedure. It may be that at some stage we reach a
\[ K^{(s-1)} \]
which is annullèd by \( \varphi_e c^{e^s} \). In that case
\[ z_1 = \varphi_1 c + \varphi_2 c^{e^2} + \cdots + \varphi_e c^{e^e} \]
is suitable for \( (1) \). We suppose in what follows that our procedure does not terminate in a finite number of steps, so that we are led to form an infinite series
\[ z_1 = \varphi_1 c + \varphi_2 c^{e^2} + \cdots . \]

We shall then be working in a field \( \mathfrak{F}' \) which is the union of all \( \mathfrak{F}_i \) and we understand \( c \) to be an arbitrary constant with respect to \( \mathfrak{F}' \).

We shall prove that the \( \rho_k \) have a common denominator. This will imply that the \( \rho_k \) become infinite with \( k \). It will be seen also that the \( z_i \) of \( (1) \) annul \( K \).

12. We begin by showing that the degrees of the \( L^{(s)} \) as polynomials in the \( u_{kn} \) do not increase with \( k \). Let us compare the degree of \( L' \) with that of \( L'' \). Let \( f_j \) and \( f_i \leq f_j \), be as in the stipulation of \( \S 9 \) relative to \( \varphi_3 \), with \( f \) the degree of \( L' \).

In \( (16) \), \( (\sigma'' - \sigma')/d_i \) is less for \( d_i > f_j \) than for \( d_i \leq f_i \). On the other hand, \( (\sigma' - \sigma''')/d_i \) obtains its maximum value \( \rho_2 \) for some \( d_i \) equal to \( f_i \). This shows that the first member of \( (16) \) cannot be as great as \( \rho_2 \) for \( d_i > f_i \). Referring now to the description of the coefficients in \( (5) \), which description is similar to that of the coefficients in \( L'' \), we see that the degree of \( L'' \) does not exceed \( f_i \).

Thus there is a positive integer \( e \) such that, for \( k \geq e \), the \( L^{(S)} \) are all of the same degree, say \( m \). Consider any \( k \geq e \), the corresponding \( L^{(2)} \), and the partial derivatives of all orders of \( L^{(2)} \) with respect to the \( u_{kn} \). We shall prove that if \( R \) is any such derivative of order less than \( m \), \( R \) is in \( \{ L^{(S)}(u_{kn}) \} \). Let \( \{ L^{(2)} \} \) have the essential prime divisors \( \Omega_1, \cdots, \Omega_p \). If one refers to the stipulation made in regard to the various \( \varphi_i (\S 9) \), and considers that the degree of \( L^{(2)} \) equals that of \( L^{(3)} \), one sees that \( R \) is annullèd by a generic zero of every \( \Omega_i \). Thus \( R \) is in every \( \Omega_i \) and so in \( \{ L^{(2)} \} \).

Let, now, \( R \) be a partial derivative of \( L^{(2)} \) of order \( m - 1 \), distinct from zero. Then \( R \) is linear in the \( u_{kn} \). Let \( L^{(2)} \) be decomposed into factors over \( \mathfrak{F}_k \) which are algebraically irreducible. Let \( Z \) be an irreducible factor of the same order in \( u_{kn} \) as \( L^{(2)} \). Then \( R \) is in \( \{ Z \} \) so that the remainder of \( R \) with respect to \( Z \) is zero. As the order of \( R \) in \( u_{kn} \) does not exceed that of \( Z \), \( R \) is divisible by \( Z \). As \( R \) is linear, \( Z \) is the product of \( R \) by an element of \( \mathfrak{F}_k \). We have thus, for some \( g \),
\[ L^{(2)} = QR^g \]
where \( Q \) except perhaps in the case in which it is an element of \( \mathfrak{F}_k \) has an order which is less than the common order, call it \( h \), of \( L^{(2)} \) and \( R \) in \( u_{kn} \). We have

\( ^1 \) We work in \( \mathfrak{F}_k \{ u_k \} \).
\[ \frac{\partial^2 L^{(k)}}{\partial u_k} = \mu Q \]

with \( \mu \) in \( \mathfrak{s}_k \). As \( \mu Q \) is reduced with respect to \( R \), it is not in \{ \( R \) \} and thus not in \{ \( L^{(2)} \) \}. It follows that \( g \) in (21) equals \( m \), so that

\[ L^{(k)} = \lambda R^n \]

with \( \lambda \) in \( \mathfrak{s}_k \).

In the expression for \( L^{(k)} \) analogous to (5), there is a term like \( g' \) in (5), free of the \( u_{ki} \). This means that \( R \) has such a term, so that, by (22), \( L^{(b)} \) has terms of the first degree. Thus, in the equation of definition of \( \rho_{k+1} \) analogous to (4), there will be, among those \( i \) which give a maximum, certain \( i \) for which \( d_i = 1 \). In other words, the denominator of \( \rho_{k+1} \) can be taken as the common denominator of \( \sigma^{(k)} \) and the \( \sigma^{(2)} \). For that common denominator, we can use that of \( \rho_2, \ldots, \rho_k \).

This shows that the \( \rho_k \) have a common denominator, so that they approach \( \infty \) with \( k \).

13. We have to show now that the expressions in (1), obtained as above, annul \( K \). Because \( a^{(k)} \), for any \( k > 1 \), is the result of performing in \( K \) the substitutions

\[ z_i = \xi c, \quad i = 2, \ldots, n, \]

\[ z_1 = \xi c + \cdots + \varphi c \alpha, \]

it suffices to show that \( \sigma^{(k)} \) approaches \( \infty \) with \( k \). This is so because the \( \sigma^{(k)} \) increase with \( k \) and have a common denominator.

**Dimensions of components**

14. We see now that \( F \) of our theorem has a zero

\[ y_i = \eta_i + \xi c, \quad i = 2, \ldots, n, \]

\[ y_1 = \eta_1 + \xi c + \varphi \alpha + \cdots. \]

Then (24) is a zero of some \( \Sigma_1 \) of \( \S 2 \), say of \( \Sigma_k \). We shall prove that \( \Sigma_k \) is of dimension \( n - 1 \). It will suffice to prove that \( \Sigma_k \) contains no d.p. in \( y_1, \ldots, y_n \).

Let \( M \) be such a d.p. in \( \Sigma_k \). We replace each \( y_i \) in \( M \) by \( z_i + \eta_i \). Then \( M \) goes over into a d.p. \( N \) in \( \mathfrak{s}_k \{ z_2, \ldots, z_n \} \) which vanishes for \( z_i = \xi c, \ i = 2, \ldots, n \). Let \( P \) be the sum of the terms of lowest degree in \( N \). Then \( P \) vanishes for \( z_i = \xi c, \ i = 2, \ldots, n \). We have here the contradiction that \( P \), which is free of \( z_1 \), holds the general solution of \( V \) of \( \S 4 \).

Then \( \Sigma_k \) is of dimension \( n - 1 \). If, in any d.p. \( M \) of \( \Sigma_k \), we replace each \( y_i \) by its expression in (24), the term free of \( c \) which is obtained is the result of replacing each \( y_i \) in \( M \) by \( \eta_i \). Thus \( \eta_1, \ldots, \eta_n \) is a zero of \( \Sigma_k \). This, as was seen in \( \S 2 \), implies the truth of our theorem.
DEGREES OF GENERALITY

15. Suppose that $F$ is algebraically irreducible, and let $\Sigma_i$, in §2, be the prime ideal associated with the general solution of $F$. Consider any $\Sigma_i$ with $i > 1$. The manifold of $\Sigma_i$ is the general solution of a d.p. $A$. We say that if $y_k$ is an indeterminate effectively present in $A$, the order of $F$ in $y_k$ exceeds that of $A$. This follows from the fact that $F$ is in $\Sigma_i$, so that, if $F$ were not of higher order in $y_k$ than $A$, $F$ would be divisible by $A$ and $\Sigma_i$ would be identical with $\Sigma_i$.

II. Low Powers and Singular Solutions

COMPONENTS

16. Let $F$ be as in §15. We know that the components which $F$ may have in addition to its general solution are general solutions of d.p. $A_1, \ldots, A_p$. There arises the problem of determining the $A$. More than this, one will desire to know whether the $A$ are visible in some way in the structure of $F$.

There will be developed, in Chapter V, a method for determining a finite set of algebraically irreducible d.p. whose general solutions make up the manifold of $F$. However, not all of the general solutions there found need be components of $F$; it may be that some of them are contained in others of them. The problem of selecting the components is identical with that of determining the influence of the components on the structure of $F$. It is best formulated as follows, without requiring the algebraic irreducibility of $F$. Let $F$ and $A$ be d.p. in $R \{ y_1, \ldots, y_n \}$ with $A$ algebraically irreducible. Let $F$ hold the general solution of $A$, that is, let the remainder of $F$ with respect to $A$ be zero. It is required to determine whether the general solution of $A$ is a component of $F$. The solution of this problem is contained in the low power theorem presented below.

PREPARATION PROCESS

17. Let $F$ and $A$ be any two d.p. of class $n$. Let the orders of $F$ and $A$ in $y_n$ be $m$ and $l$ respectively. Let $A_j$ represent the $j$th derivative of $A$, and $S$ the separant of $A$. We shall show the existence of a nonnegative integer $t$ and of a positive integer $r$ such that $S^r F$ has a representation

\[
\sum_{j=1}^{r} C_j A_1^{p_1} A_2^{q_1} A_3^{t_1} \cdots A_m^{t_m-1} F
\]

with nonnegative $p_j$ and $i_{ji}$, where no two of the $r$ sets $i_{1j}, \ldots, i_{m-1, j}$ are identical, the $C_j$ being of orders not exceeding $l$ in $y_n$, and not divisible by $A$.

If $m \leq l$, we express $F$ in the form $C A^p$ with $C$ not divisible by $A$ and we understand this expression of $F$ to be that which is indicated in (25). In what follows, we assume that $m > l$.

We let $z$ represent $y_n$ and we start with the case of $m = l + 1$. Let $F$ be of degree $a$ in $z_{l+1}$. Then $S^r F$ can be written as a polynomial in $S z_{l+1}$ with coefficients whose orders in $z$ do not exceed $l$. Now

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*We are not assuming algebraic irreducibility for $A$. 
\[ A_1 = S_{z_1} + T \]

with the order of \( T \) in \( z \) at most \( l \). Thus \( S^aF \) can be written as a polynomial in \( A_1 - T \), and hence as a polynomial in \( A_1 \), with coefficients whose orders in \( z \) are at most \( l \). If we write each coefficient in the form \( CA^p \), \( p \geq 0 \), with \( C \) not divisible by \( A \), we have a representation (25) for \( S^aF \).

Suppose now that (25) can be produced for \( m < s \) where \( s > l + 1 \). We make an induction to \( m = s \). Let \( F \), of order \( s \) in \( z \), be of degree \( a \) in \( z \). We see as above that \( S^aF \) can be written as a polynomial in \( A_{s-1} \), with coefficients whose orders in \( z \) are less than \( s \). For a sufficiently large positive integer \( b \), the product of any of these coefficients by \( S^b \) will have a representation (25). Thus \( S^b + aF \) has a representation (25).

18. We shall show now that, for any admissible \( t \), (25) is unique. Let \( S^tF \) have two distinct representations (25). By a subtraction, we get a relation

\[ 0 = \sum_{j=1}^{s} D_j A_j^{k_j} \cdots A_{m-1}^{l_1} \]

where the \( v \) sets of exponents are distinct and where the \( D \), distinct from zero and of order no more than \( l \) in \( z \), may be divisible by \( A \).

We have

\[ A_{m-1} = S z_m + T \]

with \( T \) of order less than \( m \) in \( z \). In (26), let us replace \( z_m \) by \((u - T)/S\) where \( u \) is an indeterminate in the customary sense of algebra. Then \( A_{m-1} \) is replaced by \( u \) in (26). Continuing, we see that (26) holds if the \( A_j \) are considered as algebraic indeterminates. This contradicts the fact that the \( D \) are not zero.

19. Suppose now that \( A \) is algebraically irreducible. We see, because \( S \) is not divisible by \( A \), that for two distinct values \( t_1 \) and \( t_2 \) of \( t \), with \( t_2 > t_1 \), (25) is the same except that the \( C \) for \( t_2 \) are those for \( t_1 \) multiplied by \( S^{a-b} \).

By taking \( t \) as small as possible, we are led to a unique expression (25). In all that follows, it will be understood that the smallest admissible \( t \) is used.

When \( A \) and \( F \) are both algebraically irreducible, the smallest \( t \) can be found as follows. If \( S \) is an element of \( F \), we take \( t = 0 \). Otherwise, we first secure (25) with any admissible \( t \) and then determine the highest power \( S^a \) of \( S \) which is a factor of every \( C \). As \( F \) is algebraically irreducible, \( S^t \) must be divisible by \( S^a \). A division by \( S^a \) will thus give the unique representation sought.

**The low power theorem**

20. Let \( F \) and \( A \) be of class \( n \), of the respective orders \( m \) and \( l \) in \( y \), with \( A \) algebraically irreducible. Let \( F \) hold the general solution of \( A \). Then there is no term in (25) which is free of \( A \) and the \( A_t \). Otherwise some \( C \) would hold

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*First proved by the author in paper 31. The analytic sufficiency proof there given is reproduced in Chapter VI. The algebraic sufficiency proof, to be given now, is due to Levi, 17.
the general solution of $A$. This is impossible since the $C$ are of order at most $l$ in $y_n$, and not divisible by $A$. We can now state the

**Low power theorem:** For the general solution of $A$ to be a component of $F$, it is necessary and sufficient that (25) contain a term $C_k A^{p_k}$, free of proper derivatives of $A$, which, if (25) is considered as a polynomial in $A$, $A_1, \cdots, A_{m-1}$, is of lower degree than every other term of (25).

The assumption that $F$ and $A$ are of class $n$ is made only for convenience. Any indeterminate present in $A$ is present in $F$ and may be used as $y_n$.

The low power theorem is very easily remembered for the case of a single indeterminate $y$, with $A = y$. It then becomes: Let $F$, in $\mathfrak{s}\{y\}$, vanish for $y = 0$. For $y = 0$ to be a component of $F$, it is necessary and sufficient that $F$, considered as a polynomial in $y$ and its derivatives, contain a term in $y$ alone, that is, a term free of derivatives of $y$, which is of lower degree than every other term of $F$.

Thus $y = 0$ is a component of $y y_2 - y$, but not of $y y_2 - y_2$ or of $y y_3 - y^2$.

One of the ideas in the sufficiency proof can be seen in a simple example. In $\mathfrak{s}\{y\}$, let $F = y + y y_2, A = y$. We have

$$y + y y_2 = 0, \quad [F].$$

Differentiating, we find

$$y_1 + y y_2 + y_2^2 = 0, \quad [F],$$

$$y_2 + y y_4 + 3 y y_2 = 0, \quad [F].$$

The three congruences may be written

$$(1 + B_{10}) y + B_{11} \quad y_1 + B_{12} \quad y_2 = 0, \quad [F],$$

$$B_{20} y + (1 + B_{21}) y_1 + B_{22} \quad y_2 = 0, \quad [F],$$

$$B_{30} y + B_{31} \quad y_1 + (1 + B_{23}) y_2 = 0, \quad [F],$$

where the $B$ vanish for $y = 0$. The determinant $D$ of the coefficients of $y, y_1, y_2$, in the congruences just written, contains unity as a term and is thus not zero.

If we solve for $y$, we find that

$$y D = 0, \quad [F].$$

Then $yD$ holds $F$. Thus $D$ holds every component of $F$ which $y$ does not. As $D$ does not vanish for $y = 0$, the manifold of $y$ is not part of any larger irreducible manifold held by $F$. This makes $y = 0$ a component of $F$.

The above method can be applied to any d.p. $F$ of the type $y + C$ where the terms of $C$ are of degree at least 2. The $p$th derivative of $F$ contains $y_p$. Now, as is easy to see from a consideration of weights, when $p$ is large each term in the $p$th derivative of $C$ involves a $y_i$ with $i < p$. This leads to a system of congruences of the type met above.

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10 That is, for $j \neq k, p_k < p_l + i_{il} + \cdots + i_{m-l,j}$. If $m = l$, so that (25) has just one term, and that of the type $C_k A^{p_k}$, the condition will be regarded as fulfilled.
For \( F \) of the type \( y^p + C \) with \( p > 1 \), further elements of proof are necessary. These are provided by Levi’s theory of power products, considered in Chapter I.

We shall now treat the general case.

**Sufficiency proof**

21. Using indeterminates \( w, z, u_1, \cdots, u_s \) and the field of rational numbers, we prove the following lemma.

**Lemma:** Let

\[
C = wz^p - \sum_{j=1}^{s} u_j B_j
\]

where \( p \) is a positive integer and the \( B \) are power products, of degree \( p + 1 \), in \( z \) and its derivatives. There exists a relation

\[
z^d (w^p + D) \equiv 0, \quad [C],
\]

with \( d \) and \( s \) positive integers and with each power product in \( D \) of positive degree in the \( z_j \) and of degree \( s \) in the \( w_j, u_{ij} \).

Let \( r \) be the maximum of the weights of the \( B \). If \( r = 0 \), each \( B \) is \( z^{p+1} \) and we have immediately a relation (28) with \( d = p \) and \( s = 1 \).

We suppose now that \( r > 0 \) and refer to I, §21. Let

\[
d = r(p - 1) + 1, \quad t = d(r - 1).
\]

We say that every power product in the \( z_j \) of degree \( d \) and weight not more than \( t \) is contained in \( [z^p] \). If \( p = 1 \), this is a trivial statement. Let \( p > 1 \). In (27) of I, §21, we have, for \( d \) as above, \( a = r, b = 1 \). Then

\[
f(p, d) = t + r + 1
\]

and the truth of our statement follows.

Let \( E_1, \cdots, E_\mu \) be the power products of degree \( d \) and of weight not more than \( t \). Let \( p_j \) be the weight of \( E_j \). Let \( G \) represent \( z^p \). Consider the representation of an \( E_j \) as a linear combination of the derivatives \( G_i \) of \( G \). Each \( G_i \) is homogeneous, of degree \( p \), and isobaric, of weight \( i \). On this basis, we cast out, from the representation of \( E_j \), all \( G_i \) with \( i > p_j \); from the coefficient of a \( G_i \) with \( i \leq p_j \), we cast out all terms which are not of degree \( d - p \) and of weight \( p_j - i \). Thus we write, for \( j = 1, \cdots, \mu ,

\[
E_j = \sum_{k=0}^{p_j} H_{jk} G_k
\]

where \( H_{jk} \) is either zero or else homogeneous, of degree \( d - p \), and isobaric, of weight \( p_j - k \).

By (27),

\[\text{\textsuperscript{11}} \text{The case of } r = 1 \text{ is also trivial.}\]
(30) \[ wG = \sum_{j=1}^{\sigma} u_j B_j, \quad [C]. \]

Representing by \((wG)_k\) the \(k\)th derivative of \(wG\), we have by (30), for \(k = 0, 1, \cdots, \sigma\),

(31) \[ (wG)_k = K_k, \quad [C], \]

where each term in \(K_k\) is of the first degree in the \(u_i\) and of degree \(p + 1\) in the \(z_i\); the \(p + 1\) letters \(z_i\) in each term have a total weight no more than \(r + k\).

By I, §10, we have, for any \(k\),

(32) \[ w^{k+1}G_k = \sum_{i=0}^{k} L_{ki}(wG)_i. \]

We may suppose each \(L_{ki}\) to be a d.p. in \(w\) alone, which is homogeneous, of degree \(k\), and isobaric, of weight \(k - i\). By (29) and (32), we have for \(j = 1, \cdots, \mu\),

(33) \[ w^{t+1}E_j = \sum_{i=0}^{p_j} M_{ji}(wG)_i. \]

An \(M_{ji}\) which is not zero is homogeneous, of degree \(t\), in the \(w_k\) and homogeneous, of degree \(d - p_j\), in the \(z_k\); it is isobaric, of weight \(p_j - i\), in all of its letters. By (31) and (33),

(34) \[ w^{t+1}E_j = \sum_{i=0}^{p_j} M_{ji}K_i, \quad [C]. \]

If \(N\) is a term of some \(K_i\), the total weight of the \(p + 1\) letters \(z_i\) in \(N\) is, as has been noted, no more than \(r + i\).

We shall now write (34), with prompt explanations, in the form

(35) \[ w^{t+1}E_j = \sum c_rP_r, \quad [C]. \]

The sum in (35) depends on \(j\). Each \(c_r\) is a rational number. Each \(P_r\) is a power product, which is of the first degree in the \(u_k\), of degree \(t\) in the \(w_k\) and of degree \(d + 1\) in the \(z_k\). The total weight of \(P_r\) in the \(w_k\) and \(z_k\) is, for some \(i \leq p_j\), not more than

\((p_j - i) + (r + i) = p_j + r \leq t + r.\)

Certainly then, the total weight of the \(d + 1\) letters \(z_k\) in \(P_r\) is no more than \(t + r\).

Working with some \(P_r\), let \(Q\) be the product of the \(d + 1\) letters \(z_k\) in \(P_r\). Let \(z_q\) be the highest derivative of \(z\) in \(Q\) and let \(Q = z_qR\). The weight of \(R\) cannot exceed \(t\). Otherwise, as \(t = d(r - 1)\), some derivative in \(R\) would be of order at least \(r\). We would have \(q \geq r\) and the weight of \(Q\) would exceed \(t + r\). Then \(R\) is one of the \(E_i\).

We may now write (35)
(36) \[ w^t + 1 E_t \equiv \sum_{i=1}^{n} T_{ji} E_i \quad [C], \]
with each nonzero \( T \) homogeneous, of the first degree, in the \( u_{ik} \); homogeneous, of degree \( t \), in the \( w_j \); homogeneous, of the first degree, in the \( z_k \). We write (36)
\[ (T_{11} - w^t + 1) E_1 + T_{12} E_2 + \cdots + T_{1\mu} E_\mu = 0, \quad [C], \]
(37) \[ T_{\mu1} E_1 + T_{\mu2} E_2 + \cdots + (T_{\mu\mu} - w^t + 1) E_\mu = 0, \quad [C]. \]
If \( U \) is the determinant of the system (37),
\[ U E_j \equiv 0, \quad [C], \quad j = 1, \cdots, \mu. \]

We have
\[ U = (-1)^{\mu} w^{t(t+1)} + V \]
where each power product in \( V \) is of positive degree in the \( z_j \) and of degree \( \mu(t+1) \) in the \( w_j \) and \( u_{ij} \). Observing that \( z^d \) is an \( E_j \), we have (28) with \( s = \mu(t+1) \) and \( D = (-1)^{\mu} V \).

22. We now prove a theorem which gives a result somewhat stronger than the sufficiency of the condition in the low power theorem.

**Theorem:** If (25) contains a term \( C_k A^p \) which is the only term in (25) of degree as low as \( p_k \), every component of \( A \) which is held by \( F \), but not by \( C_k \), is a component of \( F \).

Let \( \mathcal{M} \) be a component of \( A \). If \( m = l \), (25) becomes \( F = C_k A^p \) and the theorem holds. Let \( m > l \). We compare (25) and (27). We let \( A \) in (25) correspond to \( z \) in (27), \( C_k \) to \( w \) and \( p_k \) to \( p \). In a term of (25) other than \( C_k A^p \), we take a power product of degree \( p + 1 \) in \( A \) and the \( A_i \) and make it a \( B \) as in (27). Corresponding to (28), there is a relation

(38) \[ A^d (C_k^i + E) = 0, \quad [S^i F], \]

where \( E \) holds \( A \).

Suppose now that \( \mathcal{M} \) is not a component of \( F \) but rather a proper part of a component \( \mathcal{M}' \) of \( F \). Then \( A \) does not hold \( \mathcal{M}' \) so that, by (38), \( C_k^i + E \) holds \( \mathcal{M}' \). As \( E \) holds \( A \), \( C_k \) holds \( \mathcal{M} \) and the theorem is proved.

We note that \( C_k \) does not hold the general solution of \( A \). The question of sufficiency, in the low power theorem, is settled.

23. **Theorem:** If (25) contains a term \( C_k A^p \) which is the only term in (25) of degree as low as \( p_k \), every zero of \( A \) which is contained in a component of \( F \) which is not held by \( A \) is a zero of \( C_k \).

Let the zero \( \eta_1, \cdots, \eta_n \) of \( A \) be contained in a component \( \mathcal{M} \) of \( F \) which is not held by \( A \). By (38), \( C_k^i + E \) holds \( \mathcal{M} \). As \( E \) holds \( A \), \( C_k \) is annulled by \( \eta_1, \cdots, \eta_n \).
Necessity proof

24. Let \( \mathcal{M} \) be the general solution of \( A \). The set \( y_1, \ldots, y_{n-1} \) is parametric for \( \mathcal{M} \) and the order of \( \mathcal{M} \) with respect to \( y_1, \ldots, y_{n-1} \) is \( l \). We shall prove the following theorem, which will settle the question of necessity in the low power theorem.

**Theorem:** Let the terms of lowest degree in \( (25) \) involve proper derivatives of \( A \) and let \( A_h \) be the highest derivative of \( A \) which appears in the terms of lowest degree. Then \( \mathcal{M} \) is not a component of \( F \) and \( \mathcal{M} \) is contained in a component \( \mathcal{M}_1 \) of \( F \) whose order with respect to \( y_1, \ldots, y_{n-1} \) is at least \( l + h \).\(^{12}\)

We shall replace \( y_n \) in \( (25) \) by \( y_n + u_0 \), where \( u_0 \) is an indeterminate, and examine the resulting d.p. in \( u_0 \) and the \( y \). Such a replacement, made in any d.p. \( B \) in the \( y \), of order \( s \) in \( y_n \) will convert \( B \) into

\[
(39) \quad B + B_0 u_0 + \cdots + B_s u_0 + \text{terms of higher degree in the } u_0,
\]

where \( B_i \) is the partial derivative of \( B \) with respect to \( y_{ni} \).

For \( B = A_i \), \( (39) \) will contain the term \( Su_0 \) and for \( B = A_i \), \( (39) \) will contain \( Su_0, i + i \).

Let \( \eta_1, \ldots, \eta_n \) be a generic point of \( \mathcal{M} \). In \( (25) \), we make the substitution

\[
(40) \quad y_i = \eta_i, \quad i = 1, \ldots, n - 1; \quad y_n = \eta_n + u_0.
\]

Then \( S'T' \), as in \( (25) \), goes over into a d.p. \( K \) in \( \mathcal{F}_0 \{ u_0 \} \), where \( \mathcal{F}_0 = \mathcal{F} < \eta_1, \ldots, \eta_n > \).

Each \( C \) in \( (25) \) will produce, under \( (40) \), a nonzero term free of the \( u_{0j} \). As each \( A_i, i = 0, \ldots, m - l, \) vanishes for the \( \eta \), while \( S \) does not, the terms of lowest degree in the \( u_{0j} \) produced by \( A_i \) will be of the first degree and will involve \( u_{0, i + i} \).

From the terms of lowest degree in \( (25) \), we select those which are of a highest degree in \( A_h \). From the terms just taken, we select those which are of a highest degree in \( A_h - 1 \). We continue through \( A_1 \). Our process isolates a single term of \( (25) \)

\[
T = C_1 A_1^{u_1} \cdots A_h^{u_h}.
\]

Under \( (40) \), \( T \) produces a term in \( u_0^{u_0} \cdots u_{0, i + h} \) which is not cancelled. Thus the sum \( W \) of the terms of lowest degree in \( K \), which sum is of positive degree, will be of order \( l + h \) in \( u_0 \).

We are going to find for \( K \) a zero

\[
(41) \quad u_0 = \xi c + \varphi_2 c^{\xi_2} + \cdots + \varphi_k c^{\xi_k} + \cdots
\]

of the type exhibited in \( (1) \).

Let \( V \) be a factor of \( W \), irreducible in \( \mathcal{F}_0 \), which is of order \( l + h \) in \( u_0 \). Let \( \xi \) be a generic point in the general solution of \( V \). It may be that \( K \) vanishes for

\(^{12}\) Note that \( y_1, \ldots, y_{n-1} \) is parametric for \( \mathcal{M} \) if \( \mathcal{M} \) contains \( \mathcal{M} \).
\( u_0 = \xi c \). If so, \( \xi c \) is a suitable series (41). In what follows, we assume that the vanishing does not occur.

We make, in \( K \), the substitution \( u_0 = cv + u_1 \). Then \( K \) goes over into an expression \( K' \), a polynomial in \( c \) and the \( u_{1i} \), which may be written as in (3). The lowest exponent of \( c \) in \( a' \) exceeds the degree of \( W \). \( W \) contributes, to the sum in (3), terms, free of \( c \), whose degree in the \( u_{1i} \) is the degree of \( W \). This justifies us in imagining that it is the present \( K' \), which is being used in §§7–13. We secure thus the zero (41) of \( K \).

25. We have thus a zero of \( S^jF \)

\[
y_i = \eta_i, \quad i = 1, \cdots, n - 1,
\]

\[
y_n = \eta_n + \xi c + \phi_0 c^a + \cdots.
\]

As the \( \eta \) do not annul \( S \), (42) gives a zero of \( F \).

Let the components of \( F \) be general solutions of d.p. \( B_1, \cdots, B_n \). Then (42) is in the general solution of some \( B_i \). To fix our ideas, we suppose that the general solution \( \mathfrak{M}_1 \) of \( B_i \) contains (42). Let \( D \) be any d.p. which holds \( \mathfrak{M}_1 \). Then \( D \) must vanish for the \( \eta \), else it could not vanish for (42). Then \( D \) holds \( \mathfrak{M} \). Thus \( \mathfrak{M}_1 \) contains \( \mathfrak{M} \).

Under (40), let \( B_1 \) go over into a d.p. \( E \) in \( u_0 \). Let \( U \) be the sum of the terms of lowest degree in \( E \). Then \( U \) is annulled by \( \xi \). Hence the order of \( U \) in \( u_0 \) is at least that of \( V \), namely \( l + h \). Thus \( B_1 \) is of order at least \( l + h \) in \( y_n \) so that the order of \( \mathfrak{M}_1 \) with respect to \( y_1, \cdots, y_{n-1} \) is at least \( l + h \). This proves our theorem, and, with it, the necessity of the condition in the low power theorem.

**An example**

26. We consider, in \( \mathfrak{S} \{y\} \), the d.p.

\[
F = B^2 + \prod_{j=1}^{m} (y_1 - y + jy^2)
\]

where \( B = yy_2 + yy_1 - 2y_1^2 \) and \( m \) is any positive integer.

We show first that \( F \) is algebraically irreducible. Suppose that \( F \) has a factor \( G \) free of \( y_2 \). Then \( G \) is a factor of \( y^2 \), the coefficient of \( y_2 \). As \( F \) is not divisible by \( y \), there is no factor free of \( y_2 \). As the equation \( F = 0 \) defines \( y_2 \) as a function of two branches of \( y \) and \( y_1 \), there are no factors of the first degree in \( y_2 \).

Thus the manifold of \( F \) consists of the general solution, \( \mathfrak{M} \), and perhaps, of components held by \( S \), the separant of \( F \). As \( S = 2yB \), and as \( B \) holds \( y \), \( B \) holds the components other than \( \mathfrak{M} \). Thus every zero of \( F \) not in \( \mathfrak{M} \) must annul one of the d.p.

\[
A_j = y_1 - y + jy^2, \quad j = 1, \cdots, m.
\]

We have, for each \( j \), with \( A'_j \) the derivative of \( A \),

\[
B = yA'_j - 2y_1A_j.
\]
The low power theorem shows us immediately that the manifold of each $A_j$ is a component of $F$.

**Further Theorems on Low Powers**

27. Levi obtained a very broad theorem, dealing with systems of d.p., which is essentially a generalization of the low power theorem, at least as far as the question of sufficiency in that theorem is concerned. We consider a special case, which involves a single d.p.

Let

$$F = y^n_1y^n_2\cdots y^n_n + D$$

where the $p_i$ are nonnegative integers whose sum is positive and where $D$ is a d.p. of the following description:

(a) Each of its terms has a degree in the $y_{nk}$ which exceeds $p_1 + \cdots + p_n$.

(b) Given any of its terms, $E$, and any $y_j$, $E$ is either divisible by $y^n_j$ or else of degree higher than $p_j$ in the $y_{jk}$.

It is easy to see, and, in fact, it will be explicitly shown in the course of our work, that, if $p_i > 0$, the manifold of $y_{i}$ is a component of $F$.

We shall prove that the zero $y_i = 0$, $i = 1, \cdots, n$, of $F$ is not contained in any component of $F$ which is not the manifold of some $y_{ik}$ with $p_i > 0$.

We treat first the case in which only one of the $p$, say $p_n$, is positive. We collect those terms of $F$ which are not of degree higher than $p_n$ in the $y_{nk}$; they are all divisible by $y^n_n$. We write

$$F = Gy^n_n + H$$

where each term of $H$ is of degree greater than $p_n$ in the $y_{nk}$. We have $G = 1 + K$ where $K$ is free of $y_n$ and vanishes for $y_i = 0$, $i = 1, \cdots, n$.

We can now apply the low power theorem, taking $A$ as $y_n$. The manifold of $y_n$ is a component of $F$. The zero $y_i = 0$, $i = 1, \cdots, n$, of $y_n$ does not annul $G$. By §23, it cannot lie in any component of $F$ other than the manifold of $y_n$.

Suppose now that the proof has been carried through for the case in which no more than $r$ of the $p$ are positive, where $r < n$. We make an induction to the case in which $r + 1$ of the $p$ are positive.

Let $p_n$ be positive. We use (43). $H$ satisfies (a) and (b) and each of its terms is of degree greater than $p_n$ in the $y_{nk}$. We have

$$G = y^n_1\cdots y^n_{n-1} + K$$

where $K$ satisfies the following two conditions:

(c) Each of its terms is of degree higher than $p_1 + \cdots + p_{n-1}$ in the $y_{ik}$, $i = 1, \cdots, n$.

(d) Given any of its terms, $E$, and any $y_j$ with $j < n$, $E$ is either divisible by $y^n_j$ or else of higher degree than $p_j$ in the $y_{jk}$.

We write (43)
\[ F = G y^n + L_1 M_1 + \cdots + L_p M_p \]

where the \( M \) are power products of degree \( p + 1 \) in the \( y^{nk} \) and where the \( L \), like \( K \), satisfy (d) above.

By §21, there exists a relation

\[ y_n^s (G^s + N) = 0, \quad [F], \]

where \( N \) is a homogeneous polynomial, of degree \( s \), in derivatives, proper or improper, of \( G \) and the \( L \), with coefficients which vanish when \( y_n = 0 \).

Suppose now that the zero \( y_i = 0, i = 1, \cdots, n \), lies in a component \( \mathfrak{M} \) of \( F \) other than the manifold of \( y_n \). Then \( G^s + N \) holds \( \mathfrak{M} \). Now \( G^s + N \) is of the form

\[ y_1^{p_1} \cdots y_n^{p_n-1} + P, \]

where \( P \) satisfies (a) and (b) above if, in those statements, \( p_n \) is taken as zero and each \( p_i \) with \( i < n \) is replaced by \( s p_i \). By the earlier cases, \( \mathfrak{M} \) is held by some \( y_i \) with \( i < n \) and is thus the manifold of such a \( y_i \). The result is established.

28. In \( \mathfrak{S} \{ y \} \), let

\[ F = y^q y_1^p + D, \]

where \( p > 0, q > 0 \), and each term of \( D \) is of degree greater than \( q \) in proper derivatives. The manifold of \( y_1 \) is a component of \( F \). The only point which this manifold can have in common with other components is \( y = 0 \).

Suppose now that each term of \( D \) is of degree greater than \( p + q \) in \( y \) and proper derivatives. We shall show that the only component of \( F \) which contains \( y = 0 \) is the manifold of \( y_1 \).\(^{13}\)

We find readily that

\[ y_1^q (y^{p^*} + N) = 0, \quad [F], \]

where each term of \( N \) is of degree greater than \( p s \).

If \( y = 0 \) were in a second component, \( \mathfrak{M} \), of \( F \), \( \mathfrak{M} \) would be held by \( y^{p^*} + N \). That d.p. has \( y = 0 \) as a component.

29. In \( \mathfrak{S} \{ y \} \), let

\[ (44) \quad F = y^p + D \]

with \( p \) positive and less than the degree of any term of \( D \). There exists a relation

\[ (45) \quad y^d (1 + N) = 0, \quad [F], \]

where \( N \) vanishes for \( y = 0 \). We are interested in the least value of \( d \) for which it is possible to have a relation \( (45) \).

It is easy to see that \( d \) cannot be less than \( p \). If \( F \) is of positive order, the

\(^{13}\) Levi, 17, where a more general result is secured.
work of §21 gives \( r(p - 1) + 1 \), with \( r \) the greatest of the weights of the terms of \( D \), as an employable value of \( d \). If \( p = 1 \), we can thus take \( d \) as unity. It is not possible to take \( d \) as \( p \) for every \( p \). For instance, let \( F = y^3 + y_1^4 \). Suppose that we have a relation

\[
y^3(1 + N) = MF + M_4F' + \cdots + M_4F^{(s)}.
\]

For the second member of (46) to have \( y^3 \) as one of its terms, it is necessary for \( M \) to have unity as a term. Then \( MF \) has \( y_1^4 \) as a term. Equating terms of degree 4 and weight 4 for both sides of (46), we find \( y_1^4 \equiv 0, \) \([y^3]\), which is easily shown to be false.

We now let \( A \) represent \( y^p \) and \( A_j \) the \( j \)th derivative of \( A \). Suppose that, for some \( m \geq 0 \),

\[
F = A + \sum_{i=0}^{m} M_i A_i,
\]

where each \( M \) vanishes for \( y = 0 \). We are going to show that \( d \) may be taken as \( p^{14} \).

We assume, as we may, that no \( M \) is zero. We may write, on the basis of (47) with a suitable range for \( i \) and \( j \),

\[
A = \sum_{i,j} C_{i,j} y_i A_j, \quad [F],
\]

in which we understand that no \( C \) is zero. If, in the second member of (48), each \( A_j \) is replaced by the \( j \)th derivative of the second member, there results a congruence

\[
A = \sum_{i,j,k} C_{i,j} y_i A_j A_k, \quad [F].
\]

For each \( C_{i,j} \) in (48), we consider \( i + j \). Let \( r \) be the maximum of these sums. Then, in (49), no \( i + j + k \) can exceed \( 2r \). If the substitution just made is carried out \( s - 1 \) times, we find a congruence

\[
A = \sum C y_k y_\nu \cdots y_\nu A_\nu A_\nu, \quad [F],
\]

\( C \) depending on the \( i \). No sum \( i_1 + \cdots + i_{s+1} \) can exceed \( sr \). By §21, if \( s = (r + 1)(p - 1) + 1 \), every \( y_k \cdots y_\nu \) will be in \([A]\). We have thus a congruence

\[
A - \sum D_i A_i A_j = 0, \quad [F].
\]

Let \( L \) represent the first member of (50). We know from what precedes that there is a relation

\[
A(1 + N) = 0, \quad [L],
\]

with \( N \equiv 0, [A] \). Q.E.D.

\(^{14}\) Levi, 17.
Terms of lowest degree

30. We prove the following theorem.

**Theorem:** Let \( A \) and \( B \) be nonzero d.p. in \( y_1, \ldots, y_n \). Let \( B \) hold \( A \). Let \( A_i \) be the sum of the terms of lowest degree in \( A \) considered as a polynomial in the \( y_{ij} \) and let \( B_i \) be the corresponding sum for \( B \). Then \( B_i \) holds \( A_i \).

A similar result holds for the terms of highest degree.

31. **Remark.** The simplest case is that in which \( B = 0, [A] \). One might expect to have then \( B_i = 0, [A_i] \). We shall show by means of an example that this need not be so. Let, in \( \mathfrak{F} \{ y \} \),

\[
A = y_1^2 + y^2; \quad B = 2y_2A - y_1A' = 2y^3y_2 - 3y^2y_1^2.
\]

Then \( A_1 = y_1^3, B_1 = B \). If we had \( B_1 = 0, [A_1] \), it would follow that \( y^2y_2 = 0, [A_1] \). The derivatives of \( y_1^3 \) have weights which exceed 2. Thus \( y_2y_2 \) would have to be a multiple of \( y_1^3 \). This proves our statement. From the expression of \( B \) in terms of \( A \), one might now conjecture that some power of \( B_1 \) is linear in \( A_1 \) and the first derivative of \( A_1 \). In that case, some power of \( y^2y_2 \) would be such a linear combination. This is impossible since \( y^2y_2 \) is not divisible by \( y_1 \). Actually, the cube of \( B_1 \) is linear in \( A_1 \) and its first two derivatives.

32. We enter into the proof. If \( A_1 \) is free of the \( y_i, B_1 \) certainly holds \( A_1 \). In what follows, we assume that the terms of \( A_1 \) are of positive degree. Then \( A_1 \) vanishes for \( y_i = 0, i = 1, \ldots, n \).

We shall prove the permisibility of assuming that \( A_1 \) contains a term involving only the \( y_{ij} \). Let \( z_i, \ldots, z_n \) and \( w_1, \ldots, w_n \) be indeterminates. Let \( y_i \), for \( i > 1 \), be replaced in \( A_1 \) by \( z_i + w_i \). Then \( A_1 \) goes over into a d.p. \( C \) in \( y_1 \), the \( z \) and \( w \). \( C \) contains terms free of the \( z_{ij} \); the sum \( D \) of such terms is found by substituting \( w_i \) for \( y_i \) in \( A_1 \) for \( i > 1 \). Let \( t \) be an integer which exceeds the order of \( D \) in \( y_1 \). On putting \( w_2 = y_1t \) in \( D \), we convert \( D \) into a non-zero d.p. \( D_1 \) in \( y_1, w_2, \ldots, w_n \). We now replace \( w_2 \) in \( D_1 \) by \( y_1t_2 \), where \( t_2 \) exceeds the order of \( D_1 \) in \( y_1 \). Continuing, we find a substitution

\[
y_i = z_i + y_1t_i, \quad i = 2, \ldots, n,
\]

which converts \( A_1 \) into a d.p. \( E \) in \( y_1 \) and the \( z_i, E \) possessing terms free of the \( z_{ij} \). The terms of \( E \) will have the same degree as those of \( A_1 \).

The substitution (51) may be applied to \( A \) and \( B \) and will give a situation in which \( E \) takes the place of \( A_1 \). This proves the legitimacy of the assumption described above, and, in what follows, \( A_1 \) will be understood to have terms involving only the \( y_{ij} \).

Now let \( \xi_1, \ldots, \xi_n \) be any zero of \( A_1 \). We wish to show that \( A \) has a zero

\[
y_i = \xi_i e, \quad i = 2, \ldots, n,
\]

(52)

\[
y_1 = \xi_1 e + \varphi e^m + \cdots,
\]

of the familiar type. If \( A \) vanishes for \( y_i = \xi_i e, i = 1, \ldots, n \), we have (52).
Otherwise, we replace each \( y_i \) with \( i > 1 \) in \( A \) by \( x_i c \) and \( y_i \) by \( x_i c + u_i \). Then \( A \) goes over into an expression \( K' \) in \( c \) and \( u_i \) which may be written as in (3). The lowest exponent of \( c \) in \( a' \) exceeds the degree of \( A_1 \). Also, because \( A_1 \) has terms in \( y_i \) alone, \( A_1 \) contributes to the sum in (3) terms, free of \( c \), whose degree in the \( u_{ij} \) equals the degree of \( A_1 \). The discussion of §§7–13 thus holds for the present \( K' \), and we have the zero (52) of \( A \).

As (52) annuls \( B \), the \( z \) annul \( B_1 \). The theorem is proved.

33. The case of the terms of highest degree, mentioned in §30, is perhaps most conveniently treated as follows. Let \( A_1 \) and \( B_1 \) be the sums of the terms of highest degree in \( A \) and \( B \) respectively. Using indeterminates \( u; z_1, \ldots, z_n \), we put in \( A \) and \( B \)

\[
y_i = z_i / u^a, \quad i = 1, \ldots, n.
\]

We have then

\[
A = C / u^m, \quad A_1 = C_1 / u^m,
\]

\[
B = D / u^m, \quad B_1 = D_1 / u^m,
\]

with \( m \) a positive integer and \( C, C_1, D, D_1 \) d.p. in \( u \) and the \( z \). \( C_1 \) and \( D_1 \) will be the sums of terms of least degree in \( C \) and \( D \). Because \( B \) holds \( A_1 \), \( uD \) holds \( C_1 \).

By what precedes, \( uD_1 \) holds \( C_1 \). Because every zero of \( A_1 \) yields zeros of \( C_1 \) with \( u \neq 0 \), \( B_1 \) holds \( A_1 \).

**Singular solutions**

34. In studying the components of a d.p. \( F \), and in examining the manner in which they make themselves visible in the structure of \( F \), we have thus far had no need to assume \( F \) algebraically irreducible. For a closer examination of the components, algebraic irreducibility is important for \( F \), and accordingly we assume it.

As we saw in Chapter II, the discussion of the manifold of \( F \) is allied to the study of the singular solutions of \( F = 0 \). The general solution of \( F \) contains all nonsingular solutions and sometimes, in addition, some or all singular solutions. If there are other components, they are made up of singular solutions.

The problem of singular solutions has two aspects. On the one hand, one will wish to know how the singular solutions are distributed among the components of \( F \). On the other, one will, in the analytic case, desire to know how the singular solutions are related analytically to the nonsingular ones. For instance, singular solutions may be envelopes of nonsingular solutions, or may be embedded among them in an interesting way.

35. Let us examine the first question. With what we already know of the components and with what will be developed in Chapter V, we shall be able to produce a set of d.p.

\[
F, A_1, \ldots, A_p
\]

whose general solutions are the components of \( F \). The general solution of a d.p.
in (54) contains all nonsingular zeros of the d.p. One will wish to determine the singular zeros which are contained in the general solution. If one has done this, and thus knows the nature of each component of $F$, one may be interested in determining the intersection of two or more components; this is a matter of finding the intersection of the general solutions of two or more d.p.

If it were possible effectively to construct bases for the various essential prime divisors of the perfect ideal determined by a given finite system of d.p., the above questions would be answered. For instance, we could get a finite system of d.p. whose manifold is the general solution of $F$ and, after that, a finite system whose manifold consists of the singular zeros in the general solution.

The problem of determining bases for the prime divisors is at present far from being solved.\footnote{A theoretical solution of the problem is presented in Chapter V. This solution is incomplete in that one does not know how far the process used in the solution must be carried to be effective.} It is thus a matter, at this time, of treating special differential equations with such methods as one can devise.

For the case of a single indeterminate, the problem of the singular zeros in a general solution, and that of the intersection of components, reduce to the following problem: \textit{Given two algebraically irreducible d.p. in $y$, $F$ and $A$, with $F$ holding the general solution of $A$, to determine whether the general solution of $A$ is contained in the general solution of $F$.}

If $F$ is of order $n$ in $y$, and $A$ of order $n - 1$, this is merely a matter of deciding whether the general solution of $A$ is a component of $F$. The low power theorem gives the decision. If the order of $A$ is less than $n - 1$, the question becomes complicated. For instance, suppose that $A$ is of order $n - 2$. It may be that $F$ has certain components $M_1, \ldots, M_s$ of order $n - 1$. The general solution $M$ of $A$ may be found, when the low power theorem is used, to be contained in some of the $M_i$. In that case, the question of testing for the presence of $M$ in the general solution of $F$ is an intricate one, which thus far has been solved only for the case of $n = 2$.\footnote{Ritt, 31.}

For the case of $n = 2$, our problem can be reduced to the following: \textit{Let $F$, of order 2, vanish for $y = 0$. It is required to determine whether $y = 0$ is contained in the general solution of $F$.} This question can always be answered after there are performed a finite number of operations in which one examines polygons, of the Newton type, associated with $F$. The discussion is too lengthy to be presented here.

For instance, let $F$ be the d.p. of §26. $F$ is annulled by $y = 0$. For $m > 3$, it follows from §24 that $y = 0$ is in the general solution of $F$. For $m = 3$, the methods of the paper cited above show that $y = 0$ is in the general solution.

We wish, in conclusion, to compare two very simple d.p. According to §28, $y = 0$ is not in the general solution of $yy_1 + y_2^2$. Consider, again, $yy_1 + y_2^2$. By §24, its general solution contains $y = 0$.

36. The second problem on singular solutions mentioned in §34 belongs to
classical analysis rather than to differential algebra. For instance, Hamburger's work on differential equations of the first order\(^{17}\) shows that if a singular solution of such an equation is not contained in the general solution, the singular solution is an envelope of nonsingular ones. If the singular solution is contained in the general solution, it is analytically embedded among nonsingular solutions. Another paper of Hamburger's deals with algebraic differential equations of any order \(n\), supposed to have a component of order \(n - 1\). Of course, the notions of component, and of general solution, as we have them, did not exist when Hamburger wrote. The component of order \(n - 1\) is shown, speaking geometrically, to consist of envelopes of curves in the general solution. The theory of algebraic differential manifolds throws new light on the analytic theory of singular solutions, and, as one sees in connection with partial differential equations,\(^{18}\) points the way in analytical investigations.

37. Just as Lagrange dealt, to an extent, with the general solution of a differential equation, so Laplace,\(^{19}\) in a paper published in 1772, treated questions resembling those of the present chapter. Dealing with a differential equation \(F = 0\) of order \(n\), in an unknown \(y\), Laplace uses the term general integral to designate a family of solutions depending on \(n\) arbitrary constants. By a solution of the given equation, he understands an equation \(A = 0\) of order lower than \(n\) which "satisfies" the given equation. What seems to be meant, in a vague way, is that \(F\) holds the general solution of \(A\). A particular integral is a solution "contained in" the general integral and a particular solution is one which is not so contained. Laplace sets the following two problems:

*Being given a differential equation of any order,*

1. *to determine whether an equation of lower order which satisfies it is contained in the general integral;*

2. *to determine all of the particular solutions of the given equation.*

The second problem corresponds to that of the determination of the components of a d.p., the problem which is solved by the low power theorem. The first problem corresponds to that of determining whether the general solution of \(A\) is contained in that of \(F\).

As one would expect, Laplace's treatment of his problems is of a heuristic nature. It does not contain the elements of a sound theory, or even serviceable conjectures. One can have only admiration, however, for his ability to imagine problems which, with the mathematics of his day, could not be soundly formulated, much less solved.

38. A paper published by Poisson in 1806 treats,\(^{20}\) in a manner somewhat different from that of Laplace, the questions raised by the latter. Poisson's method is most easily understood from his discussion of "algebraic particular solutions." These, which had been considered by Laplace, have for counter-

\(^{17}\) Hamburger, 6.

\(^{18}\) Ritt, 41.

\(^{19}\) Laplace, 16.

\(^{20}\) Poisson, 19.
parts, in the theory of manifolds, components composed of one point, for instance, the manifold of \( y \) when
\[
F = y + \left( \frac{d^2 y}{dx^2} \right)^2.
\]

Poisson considers that it is proper to call a solution \( y(x) \) of a differential equation an algebraic particular solution if and only if the equation does not have a one-parameter family of solutions \( y(x) + cz \) with \( c \) an arbitrary constant and \( z \) a function of \( x \) and \( c \). More or less, an algebraic particular solution is, for Poisson, one which cannot be analytically embedded in a one-parameter family of solutions. With this definition, Poisson is able to state, for certain classes of equations, necessary and sufficient conditions for a given solution to be an algebraic particular solution. The results of Poisson may be regarded, as may also those of Laplace, as heuristic equivalents of portions of the low power theorem. For instance, Poisson concludes that \( y = 0 \) is a particular solution of
\[
\left( \frac{dy}{dx} \right)^m \frac{d^2 y}{dx^2} = y^n
\]
if and only if \( m \geq n \). Poisson’s treatment of his problem vaguely resembles the necessity proof for the low power theorem.

39. There is an aspect of the theory of singular solutions which is not revealed by our algebraic considerations. The equation
\[(55)\]
\[
y^2 - y \frac{dy}{dx} + \left( \frac{dy}{dx} \right)^3 = 0
\]
has \( y = 0 \) in its general solution. If we solve for \( y \) in (55) in terms of \( dy/dx \), we secure two expansions proceeding according to increasing integral powers of \( dy/dx \). They are
\[(56)\]
\[
y = \frac{dy}{dx} - \left( \frac{dy}{dx} \right)^2 + \cdots,
\]
\[(57)\]
\[
y = \left( \frac{dy}{dx} \right)^2 + \cdots.
\]
Now the solution \( y = 0 \) of (57) can be shown to be an envelope of solutions of (57). Furthermore, the low power theorem can be extended to cover equations like (57), in which infinite series appear. This suggests extending the theory of differential polynomials into one of differential power series.\(^{21}\)

III. Exponents of Ideals

40. In \( \mathfrak{T} \{ y, \ldots, y_n \} \), we consider an ideal \( \Sigma \) and, with it, \( \{ \Sigma \} \). Every d.p. in \( \{ \Sigma \} \) has a power in \( \Sigma \). By I, §15, if, for some \( p \), and for every \( A \) in \( \{ \Sigma \} \), \( A^p = 0, (2\Sigma) \), the \( p \)th power of \( \{ \Sigma \} \) will be contained in \( \Sigma \). If there is a positive integer \( p \) such that \( \Sigma \) contains \( \{ \Sigma \}^p \), the least such integer is called

\(^{21}\) Ritt, 33.
the exponent of \( \{\Sigma\} \) relative to \( \Sigma \). When no such integer exists, the relative exponent is taken as \( \infty \).

Relative exponents were investigated by Kolchin\(^{22}\). He studied, in particular, for an algebraically irreducible d.p. \( A \) in \( y \), of the first order, the exponent of \( \{A\} \) relative to \([A]\). The exponent depends on the nature of the singular zeros of \( A \). We shall content ourselves with the presentation of an example.

41. Let \( A = y_1^2 - 4y \). (See Example 1 of II, §4.) We shall prove that the exponent of \( \{A\} \) relative to \([A]\) is 2. We have, subscripts of \( A \) indicating differentiation,

\[
\begin{align*}
A_1 &= 2y_1y_2 - 4y_1, \\
A_2 &= 2y_1y_3 + 2y_2^2 - 4y_2, \\
A_3 &= 2y_1y_4 + 6y_2y_3 - 4y_3, \\
&\ldots \\
A_r &= 2y_1y_{r+1} + \cdots + 2ry_ry_r - 4y_r \quad (r > 2). 
\end{align*}
\]

The unwritten terms in \( A_r \) with \( r > 2 \) are of the form \( cy_py_q \) with \( p + q = r + 2 \) and with \( p \) and \( q \) greater than 2 and less than \( r \). As \( y_1(y_2 - 2) \) is in \([A]\), \( y_2(y_2 - 2)^2 \) is in \([A]\). If \( r > 2 \),

\[
y_2(y_2 - 2)^2 = P_r(2ry_2 - 4) + c_r
\]

with \( c_r \) a constant distinct from zero. We find then, from the last equation of (58),

\[
c_ry_r = P_r(2y_1y_{r+1} + \cdots), \quad [A],
\]

the unwritten terms being as in \( A_r \).

We shall now prove that, for \( r > 2 \),

\[
y_r(y_2 - 2) = 0, \quad [A].
\]

By (59) with \( r = 3 \), we have

\[
c_3y_3 = 2y_1y_4P_3, \quad [A].
\]

We multiply by \( y_2 - 2 \) in (61), noting that \( A_1 = 2y_1(y_2 - 2) \). We obtain (60) with \( r = 3 \). If we observe that the subscripts in the unwritten terms of (59) exceed 2 and are less than \( r \), the induction necessary to establish (60) for all \( r \) is accomplished.

Then, for \( r > 2 \),

\[
y_{r+1}(y_2 - 2) + y_ry_3 = 0, \quad [A].
\]

As the first term in (62) is in \([A]\), we have, for \( r > 2 \), \( y_ry_3 = 0, \quad [A] \). Then

\[
y_{r+1}y_3 + y_ry_4 = 0, \quad [A],
\]

\(^{22}\) Kolchin, 10.
so that \( y_2^2 y_4 \) is in \([A]\) for \( r > 2 \). In this way, we see that
\[
y_2 y_4 = 0, \quad [A],
\]
for \( p > 2, q > 2 \).

Let \( P \), any d.p. in \([A]\), be written
\[
P = Q + R,
\]
where \( Q \) consists of those terms of \( P \) which involve only \( y, y_1, y_2 \). The \( y_i \) with \( i > 2 \) are in \([A]\). Thus \( Q \) is in \([A]\). We write
\[
Q = M(y_2 - 2) + N
\]
with \( N \) free of \( y_2 \). Then \( y_1 N \) is in \([A]\) so that \( N \) is divisible by \( A \). Thus
\[
P = M(y_2 - 2) + R, \quad [A].
\]
Then
\[
P^2 = M^2(y_2 - 2)^2 + 2MR(y_2 - 2) + R^2, \quad [A].
\]
Now \( R^2 \) is in \([A]\) by (63) and \( R(y_2 - 2) \) is in \([A]\) by (60). Hence
\[
P^2 = M^2(y_2 - 2)^2, \quad [A].
\]
Each term in \( M \) involves at least one of \( y, y_1, y_2 \). We know that \( y_1(y_2 - 2) \)
and \( y_2(y_2 - 2)^2 \) are in \([A]\). Also, because \( y_1^2 = 4y, \) \( (A) \), we see that \( y(y_2 - 2) \)
is in \([A]\). Hence \( P^2 = 0, [A]\).

Thus \([A]\) is contained in \([A]\).

42. It remains to be proved that \([A]\) is not identical with \([A]\). This we show by proving that \( y_3 \) is not in \([A]\). Suppose that
\[
y_3 = CA + C_1 A_1 + \cdots + C_r A_r.
\]
In the second member of (64), we put \( y = y_4^2/4, y_2 = 2 \). We find, writing
\[
B_2 = y_4 y_5, \quad B_3 = y_4 y_6 + 4 y_5, \quad B_r = y_4 y_{r+1} + \cdots + (2r-2)y_r \quad (r > 2),
\]
that
\[
y_3 = D_2 B_2 + \cdots + D_r B_r.
\]
We see immediately that \( r > 2 \). The only term in the second member of (65) which can yield a constant times \( y_3 \) is \( D_3 B_3 \). Thus \( 1/4 \) must be a term in \( D_3 \) so that \( D_3 B_3 \) must contain \( y_4 y_4 / 4 \). The term just mentioned must cancel out in (65). The only \( B \) other than \( B_3 \) which contains a term of which \( y_4 y_4 \) is a multiple is \( B_4 \) which contains \( 6y_4 \). Thus \( D_4 \) must contain \(-y_4/24\) so that \(-y_4 y_6 / 24\) appears in \( D_4 B_4 \). The only term other than \( B_4 \) which contains a factor of \( y_4^3 y_4 \) is \( B_5 \), in which \( y_6 \) appears. It follows that \( D_5 B_4 \) contains a term in \( y_4^5 y_6 \). Continuing, we produce the contradiction that \( r \) in (64) is not exceeded by any integer. This completes the proof.