CHAPTER IX
PARTIAL DIFFERENTIAL ALGEBRA

Partial differential polynomials. Ideals and manifolds

1. We use an algebraic field $\mathfrak{F}$ of characteristic zero which admits $m$ operations of differentiation. Each element $a$ of $F$ has $m$ partial derivatives $\frac{\partial a}{\partial x_i}$, $i = 1, \cdots, m$. In this, the $x$ are not necessarily variables. They may merely be symbols which distinguish the derivatives. Each of the $m$ operations satisfies (1) and (2) of I, §1. In addition,

$$\frac{\partial}{\partial x_j} \left( \frac{\partial a}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{\partial a}{\partial x_j} \right)$$

for every $i$ and $j$. We call $F$ a partial differential field.

In our work below, definitions will usually be as for the case of one operation and will be given, formally, only when there is some necessity for it.

2. We employ indeterminates $y_1, \cdots, y_n$. With each $y_i$ are associated symbols

$$(1) \quad \frac{\partial^k + \cdots + \partial^n}{\partial x_1^i \cdots \partial x_m^j} y_i$$

where the $i_j$ are any nonnegative integers; these are the partial derivatives$^1$ of $y_i$.

$\mathfrak{F}$ being given, we understand by a partial differential polynomial (p.d.p. or d.p.), a polynomial in derivatives of the $y$ with coefficients in $\mathfrak{F}$.

3. We understand marks to be attributed to the symbols $x$ and $y$ as in VIII, §8, in such a way as to effect a complete ordering.

By the leader of a p.d.p. $A$ which actually involves indeterminates,$^2$ we shall mean the highest of those derivatives of the $y$ which are present in $A$.

Let $A_1$ and $A_2$ be p.d.p. which actually involve indeterminates. If $A_2$ has a higher leader than $A_1$, then $A_2$ will be said to be of higher rank than $A_1$. If $A_1$ and $A_2$ have the same leader, and if the degree of $A_2$ in the common leader exceeds that of $A_1$, then again $A_2$ will be said to be of higher rank than $A_1$. A d.p. which effectively involves indeterminates will be of higher rank than one which does not. Two d.p. for which no difference in rank is created by what precedes will be said to be of the same rank.

As in I, §8, we see that every aggregate of p.d.p. contains a d.p. which is not higher than any other d.p. of the aggregate.

4. If $A_1$ involves indeterminates, $A_2$ will be said to be reduced with respect to

$^1$ When the $i_j$ are all zero, (1) represents $y_i$.

$^2$ We mean that $A$ is not an element of $\mathfrak{F}$. 163
A₁ if A₂ contains no proper derivative of the leader of A₁ and if A₂ is either zero or of lower degree than A₁ in the leader of A₁. A set of p.d.p.

\[(2) \quad A_1, \cdots, A_r\]

will be called a chain if either

(a) \(r = 1\) and \(A_1 \neq 0\), or

(b) \(r > 1\), \(A_1\) involves indeterminates and, for \(j > i\), \(A_j\) is of higher rank than \(A_i\) and reduced with respect to \(A_i\).

When (b) holds, the leader of \(A_j\) is higher than that of \(A_i\) for \(j > i\).

Relative rank for chains is defined exactly as in I, §4. If \(\Phi_1, \Phi_2, \Phi_3\), are chains with \(\Phi_1 > \Phi_2\) and \(\Phi_2 > \Phi_3\), then \(\Phi_1 > \Phi_3\).

We prove that, in every aggregate of chains, there is a chain which is not higher than any other chain of the aggregate. Let \(\alpha\) be the aggregate. We form a subset \(\alpha_1\) of \(\alpha\), putting a chain \(\Phi\) into \(\alpha_1\) if the first d.p. of \(\Phi\) is not higher than the first d.p. of any other chain in \(\alpha\). It may be that the chains in \(\alpha_1\) are merely elements of \(\Phi\); if so, any of them is a chain of least rank in \(\alpha\). Let us suppose that the first d.p. in the chains of \(\alpha_1\) actually involve indeterminates. These first d.p. will all have the same leader; we represent that leader by the symbol \(p_1\). If the chains in \(\alpha_1\) all consist of one d.p., any chain in \(\alpha_1\) meets our requirements. Suppose that there are chains in \(\alpha_1\) which have more than one d.p. We form the subset \(\alpha_2\) of them whose second d.p. are of a lowest rank and indicate the common leaders of these second d.p. by \(p_2\). Now \(p_2\) is not a proper derivative of \(p_1\). As we saw above, \(p_2\) is higher than \(p_1\). If the chains in \(\alpha_2\) all have just two d.p., any of these chains serves our purpose. If not, we continue. Our result will hold unless there is an infinite sequence

\[p_1, p_2, \cdots, p_q, \cdots\]

derivatives which increase steadily in rank, no \(p_q\) being a derivative of a \(p_i\) with \(i < q\). The existence of such a sequence would contradict Riquier’s theorem of VIII, §2.

5. Let \(\Sigma\) be a system containing nonzero d.p. We define a characteristic set of \(\Sigma\) to be a chain in \(\Sigma\) of least rank.

If \(A_1\) in (2) involves indeterminates, a d.p. \(F\) will be said to be reduced with respect to (2) if \(F\) is reduced with respect to \(A_i\), \(i = 1, \cdots, r\).

Let \(\Sigma\) be a system for which (2), with \(A_i\) not free of the indeterminates, is a characteristic set. Then no nonzero d.p. in \(\Sigma\) can be reduced with respect to (2). If a nonzero d.n., reduced with respect to (2), is adjoined to \(\Sigma\), the char-
Let $G$ be any d.p. There exist nonnegative integers $s_i$, $t_i$, $i = 1, \ldots, r$, such that, when a suitable linear combination of the $A$ and their derivatives is subtracted from

$$S^n_i \cdots S^n_r I^r_1 \cdots I^r_r G,$$

the remainder, $R$, is reduced with respect to (2).

Let $p_i$ be the leader of $A_i$. We limit ourselves, as we may, to the case in which $G$ involves derivatives, proper or improper, of the $p_i$. Such derivatives will be called $p$-derivatives. Let the highest $p$-derivative in $G$ be $q$ and let $q$ be a derivative of $p_i$. For the sake of uniqueness, if there are several possibilities for $j$, we use the largest $j$ available. To fix our ideas, we assume $q$ higher than $p_r$. Then

$$S^q_j G = CA'_j + B$$

where $A'_j$ is a derivative of $A_j$ with $q$ for leader and where $B$ is free of $q$. Because $A'_j$ and $S_j$ involve no derivative higher than $q$, $B$ involves no $p$-derivative which is as high as $q$. For uniqueness, we take $q$ as small as possible.

If $B$ involves a $p$-derivative which is higher than $p_r$, we give $B$ the treatment accorded to $G$. After a finite number of steps, we reach a d.p. $D$ which differs by a linear combination of derivatives of the $A$ from a d.p.

$$S^n_1 \cdots S^n_r G.$$

$D$ contains no $p$-derivative which is higher than $p_r$.

We find then a relation

$$I^r_r D = HA_r + K,$$

where $K$ is reduced with respect to $A_r$. $K$ may involve $p_r$. Aside from $p_r$, the only $p$-derivatives present in $K$ are derivatives of $p_1, \ldots, p_{r-1}$. Such $p$-derivatives are lower than $p_r$. Let $q_1$ be the highest of them.

Suppose that $q_1$ is higher than $p_{r-1}$. We give $K$ the treatment received by $G$, obtaining a unique d.p. $L$ which differs from some

$$S^n_1 \cdots S^{n-1}_{r-1} I^{r-1}_{r-1} K$$

by a linear combination of $A_{r-1}$ and proper derivatives of $A_1, \ldots, A_{r-1}$. The d.p. $L$ is reduced with respect to $A_r$ and $A_{r-1}$. Aside from $p_r$ and $p_{r-1}$, the $p$-derivatives in $L$ are derivatives of $p_1, \ldots, p_{r-2}$, and all such $p$-derivatives are lower than $p_{r-1}$.

Continuing, we determine, in a unique manner, a d.p. $R$ as described in our statement. We call $R$ the remainder of $G$ with respect to (2).

7. Ideals of p.d.p. are defined as in I, §7. In (b) of I, §7, one requires that the $m$ partial derivatives of any d.p. in $\Sigma$ belong to $\Sigma$.

We define basis as in I, §12. The basis theorem,\textsuperscript{3} the decomposition theorem

\textsuperscript{3} In dealing with I, §10, one uses the fact that

$$u^2 \frac{\partial v}{\partial x_i} = 0, \left( u, \frac{\partial u}{\partial x_i} \right).$$
of I, §16, and the theorem on relatively prime ideals of I, §19, go over immediately to the case of several differentiations.

Manifolds are defined as in II, §1. The decomposition theorem of II, §3, then carries over.

The analytic case is formulated as follows. \( \mathcal{F} \) is a set of functions of \( m \) complex variables \( z_1, \ldots, z_m \). There is given an open region \( A \) in the space of the \( z \). The functions in \( \mathcal{F} \) are meromorphic at each point of \( A \). An analytic zero consists of functions which are analytic in an open region contained in \( A \).

To illustrate the decomposition theorem, we let

\[
A = z - (px + qy) + p^2 + q^2,
\]

where \( p = \partial z / \partial x, \ q = \partial z / \partial y \). Putting \( A = 0 \), and differentiating with respect to \( z \), we find

\[
-(rx + sy) + 2(pr + qs) = 0,
\]

where \( r = \partial^2 z / \partial x^2, \ s = \partial^2 z / \partial x \partial y \). Similarly,

\[
-(sx + ty) + 2(ps + qt) = 0,
\]

where \( t = \partial^2 z / \partial y^2 \). From (4) and (5) we obtain

\[
(rt - s^2)(x - 2p) = 0; \quad (rt - s^2)(y - 2q) = 0.
\]

Thus, either \( rt - s^2 = 0 \) or \( z = (x^2 + y^2) / 4 \). The latter zero of \( A \) does not annul \( rt - s^2 \). Thus the manifold of \( A \) is reducible. The zero \( (x^2 + y^2) / 4 \) is a component of \( A \). As one will see later, there is one other component, the general solution of \( A \).

8. The question of generic zeros is treated as in II, §6. Given a prime ideal \( \Sigma \) of p.d.p. in \( y_1, \ldots, y_n \), distinct from the unit ideal, one finds a zero \( \eta_1, \ldots, \eta_n \) of \( \Sigma \) which annuls no d.p. not contained in \( \Sigma \). The abstract theorem of zeros of II, §7, then carries over. The analytic case will be treated later.

The theoretical method of V, §28, for resolving a finite system of d.p. into finite systems equivalent to prime ideals is seen to hold for p.d.p.

**General solutions**

9. Let \( F \) be an algebraically irreducible p.d.p. and \( S \) its separant. We see as in II, §12, that the totality \( \Sigma_1 \) of those d.p. \( A \) which are such that

\[
SA = 0, \quad \{ \ F \},
\]

is an ideal. We shall prove that \( \Sigma_1 \) is prime. Let \( p \) be the leader of \( F \). Let \( AB \) be in \( \Sigma_1 \). There exist relations

\[
S^a A = R, \quad S^b B = T, \quad \{ F \},
\]

where \( R \) and \( T \) involve no proper derivatives of \( p \). Then \( SRT \) is in \( \{ F \} \). Let then
where the \( F_i \) are distinct partial derivatives of \( F \). The leaders of the \( F_i \) are distinct. We may thus, and shall, assume that the \( F_i \) increase in rank as their subscripts increase. Let \( p' \) be the leader of \( F_q \). We have
\[
F_q = Sp' + U,
\]
where the leader of \( U \) is lower than \( p' \). We replace \( p' \) in \( F_q \) and in the \( M \) by \( -U/S \). The proof is completed as in II, §12.

As in II, §§13, we prove that \( \Sigma_i \) consists of those d.p. which have zero remainders with respect to \( F \). In particular, \( \Sigma_i \) does not contain \( S \).

As in II, §§14, 15, we find that \( \{ F \} \) has a decomposition into essential prime divisors
\[
\{ F \} = \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_r
\]
in which \( \Sigma_1 \) is the only divisor which does not contain \( S \).

A change of marks may give \( F \) a new separant. Any such separant involves only derivatives present in \( F \) and is not divisible by \( F \). Hence, for the original marks, such a separant has a remainder which is not zero. Thus, in (7), \( \Sigma_i \) contains no separant of \( F \), while \( \Sigma_2, \cdots, \Sigma_r \) contain every separant.

We call the manifold of \( \Sigma_1 \) the general solution of \( F \).

**Components of a Partial Differential Polynomial**

10. Let \( F \) be a nonzero p.d.p. We shall prove that every essential prime divisor of \( \{ F \} \) has a characteristic set consisting of a single d.p. Such a d.p., call it \( A_i \), can be taken as algebraically irreducible; the prime divisor consists of those d.p. which have zero remainders with respect to \( A_i \). It will follow that *every component of a nonzero p.d.p. is the general solution of some p.d.p.*

11. Let
\[
A_1, \cdots, A_r
\]
be a chain with \( A_1 \) not an element of \( \mathcal{R} \). Let \( A_i \) have \( S_i \) for separant and \( I_i \) for initial. Let \( G \) be any p.d.p. We shall prove that there exists a power product \( J \) of the \( S_i \) and \( I_i \) such that \( JG \) is a polynomial in the \( A_i \) and their partial derivatives, with coefficients which are d.p. reduced with respect to (8).

Let \( p_i \) be the leader of \( A_i \). We limit ourselves, as we may, to the case in which \( G \) involves \( p \)-derivatives. Let the highest \( p \)-derivative in \( G \) be \( q_1 \) and let \( q_1 \) be a derivative of \( p_i \). For uniqueness, we use the largest \( f \) available. To fix our ideas, we assume \( q_1 \) higher than \( p_i \). For some partial derivative \( A_i' \) of \( A_i \), we have
\[
A_i' = S_q q_1 + T,
\]

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*We obtain \( B \) as in (12) of II, §13, with \( B \) free of proper derivatives of \( p \).

*The case of \( m > 1 \) is essentially different from that of \( m = 1 \). For instance, for \( m = 1 \), every irreducible manifold in one indeterminate is a general solution. This is not so for p.d.p.
where \( T \) involves no derivative as high as \( q_1 \). Let \( G \) be of degree \( a \) in \( q_1 \). Then \( S^jG \) can be written as a polynomial in \( A'_1 - T \), and hence as a polynomial in \( A'_1 \), with coefficients in which all \( p \)-derivatives are lower than \( q_1 \). Suppose that, among the coefficients just mentioned, there are one or more which involve \( p \)-derivatives higher than \( p_r \). Let \( q_s \) be the highest such \( p \)-derivative. We give the coefficients which involve \( q_s \), with respect to \( q_s \), the treatment accorded above to \( G \) with respect to \( q_1 \). We see now that there is a power product \( J_1 \) in one or two of the \( S \) such that \( J_1G \) is a polynomial in two derivatives of \( A_1 \), with coefficients involving no \( p \)-derivative as high as \( q_s \). We reach ultimately a \( J_uG \), with \( J_u \) a power product in the \( S_t \) in the coefficients of which the \( p \)-derivatives actually present are not higher than \( p_r \). Some \( T_rJ_uG \) is a polynomial in \( A_r \), and proper derivatives of \( A_r \) with coefficients which are reduced with respect to \( A_r \). Aside from \( p_r \), the only \( p \)-derivatives present in the coefficients are derivatives of \( p_{r_1}, \cdots, p_{r_{l-1}} \). How to complete the proof is now obvious.

Let us examine the expression found for \( JG \). In our discussion, there appeared a finite sequence of derivatives

\[
q_1, q_2, \cdots, q_t
\]

with \( q_i \) higher than \( q_{i+1} \), \( i = 1, \cdots, t - 1 \), each \( q_i \) being the leader of a derivative \( B_u \), proper or improper, of some \( A_1 \). \( JG \) is a polynomial in the \( B_u \), with coefficients reduced with respect to (8).

12. Let \( F \) be a nonzero d.p. and let (7) be a decomposition of \( \{ F \} \) into essential prime divisors. Suppose that some \( \Sigma_1 \) in (7) has a characteristic set consisting of more than one d.p. We let \( \Lambda \) stand for such a \( \Sigma_1 \) and consider a characteristic set (8) of \( \Lambda \).

Treating \( F \) as \( G \) was treated in §11, we obtain a \( J \) as in §11 and let \( H = JF \). Then \( H \) is a polynomial in \( A \) and their partial derivatives.

Let \( \eta_1, \cdots, \eta_n \) be a generic zero of \( \Lambda \) contained in an extension of \( \mathfrak{z}_0 \) of \( \mathfrak{z} \). We make in \( H \) and in the \( A \) the substitution

\[
y_i = \eta_i + z_i, \quad i = 1, \cdots, n,
\]

using the same marks for \( z_i \) as for \( y_i \). Each \( A_i \) goes over into a d.p. \( C_i \) over \( \mathfrak{z}_0 \). Let us study \( C_i \) as a polynomial in the \( z \) and their derivatives. \( C_i \) admits the zero \( z_j = 0, j = 1, \cdots, n \). We examine the terms of the first degree in \( C_i \). To \( p_i \), the leader of \( A_i \), there corresponds a derivative \( r_i \) of some \( z \). The coefficient of \( r_i \) in \( C_i \) is what \( S_i \) becomes when the \( \eta \) are substituted into it. Because \( S_i \) is not in \( \Lambda \), \( S_i \) does not vanish for the \( \eta \). Thus \( C_i \) contains effectively terms of the first degree. We represent the sum of these terms by \( D_i \). The leader of \( D_i \) is \( r_i \).

We now consider \( H \). Let \( K \) represent what \( H \) becomes under (10). Our object is to describe the terms of lowest degree in \( K \) considered as a polynomial in the \( z \) and their derivatives.

Referring to the final remarks of §11, we consider \( H \) as a polynomial in the \( B_{i, i = 1, \cdots, t} \). Let \( L \) be the sum of those terms of \( H \) which are of a lowest
total degree in the $B$. Then every term of $L$ is of the form $MN$ with $M$ reduced with respect to (8) and $N$ a power product in the $B$. Under (10), let $M$ and $N$ go over into $P$ and $Q$ respectively. Then $P$ contains an effective term which is in $\mathfrak{F}_0$, while the terms of $Q$ which are of a lowest total degree in the $z$ and their derivatives constitute a product of powers of the $D$ and their derivatives. Let us select, from $L$, those terms which are of a highest degree in $B_1$. From these latter terms we select those which are of a highest degree in $B_2$. Continuing, we are led to a definite term $MN$ of $L$ which goes under (10) into an expression $PQ$. Let
\[ N = B_a^\alpha B_b^\beta \cdots B_c^\gamma, \]
where $a < b < \cdots < c$ and $\alpha, \beta, \cdots, \gamma$ are positive. If $s_i$ is allowed to represent that derivative of $z$ whose marks are those of $q_i$ in (9), we find that $PQ$ contains effectively a term in $s_a^\alpha s_b^\beta \cdots s_c^\gamma$. This term is one of the terms of lowest degree in $K$.

Thus the leader of $W$, the sum of the terms of lowest degree in $K$, is a derivative, proper or improper, of the leader of some $D$.

13. Changing the notation if necessary, we assume that the leader of $W$ is a derivative of $z_1$. We decompose $W$ into irreducible factors in $\mathfrak{F}_0$ and consider an irreducible factor $V$ which effectively involves the leader of $W$.

$V$ is a d.p. over $\mathfrak{F}_0$. Let $\xi_1, \cdots, \xi_n$ be a generic point in the general solution of $V$, contained in an extension $\mathfrak{F}_1$ of $\mathfrak{F}_0$.

Then $W$ vanishes for the $\xi$. On the other hand, not every $D_i$ can so vanish. Let us assume that $D_1$ vanishes. We shall prove that $D_2$ does not. By the final statement of §12, the leader of $V$ is not lower than that of $D_1$. If $D_1$ had a lower leader than $V$, $D_1$ would be reduced with respect to $V$ and would not vanish for the $\xi$. Thus $D_1$ has the same leader as $V$. Then $D_1$ is divisible by $V$. As $D_1$ is linear, $D_1$ is the product of $V$ by an element of $\mathfrak{F}_0$. Thus the general solution of $V$ is the general solution of $D_1$. By §12, the leaders of $A_i$ and $D_i$ have the same marks for every $i$. Thus the leader of $D_2$ is not a derivative of that of $D_1$. Then the remainder of $D_2$ with respect to $D_1$ is not zero so that $D_2$ does not vanish for the $\xi$.

14. We say that $K$ is annulled by expressions
\[ z_i = \xi_1 c, \quad i = 2, \cdots, n, \]
(11)
\[ z_1 = \xi_1 c + \varphi_2 c^2 + \cdots + \varphi_n c^n + \cdots, \]
with $\varphi > 1$.

If $K$ vanishes for $z_i = \xi_1 c$, $i = 1, \cdots, n$, we have the desired expressions. Let the vanishing fail to occur. We put in $K$
\[ z_i = \xi_1 c, \quad i = 2, \cdots, n; \quad z_1 = \xi_1 c + u_1, \]
where $u_1$ has the same marks as $z_1$. The work of III, §§6–13, carries over with very slight changes. Where, in Chapter III, one uses derivatives of an indeterminate up to a certain order, one employs here a set of partial derivatives.
Leaders serve here as derivatives of highest order do in Chapter III. In treating (11) of III, §10, we represent the derivatives of $u_i$ appearing in $K'$ by $v_1, \cdots, v_\rho$ and the corresponding derivatives of $u_2$ by $w_1, \cdots, w_\rho$. Assuming that, for certain $l$,

$$\frac{\partial^h \cdots \partial^l L'(u_i)}{\partial^{l_1} v_1 \cdots \partial^{l_\rho} v_\rho}$$

does not vanish for $u_1 = \varphi_2$, we prove that $w_1^{l_1} \cdots w_\rho^{l_\rho}$ is present in $K''$.

15. The series (11) being obtained, we find that $H$ is annulled by expressions

\begin{equation}
\begin{aligned}
y_i &= \eta_i + \xi_i c, & i &= 2, \cdots, n, \\
y_1 &= \eta_i + \xi_i c + \varphi_2 c^{\gamma_2} + \cdots.
\end{aligned}
\end{equation}

These expressions do not annul $J$, since the $\eta$ do not. Thus $F$ vanishes for (12). Because the $D$ of §12 do not all vanish for the $\xi$, the $C$ do not all vanish for (11), so that the $A$ in the characteristic set (8) of $\Lambda$ of §12 do not all vanish under (12). Now some $\Sigma_i$ in (7) must admit (12) as a zero. Such a $\Sigma_i$ is necessarily distinct from $\Lambda$. On the other hand, such a $\Sigma_i$ must admit $\eta_1, \cdots, \eta_n$ as a zero, and thus is contained in $\Lambda$. As this is impossible, it is established that every prime ideal in the second member of (7) has a characteristic set consisting of one d.p.

16. Suppose now that $F$ of §10 is algebraically irreducible. Let $\Sigma_i$ in (7) be the prime ideal associated with the general solution of $F$. Consider any $\Sigma_i$ with $i > 1$. Its manifold is the general solution of a d.p. $A$. We say that $F$ effectively involves some proper derivative of the leader of $A$.

If this were not true, $F$ would be divisible by $A$, since $F$ is in $\Sigma_i$ and the remainder of $F$ with respect to $A$ is zero.

Let $y_i$ be any indeterminate of which some derivative appears effectively in $A$ and let $r$ be the maximum of the orders of the derivatives of $y_i$ in $A$. Marks can be chosen for which the leader of $A$ is a derivative of $y_i$ of order $r$. Thus $F$ is of higher order than $A$ in every indeterminate appearing in $A$.

THE LOW POWER THEOREM

17. Let $F$ and $A$ be two p.d.p. in $\mathfrak{F}\{y_1, \cdots, y_n\}$, neither an element of $\mathfrak{F}$. Let $S$ be the separant, and $p$ the leader, of $A$. Proceeding as in III, §17, and as in IX, §11, one proves the existence of a nonnegative integer $t$ such that $S^t F$ has a representation

\begin{equation}
\sum_{i=1}^{l} C_i A^{i_1} A^{i_2} \cdots A^{i_l}
\end{equation}

where the $A_i$ are distinct proper derivatives of $A$ and no two sets $i_1, \cdots, i_n$ are identical; the $C$ involve no proper derivative of $p$ and are not divisible by $A$.

If $F$ involves no proper derivative of $p$, there are no $A_i$ in (13); otherwise the
leader of $A_4$ is the highest of the derivatives of $p$ which appear in $F$. For a given admissible $t$, the representation (13) of $S^t F$ is unique.

In what follows, we assume $A$ to be algebraically irreducible and we use the smallest admissible $t$.

The low power theorem has the wording of III, §20, except that one uses the representation in (13).

18. We use an indeterminate $y$ and the field of rational numbers. Let $p$ be any positive integer. We shall show that every power product of degree $2p - 1$ in the $\partial y / \partial x_i, i = 1, \ldots, m$, is in $[y^p]$.

We may assume that $p > 1$. We have, for every $i$, $y^{p-1} \partial y / \partial x_i = 0$, $[y^p]$. Thus, for every $i$ and $j$,

$$(p - 1) y^{p-2} \partial y / \partial x_j \partial x_i + y^{p-1} \partial^2 y / \partial x_j \partial x_i = 0, \quad [y^p].$$

We multiply by any $\partial y / \partial x_k$. Then, for any $i, j, k$,

$$y^{p-2} \partial y / \partial x_i \partial x_j \partial x_k = 0, \quad [y^p].$$

Continuing, we verify our statement.

Let $k$ be any positive integer. We consider the derivatives of $y$ of order $k$, and form power products in these derivatives. We shall show that every such power product which is of degree $2^k m^k - 1$ is in $[y^p]$.

For $k = 1$, we observe that $2^k m^k - 1 = 2p > 2p - 1$, and use the result proved above. We suppose the proof carried through for $k < q$, where $q > 1$, and consider the case of $k = q$. Among $2^q m^q - 1p$ derivatives of order $q$, there must be at least $2^q m^q - 2p$ which are derivatives of order $q - 1$ of some one $\partial y / \partial x_j$. By the case of $k = q - 1$, a product of $2^q m^q - 2p$ derivatives as just mentioned is in $[(\partial y / \partial x_j)^{2q}]$, thus in $[y^p]$.

The weight of a product of powers of derivatives of $y$ will be understood to be the sum of the orders of the derivatives in the product.

Let $a$ be a positive integer. Let

$$f(a, p, m) = p(a + 1) \frac{(2m)^a + 1 - 1}{2m - 1}.$$  

We shall show that a power product in $y$ and its derivatives whose degree is $f(a, p, m)$ and whose weight does not exceed $af(a, p, m)$ is in $[y^p]$.

Let $P$ be a power product of degree $f(a, p, m)$ which is not in $[y^p]$. For each nonnegative integer $k$, the product $P$, by what precedes, must involve fewer than $(2m)^k p$ derivatives of order $k$. Thus $P$ involves fewer than

$$\frac{(2m)^a + 1 - 1}{2m - 1} p = \frac{f(a, p, m)}{a + 1}$$

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*In §§18, 19, we do not use marks; the order of a partial derivative is the only index of rank which is employed.

* We count each derivative of order $k$ as many times as it appears in $P$.  

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derivatives of order not exceeding \(a\). Therefore \(P\) has more than \(f - f/(a + 1)\) derivatives of orders exceeding \(a\). Then the weight of \(P\) exceeds \(af\).^{8}

19. We can now carry over the lemma of III, §21. Let \(r\) be the maximum of the weights of the \(B\). The cases of \(r = 0\) and \(r = 1\) are trivial. We therefore assume that \(r > 1\) and put

\[
    d = f(r - 1, p, m), \quad t = d(r - 1).
\]

Every power product in \(z\) and its derivatives which is of degree \(d\) and of weight not more than \(t\) is in \([x^p]\). The work of III, §21, needs only minor changes. Where one uses there the \(i\)th derivative of a d.p., one employs here appropriate partial derivatives of order \(i\). The lemma having been extended, one finds the theorems of III, §§22, 23, to hold for p.d.p.

20. The necessity proof can be conducted as follows. We assume that the terms of lowest degree in (13) involve proper derivatives of \(A\). If we let \(A\) take the place of the chain (8), (13) is an expression for \(SF\) like that of \(JG\) in §10, with the difference that the \(C\) are not reduced with respect to \(A\). For our purposes, it is enough that the \(C\) do not hold the general solution of \(A\).

We let \(\eta_1, \cdots, \eta_n\) be a generic zero in \(M\), the general solution of \(A\), and make the substitution (10) in \(SF\) and in \(A\). Then \(A\) goes over into a d.p. \(E\) in the \(z\). To \(p\), the leader of \(A\), there corresponds a derivative \(r\) of some \(z\). \(E\) has terms of the first degree and their sum has \(r\) for leader.

The substitution (10) converts \(SF\) into a d.p. \(K\) in the \(z\). Considering \(K\) as a polynomial in the \(z\) and their derivatives, we let \(W\) be the sum of the terms of lowest degree in \(K\). The leader of \(W\) is seen to be a proper derivative of \(r\). We then proceed as in §§13, 14 and find expressions (12) which annul \(F\) but neither \(S\) nor \(A\). Those expressions furnish a zero in a component \(M'\) of \(F\) which is not held by \(A\). Then \(\eta_1, \cdots, \eta_n\) is in \(M'\) and \(M\) is not a component of \(F\).

**Characteristic sets of prime ideals**

21. Let \(\Sigma\) be a nontrivial prime ideal for which

\[
(14) \quad A_1, A_2, \cdots, A_r
\]

is a characteristic set. One shows, as in V, §1, that when the \(A\) are regarded as ordinary polynomials in the symbols which they involve, (14) is a characteristic set\(^9\) for a prime p.i. \(\Lambda\). One then proves as in V, §4, that every zero of the p.d.p. (14) which annuls no separant is a zero of \(\Sigma\).

22. From this point on we limit ourselves to the consideration of the analytic case. Through §25, it will be assumed that the first mark of each \(z\) is unity.

A being the region in which the functions in \(\Sigma\) are given, we represent by

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* The result is due to Kolchin.
* As we shall see below, we do not have in this a sufficient condition for (14) to be a characteristic set of a prime ideal.
\[ \xi_1, \ldots, \xi_m \text{ or, more briefly, by } \xi, \text{ a point in } A \text{ at which the coefficients in (14) are analytic. We use the symbol } [\eta] \text{ to designate any set of numerical values which one may choose to associate with the derivatives appearing in (14).}

We wish to show that there are sets } \xi, [\eta] \text{ which annul every } A \text{ but none of the separatants of the } A. \text{ If we consider the } A \text{ as ordinary polynomials, Hilbert's theorem of zeros, as derived for the analytic case in IV, §14, holds for them. As no power of the product of the separatants is linear in the } A, \text{ we can find a system of analytic functions of } x_1, \ldots, x_m \text{ which annul the } A \text{ when substituted for the various derivatives, without annulling any separatant. The existence of a set } \xi, [\eta] \text{, described as above, follows. We shall deal with such a set.}

Let } p_i \text{ be the leader of } A_i. \text{ The equation } A_1 = 0, \text{ treated as an algebraic equation for } p_1, \text{ determines } p_1 \text{ as a function of the } x \text{ and the derivatives lower than } p_1 \text{ in } A_1, \text{ the function being analytic for } x_i \text{ close to } \xi \text{ and for the derivatives lower than } p_1 \text{ close to their values among the } [\eta]. \text{ The value of the function } p_1 \text{ for the special arguments just mentioned will be the value for } p_1 \text{ in } [\eta]. \text{ Let the expression for } p_1 \text{ be substituted into } A_2. \text{ We can then solve } A_2 = 0 \text{ for } p_2, \text{ expressing } p_2 \text{ as a function of the } x \text{ and of the derivatives other than } p_1 \text{ and } p_2 \text{ appearing in } A_1 \text{ and } A_2. \text{ We substitute the expressions for } p_1 \text{ and } p_2 \text{ into } A_3 \text{ and continue in this manner for all d.p. in (14).}

We find thus a set of expressions for the } p, \text{ each } p \text{ being given as a function of the } x \text{ and of the derivatives other than } p_1, \ldots, p_r \text{ in (14). We write}

\[ p_i = g_i, \quad i = 1, \ldots, r. \tag{15} \]

If the equations in (15) are considered as differential equations for the } y, \text{ they will form an orthonomic system. We shall prove that if (15) is extended into an orthonomic system whose first members give complete systems of monomials (VIII, §11), the extended orthonomic system is passive.}

We consider the prime p.i. } \Lambda \text{ of §21. The parametric indeterminates in } \Lambda \text{ will be those which correspond to the parametric derivatives in (15). We form a resolvent for } \Lambda \text{ with}

\[ w = b_1p_1 + \cdots + b_rp_r, \tag{16} \]

where the } b \text{ are integers. Let the resolvent be

\[ B_0w^s + \cdots + B_s = 0, \tag{17} \]

and let the expressions for the } p \text{ be

\[ p_i = \frac{E_{i0} + \cdots + E_{i, s-1}w^{s-1}}{D}. \tag{18} \]

Suppose that, in (16), the } p \text{ are replaced by the } g \text{ of (15). Then } w \text{ in (16) becomes a function of the arguments in the } g, \text{ analytic at } \xi, [\eta]. \text{ We wish to see that the functions } g_i \text{ and } w \text{ satisfy (17) and (18). We can form a zero of the characteristic set (14) of } \Lambda, \text{ in which the leaders of the } A \text{ are put equal to the } g \text{ and in which the other letters in the } A \text{ are represented by the complex variables}
of which the $g$ are functions. This zero of (14) annuls no separant; it is thus a zero of $\Lambda$. This is enough to show that the $g$ and $w$ satisfy (17) and (18).

We consider each $g$ in (15) to be expressed by the second member of (18), where $w$ is a function of the $x$ and the parametric derivatives, analytic when the arguments are close to their values\(^{10}\) in $\xi_1, [\eta]_1$.

Let us show how an orthonomic extension $\sigma$ of (15), described as in VIII, §11, is formed. We can calculate each $\partial p_i/\partial x_j$ from (18). In this calculation $\partial w/\partial x_j$ appears, and can be found from (17). Higher derivatives of the $p$ are calculated similarly. If principal derivatives appear in an expression for a $\delta p$, we can get rid of them step by step. We secure in this way the desired extension $\sigma$. Its equations will be of the form (IV, §14)

\[
\delta y = \frac{F_0 + \cdots + F_{-1}w^{-1}}{T},
\]

where $T$ involves only parametric derivatives. There may be, in the second members of (19), parametric derivatives which do not appear in (15). Such derivatives enter rationally and integrally. We shall allow these derivatives to vary in the neighborhood of any set of numerical values $[\xi]$.

If we refer now to VIII, §20, we see that every $\mu$ has an expression like the second member of (19). To establish the passivity of $\sigma$ for the neighborhood of $\xi_1, [\eta]_1, [\xi]_1$, we have to show that every $\mu$, as a function of the $x$ and of the parametric derivatives, is identically zero.

Consider some $\mu_1$, say $\mu_1$. Let $Z$ be the numerator in the expression for $\mu_1$ and let $P$ represent the first member of (17). Suppose that $Z$ is not identically zero. Then the resultant $W$ of $P$ and $Z$ with respect to $w$ is not zero. If we can show that $W$ is in $\Sigma$, we will have a contradiction. Working in the abstract, let us form a generic zero of $\Sigma$; with it is associated a quantity $w$ as in (16). The generic zero and $w$ satisfy (19) and thus annul $Z$. Then the generic zero annuls $W$. Hence $Z$ is identically zero and $\sigma$ is passive at $\xi_1, [\eta]_1, [\xi]_1$.

23. Let (14), with $A_1$ not a function in $\Sigma$, be a chain. We shall find necessary and sufficient conditions for (14) to be a characteristic set of a prime ideal.

As a first necessary condition, we have the condition that (14), when regarded as a set of polynomials, be a characteristic set for a prime p.i. This implies the existence of $r$ functions $g_i$, as in (15), which annul the $A$ when substituted for the $p_i$ without annulling any separant.

Let $\xi_1, [\eta]_1$ be some set of values as above, for which no separant vanishes. A second necessary condition is that the extended system (19) be passive for the neighborhood of $\xi_1, [\eta]_1, [\xi]_1$.

We shall prove that, if (14), considered as a set of polynomials, is a characteristic set of a prime p.i., and if, for some set $\xi_1, [\eta]_1, [\xi]_1$, (19) is passive, then (14) is a characteristic set of a prime ideal.

Let (14) satisfy the stated conditions. As the expressions for the $\mu$ vanish

\(^{10}\) It may be that $D$ vanishes at $\xi_1, [\eta]_1$, but this is not a matter for concern.
identically, (19) developed for any values at all $\xi$, $[\eta]$ which annul no separant will be passive.

The passivity of (19) implies that (14) has zeros which annul no separant. We shall prove that the system $\Sigma$ of d.p. which vanish for all zeros of (14) annulling no separant is a prime ideal for which (14) is a characteristic set.

Let $GH$ be in $\Sigma$. Let $J_1G = G_1$, $J_2H = H_1$, $[A_1, \cdots, A_r]$, where the $J$ are power products in the separants and $G_1$, $H_1$ involve no proper derivatives of the $p$. There may be, in $G_1$ and $H_1$, parametric derivatives not present in (14). But (14), considered as a set of polynomials, will be a characteristic set for a prime p.i. $\Lambda$, even after the adjunction of the new parametric derivatives to the indeterminates in (14).

Let us consider any zero of $\Lambda$ which annuls no separant in (14). By the passivity of (19) for arbitrary sets $\xi$, $[\eta]$, $[\tau]$, the mentioned zero furnishes, at a point free to vary in a region in $\Lambda$, initial conditions for a zero $\eta_1, \cdots, \eta_n$ of the d.p. in (14) which annuls no separant. $\Sigma$ admits $\eta_1, \cdots, \eta_n$ as a zero. It follows that the zero of $\Lambda$ annuls $G_1H_1$. Then $G_1H_1$, considered as a polynomial, is in $\Lambda$. Suppose then that $G_1$ is in $\Lambda$. Then $G$ is in $\Sigma$. Thus $\Sigma$, which we know to be an ideal, is prime. To prove that (14) is a characteristic set for $\Sigma$, it suffices to show that $\Sigma$ contains no nonzero d.p. reduced with respect to (14); such a d.p., by what precedes, would, considered as a polynomial, belong to $\Lambda$.

24. Given a set (14) which satisfies the first condition in §23, we can determine, with a finite number of rational operations and differentiations, whether or not (19) is passive. If (19) is not passive, we secure a d.p. involving only parametric derivatives which vanishes for all zeros of (14) which annul no separant.$^{11}$

Algorithm for decomposition

25. Let $\Phi$ be any finite system of p.d.p., not all zero. As in Chapter V, we can obtain, by a finite number of differentiations, rational operations and factorizations, a set, equivalent to $\Phi$, of finite systems $\Lambda_1, \cdots, \Lambda_\ell$ which have the following properties:

(a) The characteristic sets of the $\Lambda_i$ are not higher than those of $\Phi$.

(b) If the characteristic set of a $\Lambda_i$ involves indeterminates, the remainder of any d.p. of $\Lambda_i$ with respect to the characteristic set is zero.

(c) The characteristic set of a $\Lambda_i$, considered as a set of polynomials, is a characteristic set of a prime p.i.

Suppose that $\Lambda_1$ has a characteristic set (14) with $A_1$ not in $\mathfrak{F}$. If (19) is not passive, $\Lambda_1$ is equivalent to

$$\Lambda_1 + G, \Lambda_1 + S_1, \cdots, \Lambda_1 + S_r,$$

where the $S$ are the separants, and $G$, involving only parametric derivatives, $^{11}$ In (19) and in the analogous expressions for the $\mu$, we may use a single $T$. If $Z$ of §22 does not vanish, TW will serve our purpose.
vanishes for every zero of (14) which annuls no $S$. Now all of the systems just obtained have characteristic sets lower than (14). If (19) proves passive, $\Lambda_1$ is equivalent to

$$\Sigma, \Lambda_1 + S_1, \cdots, \Lambda_1 + S_r,$$

where $\Sigma$ is the prime ideal for which (14) is a characteristic set.

It is clear that, by this process, we arrive in a finite number of steps at a finite number of chains which are characteristic sets of a set of prime ideals equivalent to $\Phi$.

The above constitutes an elimination theory for systems of algebraic partial differential equations.

26. The assumption that the first mark of each $x$ is unity prevents us from using, in the case of one independent variable, the ordering employed in the earlier chapters. Thus, when the first mark is unity, no derivative of $y_2$ will be higher than every derivative of $y_i$. Now, in the case of $m = 1$, no two $p$ in (15) are derivatives of the same $y$. Thus, with any marks, when $m = 1$, the equations (15) are a set of ordinary differential equations for which the standard existence theorem can be used. We see that, when $m = 1$, (14) will be a characteristic set of a prime ideal if it is a characteristic set of a prime p.i.; one may use any marks which effect a complete ordering. In this way, the theory of characteristic sets of prime ideals is so framed as to include, in the case of $m = 1$, our earlier considerations.

**The theorem of zeros**

27. We treat the theorem of zeros in the analytic case. Let there be given p.d.p. $F_1, \cdots, F_p$, and a $G$ which vanishes for every analytic zero of the $F$. We have to show that $G$ is contained in $\{ F_1, \cdots, F_p \}$. Let $\Sigma$ be an essential prime divisor of the perfect ideal. Suppose that $\Sigma$ does not contain $G$. Let (14) be a characteristic set for $\Sigma$. Let $R$ be the remainder of $G$ with respect to (14) and let $K = R S_1 \cdots S_r$. Then $K$, as a polynomial, is not in $\Lambda$ of §21. A zero of $\Lambda$ which does not annul $K$ furnishes initial conditions for a zero of (14) which is a zero of $\Sigma$ and does not annul $G$. 