CHAPTER VII
INTERSECTIONS OF ALGEBRAIC DIFFERENTIAL MANIFOLDS

DIMENSIONS OF COMPONENTS OF INTERSECTIONS

1. B.L. van der Waerden has shown\(^1\) that if two irreducible algebraic manifolds in the space of \(y_1, \cdots, y_n\) have the respective dimensions \(p\) and \(q\), every component of their intersection is of dimension at least \(p + q - n\). For algebraic differential manifolds, there is no such regularity. We shall exhibit, for the case of \(n = 3\), two irreducible manifolds of dimension 2 whose intersection consists of a single point.

2. Working with \(u, v, y\), we let
\[
F = u^5 - v^5 + y(uv_1 - vu_1)^2.
\]
We take \(\mathbb{F}\) as the field of complex numbers. \(F\) is algebraically irreducible. We shall find its components. A component other than \(\mathcal{M}\), the general solution, must be held by the coefficient of \(y\), therefore by \(u^5 - v^5\). Let
\[
A_j = u - \omega^j - v, \quad j = 1, \cdots, 5,
\]
where \(\omega = e^{2\pi i/5}\). As
\[
w_1 - vu_1 = v_1A_j - vA_j', \quad j = 1, \cdots, 5,
\]
it follows from the low power theorem that, for each \(j\), the manifold of \(A_j\) is a component of \(F\). Thus \(F\) has six components, each of dimension 2.

The manifold \(\mathcal{M}'\) of \(y\) is two-dimensional. We shall show that \(\mathcal{M}\) and \(\mathcal{M}'\) have precisely one point in common, the point \(u = v = y = 0\).

We show first that \(u = v = y = 0\) is in \(\mathcal{M}\). Let \(\bar{u}, \bar{v}, \bar{y}\) be any point of \(\mathcal{M}\) which does not annul \(u^5 - v^5\). If \(c\) is an arbitrary constant with respect to \(\mathbb{F} < \bar{u}, \bar{v}, \bar{y}>\), it follows from the homogeneity of \(F\) and \(u^5 - v^5\) that \(c\bar{u}, c\bar{v}, c\bar{y}\) is in \(\mathcal{M}\). Then every d.p. which holds \(\mathcal{M}\) vanishes for \(u = v = y = 0\) and our statement is proved.

Now let \(\bar{u}, \bar{v}, 0\) be a point of \(\mathcal{M}\).

For each \(j\), we write \(u^5 - v^5 = A_jB_j\). For every zero of \(F\) with \(y = 0\), in particular, for every point of \(\mathcal{M}\) with \(y\) equal to zero, \(u^5 - v^5\), and therefore some \(A_j\), vanishes. By III, §23, a zero of an \(A_j\) which lies in \(\mathcal{M}\) annuls \(B_j\), and therefore annihilates some \(A_k\) with \(k \neq j\). It follows that \(\bar{u} = \bar{v} = 0\).

The anomaly which we have just found has nothing to do with “points at infinity.” It would be futile to try to remove it by creating a “projective space.”

ORDERS OF COMPONENTS OF AN INTERSECTION

3. We consider, as in V, §27, a finite system \(\Psi\) of nonzero d.p. in \(y_1, \cdots, y_n\). Let \(y_i\) be involved in \(\Psi\) up to the order \(m_i\).

Let \( \mathcal{M} \), the manifold of a prime ideal \( \Gamma \), be a component of \( \Psi \) of dimension \( q \). We are interested in securing a bound for the order of \( \mathcal{M} \) when \( q = 0 \), and a bound for the order of \( \mathcal{M} \) relative to any given parametric set when \( q > 0 \) (II, §35). We shall secure a bound for the order, or for the relative order, in terms of the \( m_i \).

The result which we shall obtain may be regarded as a counterpart, for systems of algebraic differential equations, of Bézout's theorem on the number of solutions of a system of algebraic equations.

4. Suppose first that \( q = 0 \).

We consider the system \( \Phi \) of d.p. in indeterminates \( u_{ij}, v_i \), obtained from \( \Psi \) as in V, §27. Some prime ideal \( \Sigma' \) as in V, §26, goes over into \( \Gamma \) by the substitution\(^3\) (29). \( \Sigma' \) is contained in a prime ideal \( \Sigma \), described as in V, §20. We wish to see that \( \Sigma \) is of dimension zero. Suppose that \( \Sigma \) is of positive dimension; it will have a parametric set \( u_1, \ldots, u_s; v_1, \ldots, v_r \). Suppose that \( v \) are actually present in this set. Then \( v_i \) corresponds to some \( y_{ij} \) in \( \Psi \). As \( \Gamma \) has a d.p. in \( y_j \) alone, \( \Sigma \) has a d.p. in \( v_i \) alone. Thus there are no parametric \( v \).

Let us now consider \( u_i \). It corresponds to some \( y_{ij} \). \( \Gamma \) has a nonzero d.p. \( N \) in \( y_j \) alone. We assume \( N \) to be algebraically irreducible. Suppose that \( N \) involves derivatives of \( y_j \) of order less than \( s \). Let \( y_{ij} \) be the lowest such derivative. Then some linear combination of \( N \) and its derivative is free of \( y_{ij} \). Continuing, we find a nonzero d.p. \( P \) in \( \Gamma \), involving \( y_j \) alone, in which the derivatives of \( y_j \) are of orders at least \( s \). To \( P \), there corresponds in \( \Sigma \) a d.p. in \( u_i \) alone.

Thus \( \Sigma \) is of dimension zero. The set (18) thus becomes

\[
\begin{align*}
&u_{h+1}, \ldots, u_m; \quad u'_{i1}, \ldots, u'_{ih}; \quad v_1, \ldots, v_r.
\end{align*}
\]

We consider the system (20) which corresponds to \( \Sigma \). The second members are expressions in \( u_{i1}, \ldots, u_{ih}; w \).

We seek a bound for \( h \). If \( y_i \) occurs up to the order \( m_i \) in \( \Psi \), \( y_i \) yields \( m_i \) letters \( u \). Thus

\[
h \leq m_1 + \cdots + m_n.
\]

Let \( \bar{u}_1, \ldots, \bar{u}_m; \bar{v}_1, \ldots, \bar{v}_r; \bar{w} \) be a generic zero of \( \Sigma \). Let \( \zeta \) be a derivative of any order of one of the quantities \( \bar{u}, \bar{v} \). Then \( \zeta \) has an expression which is rational in \( \bar{u}_1, \ldots, \bar{u}_h; \bar{v} \). If \( \zeta \) is one of the quantities just written, the expression is \( \zeta \) itself. Otherwise, we use (20) and \( R = 0 \); a sufficient number of differentiations and substitutions gives the desired expression for \( \zeta \). In particular, proper derivatives of \( w \) which appear during the differentiations of the \( \epsilon \) are obtained by differentiation from \( R = 0 \).

Let us consider now any \( h + 1 \) of the letters \( u_{ij}, v_{ij} \). They furnish \( h + 1 \) quantities \( \zeta \), with expressions as just described. Using these \( h + 1 \) expressions, and the relation \( R = 0 \) for \( \bar{u}_1, \ldots, \bar{u}_h; \bar{v}, \bar{w} \), we obtain, by an elimination, a non-

\footnote{We use, at present, equation numbers of Chapter V. The \( u_i \) of \( \Sigma' \) are the \( u_{ij} \) of \( \Phi \).}
zero polynomial in the \( h + 1 \) letters \( u_{ij}, v_{ij} \) which is a d.p. in \( \Sigma \). It follows that, given any \( h + 1 \) distinct \( y_{ij}, \Gamma \) contains a nonzero d.p. which involves only those \( y_{ij} \). This means, by II, §35, that the order of \( \Gamma \) cannot exceed \( h \).

We may thus state the following theorem:

**Theorem:** Let \( \Phi \) be a finite system of nonzero d.p. in \( \mathcal{F}\{ y_1, \ldots, y_n \} \). Let \( m_1 \) be the maximum of the orders of those derivatives of \( y_1 \) which appear in \( \Phi \). If a component \( \mathcal{M} \) of \( \Phi \) is of dimension zero, the order of \( \mathcal{M} \) is at most \( m_1 + \cdots + m_n \).

5. We now suppose that \( q > 0 \). We write the indeterminates as \( u_1, \ldots, u_q; y_1, \ldots, y_p \), with the \( u \) parametric for \( \Gamma \). Let \( A_1, \ldots, A_p \), with \( A_i \) of order \( r_i \) in \( y_1 \), be a characteristic set for \( \Gamma \). Let \( B_i, i = 1, \ldots, p \), be a nonzero d.p. in \( \Gamma \) involving only \( y_1 \) and the \( u \). Let \( C \) be a d.p. which is not in \( \Gamma \) and which holds every component of \( \Psi \) other than \( \mathcal{M} \).

Let \( \bar{u}_1, \ldots, \bar{u}_q; \bar{y}_1, \ldots, \bar{y}_p \) be a generic zero of \( \Gamma \). Let the \( \bar{u} \) be substituted for the \( u \) in \( \Psi \). Then \( \Psi \) becomes a system \( \Psi' \) in \( y_1, \ldots, y_p \) over \( \mathcal{F}<\bar{u}_1, \ldots, \bar{u}_q> \). Each \( B_i \) becomes a nonzero \( B'_i \) and \( C \) a nonzero \( C' \).

Let the components of \( \Psi \) other than \( \mathcal{M} \) be manifolds of prime ideals \( \Gamma_1, \ldots, \Gamma_p \). Then \( \Psi' \) is equivalent to \( \Gamma'_1, \ldots, \Gamma'_p \), each accented system resulting from the corresponding unaccented one when the \( u \) are replaced by the \( \bar{u} \).

The totality of d.p. in \( \mathcal{F}<\bar{u}_1, \ldots, \bar{u}_q> \{ y_1, \ldots, y_p \} \) which vanish for \( \bar{y}_1, \ldots, \bar{y}_p \) is a prime ideal \( \Delta \) held by \( \Psi' \). Each \( \Gamma'_i \) is held by \( C' \), while \( \Delta \) is not. Thus, the manifold of \( \Delta \) is contained in a component \( \mathcal{M}' \) of \( \Psi' \) which is held by \( \Gamma' \). \( \mathcal{M}' \) must be of dimension zero, since each \( B'_i \) is in \( \Gamma' \). Then \( \Delta \) is of dimension zero and, by II, §36, its order \( h \) does not exceed the order of \( \mathcal{M}' \).

Now \( \Delta \) contains no d.p. involving only \( y_{ij} \) with \( j < r_i \); otherwise, there would be a nonzero d.p. reduced with respect to \( A_1, \ldots, A_p \) which vanishes for the generic zero of \( \Gamma \). By II, §35, \( h \geq r_1 + \cdots + r_p \). By §4, if the highest derivative of \( y_1 \) in \( \Psi \) is of order \( m_i \), the order of \( \mathcal{M}' \) does not exceed \( m_1 + \cdots + m_p \). Thus

\[
r_1 + \cdots + r_p \leq m_1 + \cdots + m_p.
\]

We may thus formulate the following theorem:

**Theorem:** Let \( \Phi \) be a finite set of d.p. in \( \mathcal{F}\{ u_1, \ldots, u_q; y_1, \ldots, y_p \} \), the \( u \) being a parametric set for a component \( \mathcal{M} \) of \( \Phi \). Let \( m_i, i = 1, \ldots, p \), be the maximum of the orders of those derivatives of \( y_1 \) which appear in \( \Phi \). Then the order of \( \mathcal{M} \) relative to \( u_1, \ldots, u_q \) cannot exceed \( m_1 + \cdots + m_p \).

In what precedes, the condition that \( \mathcal{M} \) be a component of \( \Phi \) is essential. For instance, taking \( \Phi \) as \( y_{10} + y_{20} \), the manifold of \( y_{1n} + y_{10} + y_{20} \), which is of order \( n \), is, for every \( n \), held by \( \Phi \).

6. Jacobi examined, from the heuristic standpoint, the problem of determining the number of arbitrary constants in the solution of a system of \( n \) differential equations in \( n \) unknowns.\(^8\) Taking the system in the form

\(^8\) See Ritt, 29.
\[ u_i = 0, \quad i = 1, \ldots, n, \]
where each \( u \) involves the unknowns \( y_1, \ldots, y_n \), a certain number of their derivatives and the independent variable \( x \). Jacobi considers the derivatives of \( y_j \) appearing in \( u_i \) and denotes the maximum of the orders of those derivatives by \( a_{ij} \). He forms all sums

\[ a_{ij_1} + \cdots + a_{ij_n} \]

where \( j_1, \ldots, j_n \) is a permutation of \( 1, \ldots, n \). He arrives at the conclusion that the number of arbitrary constants in the solution of (1) does not exceed the greatest sum (2).

After our study of algebraic differential manifolds, it is unnecessary to insist on the fact that the notion of the number of constants in the solution of a general system never was a notion which was definite in advance. For algebraic systems, the concept is made definite by the theory of orders of irreducible manifolds which has been developed here. It is thus not surprising that Jacobi's work on this question, in spite of its daring and ingenious quality, should not have firm logical structure.

One would be disposed to regard Jacobi's work as conjectural and to expect that his bound would be found valid in a rigorous theory. We shall see later that Jacobi's bound, like weaker ones given before his time, does not have the broad applicability which one might anticipate for it. We shall treat now a situation in which Jacobi's bound is found to hold.

We deal with two nonzero d.p. \( A \) and \( B \) in \( y \) and \( z \). We represent by \( a \) and \( b \) the respective orders of \( A \) in \( y \) and \( z \); by \( c \) and \( d \) the orders of \( B \) in \( y \) and \( z \). Let

\[ h = \text{Max} (a + d, b + c). \]

We prove the following theorem:

**Theorem:** If \( \mathcal{M} \), of dimension zero, is a component of the system \( A, B \), the order of \( \mathcal{M} \) is at most \( h \).

We assume that \( \mathcal{M} \) is of order greater than \( h \) and produce a contradiction. Fixing our ideas, we assume that \( b \geq d \).

There exist nonzero d.p. \( C \) whose orders in \( y \) and \( z \) do not exceed \( c \) and \( d \) respectively and which hold \( \mathcal{M} \), such that the system \( A, C \) has no component of dimension unity containing \( \mathcal{M} \). \( B \) is such a d.p. From among all such d.p. \( C \), we select one which, for the order \( y, z \) of the indeterminates, is of a least rank. The d.p. selected will be denoted by \( D \).

We are going to prove that \( D \) is free of \( z \). We assume that \( z \) is present in \( D \) and force a contradiction.

Let \( D \) be of order \( e \) in \( y \) and of order \( f \) in \( z \). Let \( S \) be the separant of \( D \). There is a relation

\[ \text{A better bound can be given in the case in which one of } y \text{ and } z \text{ is absent from one of } A \text{ and } B. \text{ For instance, if } B \text{ is free of } z, \text{ it can be shown as below that the order of } M \text{ does not exceed } b + c. \]
\[ S'A = E, \quad [D], \]
where \( E \) has an order in \( z \) not exceeding \( f \) and an order in \( y \) not exceeding
\[ \text{Max} (a, e + b - f). \]

Suppose that \( S \) does not hold \( \mathfrak{M} \). Let \( \mathfrak{M}' \) be a component of the system \( D \), \( E \) which contains \( \mathfrak{M} \). Then \( \mathfrak{M}' \) is a component of \( A, D \) and is thus of dimension zero. Applying the theorem of \( \S 4 \) to \( D, E \) and using II, \( \S 36 \), we find that the order of \( \mathfrak{M} \) does not exceed \( ^c \)
\[ \text{Max} (a, e + b - f) + f = \text{Max} (a + f, e + b). \]

As \( f \leq d \) and \( e \leq c \), the order of \( \mathfrak{M} \) cannot exceed \( h \).

Thus \( S \) holds \( \mathfrak{M} \). If \( D \) is of degree \( q \) in \( z_r \), then
\[ qD = z_r S + T, \]
where \( T \), like \( S \), is of lower rank than \( D \) in \( z \). Also \( T \) holds \( \mathfrak{M} \).

By I, \( \S 29 \), we may assume that \( \bar{\mathfrak{F}} \) has a nonconstant element.\(^4\) We shall prove the existence of an element \( \mu \) in \( \bar{\mathfrak{F}} \) such that all components of the system
\[ A, \quad S + \mu T \]
which contain \( \mathfrak{M} \) are of dimension zero. As \( S + \mu T \) will be of lower rank than \( D \) in \( z \), our statement that \( D \) is free of \( z \) will be proved.

By (3), the system \( A, D \) holds the system
\[ A, S, T. \]

Let \( u \) be an indeterminate. We consider the system
\[ A, \quad S + uT \]
in \( y, z, u \). Let the essential prime divisors of the perfect ideal determined by the system (6) be \( \Sigma_1, \ldots, \Sigma_r \). Let \( \Sigma_1, \ldots, \Sigma_s \) be those \( \Sigma \) which are not held by (5). We say that each of these ideals contains a nonzero d.p. in \( y \) and \( u \) alone and a nonzero d.p. in \( z \) and \( u \) alone.

Suppose that \( \Sigma_1 \) contains no d.p. in \( y \) and \( u \) alone. Then, if the indeterminates are taken in the order \( u, y, z, \Sigma_1 \) has a characteristic set composed of one d.p., so that the manifold of \( \Sigma_1 \) is the general solution of a d.p. \( F \) (II, \( \S \S 18, 33 \)). Now \( F \) cannot involve \( u \), for \( F \) will continue to be a characteristic set for \( \Sigma_1 \) if the indeterminates are taken in the order \( y, z, u, A \), which is in \( \Sigma_1 \), does not involve \( u \). We take the remainder of \( S + uT \) with respect to \( F \) for the order \( u, y, z \). We secure a relation

\(^4\) As \( b \geq d \geq f \), we have \( e + b - f \geq e \).

\(^4\) Suppose that \( \bar{\mathfrak{F}} \) consists purely of constants. The ideal \( \{ A, B \} \) has essential prime divisors \( \Sigma_1, \ldots, \Sigma_s \). When an element \( z \) of derivative unity, is adjoined to \( \bar{\mathfrak{F}} \), we secure a larger \( \{ A, B \} \). Its essential prime divisors can easily be shown to be the prime ideals generated by the \( \Sigma \) in \( \bar{\mathfrak{F}} < z > \). The new prime ideals have the same characteristic sets as the old ones.
\[ J(S + uT) = 0, \quad [F], \]

with \( J \) a power product in the initial and separatant of \( F \). This means, since \( F \) is free of \( u \), that

\[ JS = 0, \quad JT = 0, \quad [F]. \]

It follows that each d.p. in (5) is in \( \Sigma_i \).

This proves that \( \Sigma_i \), for \( i \leq s \), contains a d.p. \( H_i \) in \( y \) and \( u \) alone. Similarly each such \( \Sigma_i \) contains a d.p. \( K_i \) in \( z \) and \( u \) alone. Let \( M \) be the product of the d.p. \( H \) and \( N \) the product of the \( K \). Let \( u \) be fixed as an element \( \mu \) in \( F \) so that \( M \) goes over into a nonzero d.p. \( U \) in \( y \) alone, \( N \) into a nonzero d.p. \( V \) in \( z \) alone.

Then those zeros of

\[ (7) \quad A, \quad S + \mu T \]

which are not zeros of (5) must annul \( U \) and \( V \). A fortiori, all zeros of (7) which are not zeros of \( A, D \) annul \( U \) and \( V \).

This shows that a component of (7) which is not contained in a component of \( A, D \) is held by \( U \) and \( V \). Then every component of (7) which contains \( \mathfrak{M} \) is of dimension zero. This proves that \( D \) is free of \( z \).

\( D \) must involve \( y \) effectively, since \( A, D \) has zeros. Denoting still by \( S \) the separatant of \( D \), we secure a relation

\[ S^2A = L, \quad [D], \]

where the orders of \( L \) in \( y \) and \( z \) do not exceed \( e \) and \( b \) respectively. We reason with \( D, L \) as with \( D, E \), above, to show that \( S \) holds \( \mathfrak{M} \). Then we follow the method above to find a system similar to (4), with \( S + \mu T \) of lower rank than \( D \). Thus it is not possible to choose a d.p. of lower rank among the d.p. \( C \). This completes the proof that the order of \( \mathfrak{M} \) does not exceed \( h \).

**Intersections of General Solutions**

7. By all the rules of play, the bound \( h \) of §6 should, when \( A \) and \( B \) are algebraically irreducible, apply to the components of dimension zero in the intersection of the general solutions of \( A \) and \( B \). The general solution of a d.p. \( F \) in \( y \) and \( z \) can be regarded as the solution of the differential equation obtained by solving for the highest derivative of one of \( y \) and \( z \) in the equation \( F = 0 \). To be sure, we would then be dealing with irrational differential equations. However, as Jacobi's considerations are detached from questions of the theory of functions, one would not expect irrationality to have a bearing on the problem. It might be suggested that Jacobi's heuristic work, as well as previous work which yielded bounds like that of §4, was intended to apply to the "general case." If so, the heuristic history of differential equations has been different from that of algebraic equations. Bézout's work of the middle eighteenth century, on the number of solutions of a system of algebraic equations, was entirely heuristic. His conjecture was validated, late in the nineteenth century, not for a "general case" but for all systems.
The actual situation is as follows. If the orders of \( A \) and \( B \) in each of \( y \) and \( z \) do not exceed unity, we get the bound of §6 for a component of the intersection of the general solutions. For higher orders, that bound need not hold. We shall show how to construct, for every \( n > 3 \), a d.p. of order \( n \) in \( y \) and in \( z \) whose general solution intersects the manifold of \( y \) in an irreducible manifold of dimension zero and order \( 2n - 3 \).

8. We consider \( A \) and \( B \), as in §6, assuming them to be algebraically irreducible, with each of \( a, b, c, d \) not greater than unity. We prove the following theorem.

**Theorem:** If \( \mathfrak{M} \), of dimension zero, is a component of the intersection of the general solutions of \( A \) and \( B \), the order of \( \mathfrak{M} \) does not exceed \( h \).

Thus the order of \( \mathfrak{M} \) does not exceed 2.

We represent by \( \mathfrak{M} \) the intersection of the general solutions of \( A \) and \( B \).

If \( a, b, c, d \) are all zero, the general solutions of \( A \) and \( B \) are their complete manifolds and we have merely to apply the theorem of §4.

Suppose now that \( a = b = 0 \) and that at least one of \( c \) and \( d \) is 1. We consider first the intersection \( \mathfrak{M}' \) of the complete manifolds of \( A \) and \( B \). Every component of \( \mathfrak{M}' \) of dimension zero has an order not exceeding unity. By II, §36, if \( \mathfrak{M} \) is not contained in a component of \( \mathfrak{M}' \) of dimension unity, the order of \( \mathfrak{M} \) does not exceed unity.

We have now to consider the case in which \( \mathfrak{M} \) is contained in a component \( \mathfrak{M}'' \) of \( \mathfrak{M}' \) of dimension unity. \( \mathfrak{M}'' \) is the general solution of a d.p. \( C \). Because \( A \) holds \( \mathfrak{M}'' \), \( C \) must be of order zero in each of \( y \) and \( z \); this implies that \( \mathfrak{M}'' \) is the manifold of \( A \). Then \( \mathfrak{M}'' \) must be a component of the manifold of \( B \). Otherwise \( \mathfrak{M}'' \) would be contained in the general solution of \( B \) and \( \mathfrak{M} \) would not be a component of \( \mathfrak{M} \).

We suppose, as we may, that \( A \) involves \( z \) effectively. As \( \mathfrak{M}'' \) is a component of \( B \) other than the general solution, we have \( d = 1 \) (III, §15). Let \( S \) be the separant of \( A \). We have, by the low power theorem, a relation

\[
S^IB = C_0A^p + C_1A^{p+q_1} + \cdots + C_rA^{p+q_r}.
\]

Here \( A_1 \) is the derivative of \( A \) and, for every \( i \), \( p_i + q_i > p \). The orders of the \( C \) in \( z \) and in \( y \) do not exceed 0 and 1 respectively, and no \( C \) is divisible by \( A \).

By III, §23, as \( \mathfrak{M} \) is in the intersection of \( \mathfrak{M}'' \) and the general solution of \( B \), \( C_0 \) must hold \( \mathfrak{M} \). The manifold of the system \( C_0, A \) is a proper part of \( \mathfrak{M}'' \) and thus, by II, §36, has components which are all of dimension zero. By §6, the order of such a component cannot exceed unity. Then the order of \( \mathfrak{M} \) does not exceed unity; this was what was to be proved.

Suppose now that at least one of \( a \) and \( b \) is unity and that at least one of \( c \) and \( d \) is unity. We take up immediately the case in which \( \mathfrak{M} \) is contained in a component \( \mathfrak{M}' \) of \( \mathfrak{M}'' \) of dimension unity; when \( \mathfrak{M} \) is not so contained, it follows

---

7 By III, §15, the components of \( B \) other than its general solution are manifolds of d.p. of orders zero in \( y \) and \( z \).
from §6 that its order does not exceed \( h \). As \( \mathfrak{M} \) is a component of \( \mathfrak{N} \), \( \mathfrak{M}'' \) is not part of \( \mathfrak{N} \). Let, then, \( \mathfrak{M}'' \) fail to be contained in the general solution of \( B \). Then some other component of \( B \) contains \( \mathfrak{M}'' \) and is thus identical with \( \mathfrak{M}'' \). By the case which precedes, the components of the intersection of \( \mathfrak{M}'' \) with the general solution of \( B \) are of dimension zero and of order at most unity. This completes the proof.

9. We are going to present a d.p. \( F \) in \( y \) and \( z \), of order 4 in \( y \) and in \( z \), whose general solution will be shown to intersect the manifold of \( y = 0 \) in an irreducible manifold of dimension zero and order 5.

Through §13, \( K_1 \) will represent, for any d.p. \( K \), the derivative of \( K \). We let

\[
\begin{align*}
A &= y_1 - z_3 y^2, \\
B &= A^4 - y_3^8, \\
C &= y_3 A_1 - 2y_4 A, \\
F &= B - y^6 C^2 = A^4 - y_3^8 - y_4^6 C^2.
\end{align*}
\]

We use the field of rational numbers. Let us see first that \( F \) is algebraically irreducible. If we consider the equation \( F = 0 \) as an algebraic equation for \( y_4 \), we secure a function \( y_4 \) of two branches. Thus, if \( F \) were factorable, it would have a factor of positive degree free of \( y_4 \). Such a factor would have to be a factor of \( y^6 A^2 \). As \( F \) is not divisible by \( y \) or by \( A \), \( F \) is algebraically irreducible.

Let us now determine the components of \( F \) other than the general solution.

Let \( \mathfrak{N} \) be such a component. As \( \partial F / \partial y_4 = 4y^6 AC \), \( \mathfrak{N} \) must be held by \( yc \) or by \( A \). Suppose that \( A \) holds \( \mathfrak{N} \). By (10) and (11), \( y_3 \) holds \( \mathfrak{N} \). In every case then, \( B \) holds \( \mathfrak{N} \).

Now \( B \) is the product of the four d.p.

\[
E^{(j)} = y_1 - z_3 y^2 - jy_3^2, \quad j = \pm 1, \quad \pm (-1)^{1/2},
\]

each of which is algebraically irreducible. For what follows, it is important to know that the manifold of each \( E \) is irreducible. From the manner in which \( z_3 \) figures in (12), one sees that a component of \( E^{(j)} \) other than the general solution is held by \( y \). Such a component, being of dimension unity, must be the manifold of \( y \). But the low power theorem shows that the manifold of \( y \) is not a component. This proves the irreducibility of the manifolds of the \( E \).

We have, for every \( j \),

\[
C = y_3 E_1^{(j)} - 2y_4 E^{(j)}.
\]

Referring to (11), and applying the low power theorem, we see that the manifold of each \( E \) is a component of \( F \).

It will be proved that the intersection of the general solution of \( F \) with the manifold of \( y = 0 \) is the manifold of the system \( y = 0, z_3 = 0 \). The latter manifold is of dimension zero and order 5.

\[\text{footnote}{For the order \( y, z \) of the indeterminates, \( F \) as it stands is in the form (25) of III, §17.}\]
10. We refer to I, §26. We use any positive integer $p$ and any power product $P$ in $y$ and its derivatives. The degree of $P$ is denoted by $d$ and its weight by $w$. The second member of (31) of I, §26, will be represented by $\delta(p, w)$. Let $U = y^p$. We shall prove that $P$ has a representation as a homogeneous polynomial in $U$ and derivatives of $U$, whose coefficients are homogeneous polynomials\(^9\) in $y$ and derivatives of $y$ of a common degree not greater than $\delta(p, w)$.

If $d \leq \delta(p, w)$, $P$ itself is the representation sought. Otherwise, by I, §26, $P$ is a linear combination of $U$ and its derivatives, with coefficients all of degree $d - p$ and none of weight exceeding $w$. If $d - p \leq \delta(p, w)$, we have the desired representation. Otherwise, the coefficients of $U$ and its derivatives will be in $[U]$. Continuing, we have $P$ expressed as in our statement.

11. Let $\Sigma$ be an ideal of d.p. in $y$ and $z$; $M$ a d.p. in $y$ and $z$; $\alpha$ a nonnegative number. We shall say that $M$ admits $\alpha$ as a multiplier with respect to $\Sigma$ if, for every $\varepsilon > 0$, there exists an integer $n_0(\varepsilon)$ such that, for every $n > n_0(\varepsilon)$,

$$M^n = P, \quad (\Sigma),$$

where $P$ is a d.p. depending on $M$ and $n$ which, arranged as a polynomial\(^8\) in the $y$, contains no term of degree less than $n(\alpha - \varepsilon)$. $P$ may be zero. If $\alpha$ is a multiplier for $M$ and if $0 \leq \gamma < \alpha$, $\gamma$ is also a multiplier.

We prove the following properties of multipliers:

(a) Let $M$ and $N$ admit $\alpha$ and $\beta$, respectively, as multipliers with respect to $\Sigma$. Let $\gamma = \min (\alpha, \beta)$. Then $M + N$ admits $\gamma$ as a multiplier.

(b) For $M$ and $N$ as in (a), $MN$ admits $\alpha + \beta$ as a multiplier.

(c) Let $M^p$, where $p$ is a positive integer, admit $\alpha$ as a multiplier. Then $M$ admits $\alpha/p$.

(d) Let $M$ admit $\alpha$ as a multiplier. Then $M_1$, the derivative of $M$, also admits $\alpha$.

(e) If $M = N$, $(\Sigma)$, $M$ and $N$ admit the same multipliers.

Proving (a), we take an $\varepsilon > 0$. Let $n_0(\varepsilon/2)$ serve as above for both $M$ and $N$ with respect to $\varepsilon/2$. We consider $(M + N)^n$ for any $n \geq 1$. Let $R = M^n N^b$ where $a + b = n$. If $a$ and $b$ both exceed $n_0(\varepsilon/2)$, we have $R = P, (\Sigma)$, where no term of $P$ is of degree less than

$$a(\alpha - \varepsilon/2) + b(\beta - \varepsilon/2),$$

which quantity is not less than $n(\gamma - \varepsilon/2)$. If $b \leq n_0(\varepsilon/2) < a$, we have $R = P, (\Sigma)$, with no term of $P$ of degree less than

$$[n - n_0(\varepsilon/2)](\alpha - \varepsilon/2).$$

The last quantity, if $n$ is large in comparison with $n_0(\varepsilon/2)$, exceeds $n(\alpha - \varepsilon)$. The truth of (a) is now clear.

The proofs of (b), (c), and (e) are trivial.

\(^9\) Over the field of rational numbers.

\(^8\) When $P$ is thus arranged, its coefficients are d.p. in $z$. The definition of multiplier thus gives a special role to $y$. 
Proving (d), we take an $\epsilon > 0$ and, relative to $M$, an $n_0(\epsilon/2)$. Let $m$ be a fixed integer which exceeds $n_0(\epsilon/2)$. We consider an $n > 0$ and use $\delta(m, n)$ as in §10. Then $M^n$ is a polynomial in $M^m$ and its derivatives, with coefficients which are d.p. in $M$ of degree not greater than $\delta(m, n)$. In this expression, every power product in $M^m$ and its derivatives is of degree not less than

$$q = \lfloor n - \delta(m, n) \rfloor / m.$$  

Now if $n$ is large, $\delta(m, n)$, as one sees from I, §26, is small in comparison with $m$, so that $q$ is only slightly less than $n/m$. Each power product in $M^m$ and its derivatives is congruent to a d.p. whose terms have degrees in the $y$s not less than $qm(\alpha - \epsilon/2)$. If $n$ is large, this quantity exceeds $n(\alpha - \epsilon)$, q.e.d.

12. We return to $F$ of §9, denoting the general solution of $F$ by $\mathfrak{M}$. We show now that a point in $\mathfrak{M}$ with $y = 0$ satisfies $z_6 = 0$. Later we shall prove that every $z$ with $z_5 = 0$ is admissible.

We determine first a d.p. $G$ which holds $\mathfrak{M}$, but no other component of $F$.

We have, by (9) and (10),

$$AB_1 - 4A_1B = 4y_5^3C.$$  

Thus, by (11) (first representation of $F$), we have when $F = 0$

$$4y_5^3B^{1/2} = y^3(AB_1 - 4A_1B).$$  

Again, letting $K = y^3C$, we have by (11), when $F = 0$, the relation $B^{1/2} = K$.

Thus, for $F = 0, B \neq 0$,

$$B^{-1/2}B_1 = 2K_1.$$  

Substituting into (15) the expression which (16) furnishes for $B_1$, we find, for $F = 0, B \neq 0$,

$$4y_5^{14} + L = 0,$$

where

$$L = -4y_5^3A_1K_1 + y^6A_2^2K_1^2 - 4y^6A_2^2B.$$  

We designate the first member of (17) by $G$. Then $G$ holds $\mathfrak{M}$.

13. In what follows, all multipliers will operate with respect to $[F, G]$.

In (11), $y_5^2$ and $y^6C^2$ contain no terms of degree less than 8 in the $y$s. Thus $A_4$ admits 8 as a multiplier so that, by (c) of §11, $A$ admits 2. Now $z_9y^2$ admits 2. By (a) of §11, $y_1$ admits 2. Then, by (d), every $y_i$ with $i \geq 1$ admits 2. From (10), using (a), (b), (d), we find that $C$ admits 4. Referring to (11) and using (e), we see now that $A^4$ admits 14 so that $A$ admits 3. By (10), now, $C$ admits 5 and we find from (11) that $A$ admits 4. We return to (10) and see that $C$ admits 6. Also, by (11), $B$ admits 18. Finally, $K$ of §12 admits 9.

By (18), $L$ admits 30. By (17), $y_3$ admits 15/7. Now $y_3 - z_9y^2 - 2z_9y_1$, which is $A_1$, admits 4. As $y_1$ admits 2, $y_2 - z_9y^2$ admits 3. Then $y_3 - z_9y^2 - 2z_9y_1$ admits 3 so that $y_4 - z_9y^2$ admits 3. As $y_3$ admits 15/7, $z_9y^2$ admits 15/7.
We infer that \([F, G]\) contains a d.p. of the type \((z_0 y^2)^m + M\) where every term of \(M\) is of degree greater than \(2m\) in the \(y_i\). It follows from III, §23, that a point in \(\mathfrak{M}\) cannot have \(y = 0\) unless \(z_5 = 0\).

14. Let \((0, \alpha)\) be a generic point in the manifold of \(y = 0, z_5 = 0\). We shall prove that \(\mathfrak{M}\) contains \((0, \alpha)\). This will imply that \(\mathfrak{M}\) contains the manifold of \(y, z_5\), and our investigation of \(F\) will be completed.

Representing by \(c\) an arbitrary constant with respect to \(\mathfrak{F} < \alpha >\) and by \(v\) a new indeterminate, we make in \(F\) the substitution\(^{11}\)

\[
y = \sum_{j=1}^{6} c^j \alpha^j a_{2,j}^{-1} + c^5 v.
\]

We represent by \(A', A'_1, B', C', F'\) the expressions into which \(A, A_1, B, C, F\) are transformed when \(z\) is replaced by \(\alpha\) and \(y\) by the second member of (19).

We find from (19)

\[
A' = c^4 v_1 + c^5 P,
\]

with \(P\) a polynomial in \(\alpha_2, \alpha_3, c, v\). Then we may write

\[
A'_1 = c^5 v_2 + c^5 Q,
\]

with \(Q\) a polynomial in \(\alpha_2, \alpha_3, \alpha_4, c, v, v_1\).

From (19), we have, remembering that \(\alpha_5 = 0\),

\[
y_3 = 6c^3 \alpha_2 \alpha_4 + \cdots; \quad y_4 = 6c^3 \alpha_2^2 + \cdots.
\]

By (20), (21), (22), we have, putting \(\beta = 6\alpha_2 \alpha_4\) and \(\gamma = 12\alpha_2^2\),

\[
C' = c^9 (\beta v_2 - \gamma v_1) + c^{10} R,
\]

with \(R\) a polynomial in \(\alpha_2, \alpha_3, \alpha_4, c\) and the \(v_j\) with \(j \leq 4\). We find thus

\[
F' = c^{24}\left[v_1^4 - \beta^8 - (2\beta v_2 - \gamma v_1)^3\right] + c^{25} T,
\]

with \(T\) of the type of \(R\).

Let \(V\) represent the coefficient of \(c^{24}\) in \(F'\). As \(\beta \neq 0\), the differential equation \(V = 0\) for \(v\) is effectively of the second order. Let then \(v = \xi\) be a zero (constructed by the abstract method) of \(V\) which does not annul \(v_1^4 - \beta^8\).

We wish to show that \(F'\) is annulled by a series

\[
v = \xi + \varphi_2 c^{\rho_2} + \varphi_3 c^{\rho_3} + \cdots
\]

of the usual type, with \(\rho_2 > 0\).

It will suffice to show that \(G = F'/c^{24}\) is annulled by a series (24). If \(G\) vanishes for \(v = \xi\), then \(v = \xi\) is an acceptable series (24). In what follows, we assume that such vanishing does not occur. We put, in \(G\), \(v = \xi + u_1\). Then \(G\) goes over into an expression \(K'\) in \(c\) and \(u_1\)

\[
K' = a'(c) + \sum b'_i (c) u_{10}^a \cdots u_{14}^a.
\]

\(^{11}\) Subscripts of \(\alpha\) indicate differentiation.
Here $\sum$ contains the terms of $K'$ which are not free of the $u_{ij}$ and $i$ ranges from unity to some positive integer. As to $a'$ and the $b'$, they are polynomials in $c$ with coefficients in $F < \alpha, \xi >$. Because $\xi$ does not annihilate $G$, $a'$ is not zero. On the other hand, because $G$ vanishes for $v = \xi$, $c = 0$, the lowest power of $c$ in $a'$ is positive. Because the bracketed terms in (23) contribute effectively to $\sum$ in (25), certain of the $b'$ contain terms of zero power in $c$.

Let $\sigma'$ be the lowest exponent of $c$ in $a'$ and $\sigma'_i$ the lowest exponent of $c$ in $b'_i$. Let

$$\rho_2 = \text{Max} \frac{\sigma' - \sigma'_i}{\alpha_{zi} + \cdots + \alpha_{zi}},$$

where $i$ has the range which it has in $\sum$. As $\sigma' > 0$ and certain $\sigma'_i$ equal 0, $\rho_2 > 0$. We may now suppose ourselves to be working with $K'$ of III, §7. We obtain the series (24).

We have shown, all in all, that $F$, for $z = \alpha$, is annihilated by a series

$$y = c + c^2\alpha_2 + \cdots + c^{6}\alpha_2^4 + c^6(\alpha_2^6 + \xi) + \cdots,$$

where the unwritten terms have rational exponents greater than 6. The series (26) does not annul $B$ for $z = \alpha$. Indeed,

$$B' = c^{\sigma_1}(\beta^6 - \beta^8) + \cdots$$

and the coefficient of $c^{\sigma_1}$ does not vanish for $v = \xi$.

It follows that every d.p. which holds $\mathcal{M}$ vanishes for $z = \alpha$ and for $y$ as in (26). This means that $y = 0$, $z = \alpha$ is in $\mathcal{M}$.

15. If, in (8) to (11), we replace $z_3$, $y_3$, $y_4$ wherever they occur by $z_{n-1}$, $y_{n-1}$, $y_2$, with $n \geq 4$, we obtain a d.p. $F$ with a general solution which intersects the manifold of $y = 0$ in that of $y = 0$, $z_{2n-3} = 0$; the proofs require only the slightest changes.

In $F$ of §9, if one replaces $z_3$ by $z$, one obtains a d.p. which is of the first order in $z$ and whose general solution intersects the manifold of $y = 0$ in that of $y = 0$, $z_2 = 0$. This, in itself, is sufficiently anomalous. However, if it is desired to secure a d.p. $F$ whose order in $z$ cannot be reduced, it suffices to replace $y_3$ and $y_4$, in (9), (10), (11), by $zy_3$ and its derivative, respectively.

**INTERSECTIONS OF COMPONENTS OF A DIFFERENTIAL POLYNOMIAL**

16. Dealing with the analytic case, we prove the following theorem:

**Theorem:** Let $F$ be a d.p. in $y_1, \cdots, y_n$. A zero of $F$ which is contained in more than one component of $F$ annihilates $\partial F/\partial y_{ij}$ for $i = 1, \cdots, n$ and for every\footnote{The $j$ for which this result is significant are those for which $y_{ij}$ appears effectively in $F$.} $j$.

Thus, in particular, if $F$ vanishes for $y_i = 0$, $i = 1, \cdots, n$, and, considered as a polynomial in the $y_{ij}$, contains a term of the first degree, the zero $y_i = 0$ belongs to only one component of $F$.

Let

$$\tilde{y}_1, \cdots, \tilde{y}_n$$

(27)
be a zero for which some $\partial F/\partial y_{ij}$ fails to vanish. We shall prove that (27) is contained in only one component of $F$.

We know that systems defining the components can be secured by choosing a sufficiently large positive integer $p$ and resolving the system of derivatives

$$F, F_1, \ldots, F_p,$$

the $F$ being considered as polynomials in the $y_{ij}$, into prime p.i. none of which holds any other. We shall show that, for any $p \geq 1$, (28) yields only one prime p.i. whose polynomials vanish when each $y_{ij}$ in (28) is replaced by $\tilde{y}_{ij}$ as determined by (27). This will prove our theorem.

Reassigning the subscripts of the $y_i$ if necessary, we assume that one or more $\partial F/\partial y_{ij}$ do not vanish for (27) and let $m$ be the greatest value of $j$ for which the vanishing does not occur. Putting the polynomials in (28) equal to zero, we secure a set of equations which we shall regard as equations to be solved for those $y_{1m+j}$ for which $0 \leq j \leq p$, in terms of $x$ and the other $y_{1k}$ in (28).

Let $\xi$ be a value of $x$ at which the coefficients in $F$ and the functions in (27) are analytic, and at which $\partial F/\partial y_{1m}$ does not vanish for (27). Let $[\eta]$ represent, collectively, the values at $\xi$ of the $\tilde{y}_{ij}$ in the zero of (28) derived from (27).

The polynomials in (28) vanish at the point $\xi$, $[\eta]$ in the space of $x$ and the $y_{ij}$ in (28). We shall examine, at $\xi$, $[\eta]$, the jacobian with respect to $y_{1m}, \ldots, y_{1m+p}$ of the polynomials in (28). In the first row of this jacobian, which row we understand to consist of partial derivatives of $F$, only the first term $\partial F/\partial y_{1m}$ fails to vanish at $\xi$, $[\eta]$. To treat the other rows, let us imagine the polynomials in (28) to be expanded in powers of the various differences $y_{ij} - \tilde{y}_{ij}$. The expansion of $F$ will contain a term $\alpha(y_{1m} - \tilde{y}_{1m})$, where $\alpha$ is the function of $x$ to which $\partial F/\partial y_{1m}$ reduces for (27). By the nature of $m$, $F_1$ must contain the term $\alpha(y_{1m+1} - \tilde{y}_{1m+1})$ and can have no term $\beta(y_{ij} - \tilde{y}_{ij})$ with $j > m + 1$. Thus, in the second row of the jacobian, the value of the second element at $\xi$, $[\eta]$ is that of $\partial F/\partial y_{1m}$, and the elements which follow have zero values. Continuing, we find the value of the jacobian at $\xi$, $[\eta]$ to be the $(p + 1)$th power of the value of $\partial F/\partial y_{1m}$.

Thus, for the neighborhood of $\xi$, $[\eta]$, $y_{1m}, \ldots, y_{1m+p}$ are determined by our equations as analytic functions $f_1, \ldots, f_{m+p}$ of $x$ and the remaining $y_{ij}$. By specializing the $y_{ij}$ in the $f$ as functions of $x$, we can construct zeros of (28). Indeed, we secure in this way all zeros of (28) which, in an area contained in a small neighborhood of $x = \xi$, approximate closely to the zero of (28) derived from (27).

Some prime p.i. in the decomposition of (28), call it $\Sigma$, is such that all its polynomials vanish when $y_{1m}, \ldots, y_{1m+p}$ are replaced by their $f$. Then $\Sigma$ must admit the $\tilde{y}_{ij}$ as a zero. If a prime p.i. $\Sigma'$ which $\Sigma$ does not hold vanishes for the $\tilde{y}_{ij}$, $\Sigma'$ has, by IV, §39, zeros which are not in the manifold of $\Sigma$ and which approximate closely to the $\tilde{y}_{ij}$. Thus, by what precedes, $\Sigma$ is the only prime p.i. in the decomposition of (28) which has the $\tilde{y}_{ij}$ as a zero. The theorem is proved.

If one allows all the $\partial F/\partial y_{ij}$ to vanish and requires the nonvanishing of one
or more partial derivatives of the second order, there is no upper bound to the number of components to which a zero of $F$ may belong. We illustrate this by an example in $\mathfrak{F}\{y\}$. Let

$$F = y_2^2 + \prod_{i=1}^{m} [(x + j)y_i - y],$$

where $m$ is any integer greater than unity. Now $(x + j)y_i - y$ has $(x + j)y_2$ as derivative, and therefore has, for every $j$, a manifold which is a component of $F$. The zero $y = 0$ belongs to every such component.

**Analogue of a theorem of Kronecker**

17. It is a theorem of Kronecker that, given any system of polynomials in $n$ indeterminates, there exists an equivalent system containing $n + 1$ or fewer polynomials. We present an analogous theorem for d.p.

**Theorem:** Let $\mathfrak{F}$ contain a nonconstant element. Let

$$F_1, \ldots, F_r$$

be any finite system of d.p. in $\mathfrak{F}\{y_1, \ldots, y_n\}$. There exists a system composed of $n + 1$ linear combinations of the $F_i$ with coefficients in $\mathfrak{F}$, whose manifold is identical with that of (29).

We introduce $r(n + 1)$ new indeterminates $u_i^{(0)}$, $i = 1, \ldots, n + 1$; $j = 1, \ldots, r$ and consider the system $\Lambda$,

$$u_i^{(0)}F_1 + \cdots + u_r^{(0)}F_r, \quad i = 1, \ldots, n + 1,$$

in the $u$ and $y$.

Consider a zero of $\Lambda$ for which $F_1 \neq 0$. For it, we have

$$u_i^{(0)} = -\frac{u_2^{(0)}F_2 + \cdots + u_r^{(0)}F_r}{F_1}, \quad i = 1, \ldots, n + 1.$$

If we differentiate the relations (30) often enough, the $u_i^{(0)}$ will be more numerous than the $y_i$. By an elimination, we obtain a d.p. $K_1$ in the $u$ which is annulled by every zero of $\Lambda$ for which $F_1 \neq 0$. We find, similarly, a $K_i$ for each $F_i$ with $i > 1$. We fix the $u_i^{(0)}$ as elements $\mu_i$ in $\mathfrak{F}$ which do not annul the product of the $K_i$. Then the manifold of the $n + 1$ d.p.

$$\mu_1F_1 + \cdots + \mu_rF_r, \quad i = 1, \ldots, n + 1,$$

in $y_1, \ldots, y_n$ is identical with that of (29).

The proof just given does not involve the notion of irreducible manifold. It is considerably shorter than the proof given in A.D.E. However, the older proof gives information on the degree to which one can approximate to the representation of a manifold with a system of $p$ equations with $1 \leq p \leq n + 1$.

---