CHAPTER I

The Simplest Concepts and the Object of the Investigation

This chapter contains a survey of the fundamental concepts and definitions of the spaces which we shall investigate. In contrast to the practice in the following chapters, the assertions are formulated without proofs. In §6 we enumerate some investigations on two-dimensional manifolds of bounded curvature which are not included in this monograph.

1. The intrinsic metric of a space.

1. Metric spaces. In what follows we shall consider only metric spaces. For brevity we write simply space, i.e., assuming each time that the space $R$ is a set on which there is defined a function $\rho(X,Y)$ of pairs of elements with the properties:

(1) \begin{align*}
1) & \quad \rho(X,Y) = 0 \text{ if and only if } X = Y; \\
2) & \quad \text{for any } X,Y,Z \in R \\
(2) & \quad \rho(X,Y) + \rho(Y,Z) \geq \rho(Z,X).
\end{align*}

As is well known, from (1) and (2) it follows that:

(3) \quad \rho(X,Y) = \rho(Y,X); \quad \rho(X,Y) \geq 0.

The elements of the set are called points of the space. The function $\rho$ is called the metric of the space, and its value is the distance between the corresponding points. Relation (2) is known as the triangle inequality.

It is natural to introduce the fundamental topological concepts into a metric space: the neighborhood of a point, convergence of sequences of points, closed and open sets, the boundary of a set, continuous mappings, continuous functions and so forth. In particular, a neighborhood of a point $A$ is any set $M \subset R$ which contains $A$ along with all points distant less than some positive amount from $A$. If we write $X_\alpha \to X$, or that “the point $X_\alpha$ converges to the point $X$,” we mean that $\rho(X_\alpha, X) \to 0$.

We note that $\rho(X,Y)$ is a continuous function of each of its arguments, a fact which follows from the triangle inequality.

The set $M \subset R$ is said to be compact (in itself) if any infinite subset
in it contains a sequence converging to a point of the same set. A space is locally compact if there is a compact neighborhood of each of its points.

2. The length of a curve. If we are given a continuous mapping $X(t)$ of the segment $a \leq t \leq b$ into the space $R$, then we say that the curve is defined in the concrete parametrization $X(t)$. It joins the points $X(a)$ and $X(b)$, which are the ends of the curve.

The same point $X(t)$ may correspond to different values of $t$. The segment $a \leq t \leq b$ decomposes into connected components $k_i$, each of which corresponds to one and the same point $X(t)$. These components are points or closed intervals in $[a, b]$. For the components and for the parameter $t$ itself, it is meaningful to say that the order of succession on $[a, b]$ is preserved. Two parametrizations $X(t)$, $Y(s)$ ($a \leq t \leq b$, $c \leq s \leq d$) are by definition equivalent if between the components $k_i$ and $k_j$ it is possible to establish a strictly monotone 1-1 relation $\phi$ under which

$$X(k_i) = Y(\phi(k_i)).$$

Any curve is a class of equivalent parametrizations.

Any curve is called simple if the preimage of each of its points $X(t)$ consists of one connected component $k(t)$. The concepts of the ends of a curve and of the simplicity of a curve do not depend on the choice of parametrization.

Suppose that $L$ is a curve in $R$ and $X(t)$ ($a \leq t \leq b$) is its parametrization. We divide up $[a, b]$ by the points $a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ and form the sum $\sum_{i=1}^{n} \rho(X_{i-1}, X_i)$ of the successive distances between the points $X_i = X(t_i)$. The length of the curve is the least upper bound of such sums under all possible subdivisions:

$$s(L) = \sup \sum_{i=1}^{n} \rho(X_{i-1}, X_i).$$

We shall list the most important properties of length. For the proof see for example [5, § 1, Chapter II].

1. Length is additive. If the curve $L$ is decomposed into two arcs $L_1$ and $L_2$, then

$$s(L_1 + L_2) = s(L_1) + s(L_2).$$

2. If the curve $L = X(t)$ ($a \leq t \leq b$) is rectifiable, then the length of its arc from $X(a)$ to $X(t)$ is a continuous function of $t$. 
3. A rectifiable curve admits a parametrization with arc length as a parameter.

4. Suppose that the curves $L_n$ converge to the curve $L$, i.e., in some collection of parametric representations $X_n(t), X(t)$, with the same interval $a \leq t \leq b$ of variation of the parameter, $X_n(t) \to X(t)$ uniformly in $t$. Then

$$s(L) \leq \lim_{n \to \infty} \inf s(L_n).$$

In particular, if the $s(L_n)$ are bounded uniformly, then $L$ is rectifiable.

5. If $\{L\}$ is a collection of curves whose lengths are uniformly bounded and which lie in a compact part of $\mathbb{R}$, then there exists a convergent sequence of curves in $\{L\}$.

In what follows the length of a curve $L$, measured in the metric $\rho$, will be denoted by $s(L)$.

3. Intrinsic metric. Suppose that $M$ is a set in the space $\mathbb{R}$. We shall call $M$ metrically connected if any two points $X, Y \in M$ may be joined by a rectifiable curve $\overline{XY}$ lying entirely in $M$. To pairs of elements of such a set one may assign the function

$$\rho_M(X, Y) = \inf_{\overline{XY} \in M} s_{\rho}(\overline{XY}).$$

It is not hard to verify that $\rho_M$ is a metric. It is natural to call it the metric induced by the choice of $M$ in $\mathbb{R}$, or the intrinsic metric of the set $M \subset \mathbb{R}$.

The concept of the ordinary intrinsic metric of a surface is a particularly important case of this last definition. In this case $\mathbb{R}$ is a Euclidean space and $M$ a surface. Any metrically connected surface $M$ has an intrinsic metric. This metric turns the surface $M$ into a metric space which in itself, independently of the imbedding of the surface in the enveloping space, is the object of study of the intrinsic geometry of the surface.

The new metric $\rho_M$ makes it possible to measure the length $s_{\rho_M}(L)$ of the curve $L$ lying in the set $M$. It is not hard to show that such a measure does not lead to any new result. It turns out that

$$s_{\rho_M}(L) = s_{\rho}(L).$$

This makes it possible to rewrite (6) in the form

$$\rho_M(X, Y) = \inf_{\overline{XY} \in M} s_{\rho_M}(\overline{XY}).$$
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Here the question is already that of the properties of the metric $\rho_{M}$ itself; this time the outer space may simply not exist.

We shall agree more generally to call any metric $\rho$ intrinsic if the distance between any two points is equal to the lower bound of the length of curves joining these points:

$$\rho(X,Y) = \inf_{\bar{XY}} s_{\bar{XY}}.$$  

In particular, every metrically connected surface, with respect to its intrinsic metric, is a space with an intrinsic metric in the above sense. It is easy to give an example of a nonintrinsic metric $\rho^*$. For example, on the real line we may set

$$\rho^*(x_1, x_2) = \min (|x_1 - x_2|, 1).$$

The shortest arc $XY$ is the shortest of the rectifiable curves joining the points $X$ and $Y$. In order that a curve in a space with an intrinsic metric should be a shortest arc, it is necessary and sufficient that its length should be equal to the distance between its endpoints. From assertions 4 and 5 of subsection 2 it follows that in a locally compact space $R$ with an intrinsic metric there is a neighborhood $V$ of each point such that any two points in it can be joined by at least one shortest arc. It is not difficult to choose the neighborhood $V$ so that these shortest arcs do not leave the limits of a given neighborhood $U$ of the point $A$.

Along a shortest arc in a space with an intrinsic metric the length of the arc coincides with the distance between points. A curve on which each point has a neighborhood for which the segment of the curve in it is a shortest arc is called a geodesic.

2. Two-dimensional manifolds of bounded curvature.

4. Concept of an angle. The most essential feature of the intrinsic geometry of a regular surface is the curvature of the surface. We have in mind the complete Gaussian curvature (i.e., the integral of the Gaussian curvature as a function of a region on the surface). For a geodesic triangle $T$ this curvature is equal to the excess $\delta(T) = \alpha + \beta + \gamma - \pi$, where $\alpha, \beta,$ and $\gamma$ are the angles of the triangle $T$. This well-known fact leads to the idea that one might arrive at the extension of the concept of curvature to general surfaces by starting with the concept of angle. Moreover, the concept of the angle between curves is itself the simplest concept of intrinsic geometry, next to the
concepts of distance and length.

Suppose that \( L = X(t), M = Y(s) \) \((0 \leq t \leq 1, 0 \leq s \leq 1)\) are two curves in the space \( R \) with the common origin \( X(0) = Y(0) = O \). From \( L \) and \( M \) take two distinct points \( X \in L, Y \in M \) and on the plane construct the triangle \( T_0 = O'X'Y' \) with the sides

\[
O'X' = \rho(O, X); \ O'Y' = \rho(O, Y); \ X'Y' = \rho(X, Y).
\]

Such a triangle exists, since the indicated distances satisfy the triangle inequality. Suppose that \( \tau(X, Y) \) is the angle in \( T_0 \) at the vertex \( O' \). This angle also exists, since \( X \) and \( Y \) do not coincide with \( O \).

The upper angle between the curves \( L \) and \( M \) at \( O \) is the upper limit

\[
\bar{\alpha} = \lim_{x, y \to O} \sup \tau(X, Y).
\]

Since \( 0 \leq \tau(X, Y) \leq \pi \), the upper limit (10) necessarily exists, with \( 0 \leq \bar{\alpha} \leq \pi \). By \( X, Y \to O \) in (10) we have in mind points \( X(t), Y(t) \) distinct from \( O \), which tend to \( O = X(0) = Y(0) \) in the sense of the values of the parameter on \( L \) and \( M \).

If the limit

\[
\alpha = \lim_{x, y \to O} \tau(X, Y)
\]

exists, then its value is called the angle between \( L \) and \( M \).

5. Simple triangles. A triangle in the space \( R \) is a figure consisting of three distinct points (the vertices of the triangle) and three shortest arcs joining these points (sides of the triangle).

Suppose that in the space \( R \) we have selected a region \( G \) open in \( R \) which is homeomorphic to an open disc on the plane. Suppose that the triangle \( T \) lies in this region and that its sides form a simple closed
contour. Then they bound a region in $G$. We shall attach it to $T$ and say that $T$ is a **triangle homeomorphic to a disc**. Here $T$ appears as a set (in the selected region $G$) with distinguished points, namely the vertices of the triangle.

Suppose that $T$ is a triangle homeomorphic to the disc, distant from the boundary of $G$ by more than one fourth of its perimeter. Then every curve joining two points on the boundary of $T$ and otherwise lying outside $T$ will, provided it is shorter than half the perimeter of $T$, lie in the region $G$ and therefore include from outside a definite piece of the boundary of $T$ (Figure 1). We shall say that such a triangle $T$ is **convex relative to the boundary** if no two points of its contour can be joined by a curve, lying outside $T$, which is shorter than the part of the boundary joining the points (where we mean, of course, that part of the boundary which is included by the curve).\(^1\)

A **simple triangle** in the region $G$ is a triangle homeomorphic to the disc and convex relative to the boundary.

Two simple triangles are said to be **nonoverlapping** if they do not have common interior points.

6. **Fundamental definition.** A space $R$ satisfying the following three requirements will be called a two-dimensional manifold of bounded curvature.

1. $R$ is a metric space with an intrinsic metric.
2. Each point has a neighborhood in $R$ homeomorphic to the disc.
3. To each point in $R$ there corresponds a neighborhood $G$ homeomorphic to the open disc, within the limits of which, for any finite system $(T)$ of pairwise nonoverlapping simple triangles, the sum of the excesses \(\delta(T) = \bar{\alpha} + \bar{\beta} + \bar{\gamma} - \pi\) computed at the upper angles \(\bar{\alpha}, \bar{\beta}, \bar{\gamma}\) of these triangles is bounded above by a number depending only on the given neighborhood $G$.

\[
(12) \quad \sum_{T \in (T)} \delta(T) \leq C(G) < \infty.
\]

Such a space is the fundamental object of investigation in the present paper.

**Remark.** (1) The requirement of an intrinsic character for the metric is necessary, if we wish to include the intrinsic geometry of surfaces.

\(^1\) In § 1 of Chapter III the concept of boundary convexity is extended to certain other types of figures and triangles.
This requirement excludes from consideration only metrically disconnected surfaces; for example a cylinder with a directrix in the form of a curve not rectifiable on any part of it.

(2) The requirement of the existence for each point of a neighborhood homeomorphic to the disc is also natural if we wish to be close to the intrinsic geometry of surfaces. The distinctive feature of this requirement consists of the fact that it is laid not on the pieces of the surfaces as a set in the enveloping Euclidean space but on the topological structure in the sense of the intrinsic geometry itself. This also excludes from consideration certain surfaces, for example a cone with a directrix in the form of a curve which is not rectifiable on any part of it. (This surface is metrically connected. But in its intrinsic geometry it is isometric to a continuum of isolated segments with a unique common endpoint.)

(3) From the first requirement of the definition given above it follows that the space $R$ is connected, and from the second requirement it follows that it is locally compact. From the local compactness the connectedness and the metrizability of the topological space, it follows from the theorems of P. S. Uryson and P. S. Aleksandrov [77] that the space $R$ has a countable basis. Thus in view of the first two requirements $R$ is a two-dimensional manifold.

(4) The most essential requirement is the third restriction, which we call the condition of boundedness of the curvature. If we wish to preserve concepts connected with curvature it is natural to impose a requirement of some such form. In the formulation adopted, requirement (3) is imposed in the weakest possible form. It contains only a one-sided estimate of the excesses; the excesses are considered in the sense of the always-existing upper angles; the requirement is imposed for triangles of the simplest possible form, and only locally.

(5) As we shall see in what follows, the requirements imply very comprehensive properties of the spaces. In particular, in Chapter IV we shall verify the fact that bounds of the type of inequality (12) are true in a much wider sense: for the absolute values of the excesses; for a wider class of triangles; for excesses measured with variously defined angles; in any region $G$ with a compact closure.

3. **Approximation by polyhedral metrics.**

7. **The polyhedral metric.** A special case of a two-dimensional manifold of bounded curvature is a space with a polyhedral metric. This
is a space in which each point has a neighborhood isometric to the lateral surface of a cone. An entire space of such a type, if it is closed, and a selected portion of such a space, bounded by simple closed polygons, can be defined by evolutes of plane triangles, i.e., by complexes of triangles with identified sides, the identification being carried out with preservation of the correspondence between pieces of sides of equal length. (For details on the polyhedral metric see [5, pp. 23-27].)

In a polyhedral metric there exists an angle between shortest arcs; the angle $\bar{\alpha}$ of a sector and the complete angle $\theta$ around a point have an evident meaning, and also the curvature at a point as the difference $2\pi-\theta$. Points with curvature different from zero are called vertices of the metric or essential vertices in distinction from vertices of the evolute, among which there may be nonessential vertices (with the angle $\theta = 2\pi$). The curvature $\omega$ of any set is defined as the sum of the curvatures of the vertices belonging to that set. The positive and negative parts $\omega^+,$ $\omega^-$ of the curvature are defined as follows: the first as the sum of the curvatures for vertices with positive curvatures; the second as the absolute value of the sum of the negative curvatures. We always have $\omega = \omega^+ - \omega^-$. Finally, the absolute curvature is $\Omega = \omega^+ + \omega^-.$

In a polyhedral metric for a simple polygonal line it is natural to introduce the concept of one-sided (right or left) rotation at each vertex as the difference $\pi - \bar{\alpha}$, where $\bar{\alpha}$ is the angle of the corresponding sector between the links of the line. Among the vertices of the polygonal line are included all the essential vertices of the metric through which the line passes. The rotation $\tau$ of the entire line is defined as the sum of the rotations at the vertices. Its positive and negative parts are as follows: $\tau^+$ is the sum of the positive rotations at the vertices and $\tau^-$ the absolute value of the sum of the negative rotations. Always $\tau = \tau^+ - \tau^-.$

A shortest arc in a polyhedral metric evidently can not have a positive rotation on either side. Thus it can pass only through vertices with negative curvature. For a simple closed polygon bounding a region $G$ homeomorphic to the open disc, application of Euler's theorem immediately leads to the relation

\begin{equation}
\tau + \omega = 2\pi,
\end{equation}

where $\omega$ is the curvature of the region $G$ and $\tau$ the rotation of the boundary on the side of the region itself. This is the Gauss-Bonnet
theorem for a polyhedron.

The concepts listed here for the polyhedral metric may now be introduced for arbitrary two-dimensional manifolds of bounded curvature. In the case of a polyhedral metric their new meaning coincides with the definitions just given.

8. Convergence of metrics. Theorems on approximation. Suppose that a sequence of metrics \( \rho_n \) is given on one and the same set \( M \). If uniformly for all points \( X, Y \in M \)

\[
\lim_{n \to \infty} \rho_n(X, Y) = \rho(X, Y)
\]

and if the limiting function \( \rho(X, Y) \) is also a metric, then we say that the metric \( \rho_n \) converge to the metric \( \rho \). If we speak of the convergence of the metric spaces \( R_n \) to the space \( R \), we suppose fixed some 1-1 mapping \( \phi_n \) of each space \( R_n \) onto \( R \) and we refer to the uniform convergence mentioned above of the metric \( \rho_n \) on \( R \) as carried over by the mappings \( \phi_n \) to the metric \( \rho \) of the space \( R \).

In Chapter III we prove that in a two-dimensional manifold of bounded curvature each point has a neighborhood in the form of a convex polygon homeomorphic to the disc, within which \( \rho \) admits uniform approximation by polyhedral metrics \( \rho_n \), the absolute curvature of the latter being bounded uniformly.

This theorem opens up the possibility of investigating our spaces by approximation by polyhedral metrics. One of the first results in this direction is the proof of the existence of an angle between arbitrary shortest arcs (subsection 6 of Chapter IV).

In Chapter IV it is proved that every two-dimensional metrized manifold, whose metric admits locally uniform approximation by polyhedral metrics with positive curvatures uniformly bounded, certainly is a two-dimensional manifold of bounded curvature. This result together with the preceding remarks, gives an exhaustive characterization of our spaces as the closure of the class of spaces with polyhedral metrics. Thus we already have the possibility, mentioned in the Introduction, of uniform approximation by means of two-dimensional Riemannian spaces with uniformly bounded absolute integral curvatures.

4. Quantitative characteristics of figures. In this section we shall only mention the basic concepts. Their precise definitions and properties will be given below in the appropriate chapters of this paper.
9. The curvature of sets. The existence of an angle between shortest arcs makes it possible to define the excess $\delta(T)$ of a triangle, i.e., the difference between the sum of its angles and $\pi$. The function $\delta(T)$ is basic for the introduction of four functions of a set, completely additive on the ring of Borel sets (§ 2 of Chapter V). These are the curvature $\omega$, its positive and negative parts $\omega^+$ and $\omega^-$, and the absolute curvature $\Omega$. Here $\omega = \omega^+ - \omega^-$, $\Omega = \omega^+ + \omega^-$. In the case of Riemannian metrics, these functions coincide with the corresponding integral curvatures, and in the case of polyhedral metrics with the definitions of subsection 7. As in the polyhedral metric, the curvature of a one-point set coincides with the quantity $2\pi - \theta$, where $\theta$ is the complete angle of the sector around the point in question. For a triangle $T$ its curvature $\omega(T)$ generally speaking does not coincide with $\delta(T)$.

10. Rotation of a curve. The possibility of distinguishing the two sides of a simple arc lying in a two-dimensional manifold makes it possible to consider the approximation of the simple curve $L$ by polygonal lines $L_n$ converging to $L$ from the right and from the left (subsection 2 of Chapter VI). The measurement of sectors at the vertices of the approximating polygonal lines $L_n$ makes it possible to define the left and right rotations for a wide class of curves; these are $\tau_-$ and $\tau_+$ (subsection 3 of Chapter VI). In the case of regular curves in Riemannian spaces the left and right rotations differ only in sign and coincide with the integral geodesic curvature. More generally, however, they may be essentially different.

In the case of the so-called curves with bounded variation of the rotation (subsection 1 of Chapter IX) the usual methods of measure theory may be used to pass from the rotation of open arcs of a curve to the definition of set-functions $\tau$, $\tau^+$, $\tau^-$, $\sigma$ on the curve. All of these functions, the one-sided rotation, its positive and negative parts and its variation, are completely additive on Borel sets on the curve. Here $\tau = \tau^+ - \tau^-$, $\sigma = \tau^+ + \tau^-$. The rotation of a shortest arc on any section always turns out to be nonpositive.

11. Rotation and curvature. For a simple open arc that has a rotation, the sum of its right and left rotations coincides with the curvature of the arc as a point set (subsection 4 of Chapter VI). Thus the curvature distributed along the curve splits, so to speak, into two parts: the right and left rotations of the curve. When this fact is taken
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In this context, it is possible to follow visually the connection between the
In the same chapter we consider the connection at the angles between converging pairs of shortest arcs and the limiting pair of shortest arcs. In Chapter IX we consider in detail the convergence of curves, the connection between the rotations of converging curves and the limiting curve, the cases of convergence of lengths of converging curves. The set of curves with bounded variation of rotation studied in Chapter IX may be regarded as a special kind of closure of the class of simple polygonal lines on the surface, in the same way as two-dimensional manifolds of bounded curvature form the closure of the class of polyhedral metrics. The case when the limiting curve contains a singular point with complete angle $\theta = 0$ constitutes an exception.

5. Cutting and pasting.

15. Polygons. In the spaces we are considering, polygons are compact connected sets bounded by a finite number of simple closed polygonal lines. In particular, if the space in the large is a closed manifold, then the entire space is regarded as a polygon. The definition just adopted excludes figures which could be called infinite polygons, open polygons, disconnected polygons, polygons with multiple boundary points, and so forth. In all cases of digression from the definition made here we shall make the appropriate stipulations.

Suppose that a shortest arc $AB$ is drawn from the point $A$ on the boundary of a polygon $P$, and that with the exception of the point $A$ this arc lies inside $P$. We isolate $P$ from the enveloping manifold and "cut" $P$ along the curve $AB$. Let us explain this more precisely. Each point $X$ of $AB$, except the endpoint $B$, has a neighborhood homeomorphic to a disc, which is cut by the curve $AB$ into two half-neighborhoods. Consider all the points $X \in [AB]$ in two copies of the disc. The neighborhoods of one of them are the left half-neighborhoods of the original point $X$, and the neighborhoods of the other are the right half-neighborhoods of $X$. Consequently, this set of points of $P$ with doubling of the points of the curve $[AB]$ becomes a topological space. We shall define in it an intrinsic metric, considering the distance $\rho(X,Y)$ to be the greatest lower bound of the lengths of curves joining $X$ and $Y$ which lie in the "polygon with a cut" just constructed. (Here curves lying in $P$ and intersecting the cut $AB$ are excluded.) The metric space just constructed is called a polygon with a cut. Analogously we may construct polygons with several cuts, cuts along polygonal lines or other curves, with cuts
along curves that do not abut the boundary of \( P \).

16. **Theorems on pasting.** A construction in a certain sense inverse to the cutting just described is the *pasting* of two-dimensional manifolds with an edge. Suppose that we have a finite set of two-dimensional manifolds with edges \( \bar{R}_j \), with interior metrics \( \rho_j \). We admit also manifolds with incomplete edges. For example, we may consider two semidiscs, in each of which the points on the circumference are not counted, but all the points of the diameter with the exception of its endpoints are counted. But we assume that the points of the edge of each manifold \( \bar{R}_j \), belonging to that manifold form a finite number of connected components which are simple closed curves or open arcs.

We shall divide up the boundary curves by several points, the "vertices," into a finite number of sections, the "ribs," and we map topologically various ribs taken along with the manifolds \( \bar{R}_j \) pairwise onto one another. We identify the points of two ribs corresponding to one another under the homeomorphism. Each "pasting" of ribs is accompanied by an identification of the corresponding vertices, namely the ends of these ribs. We suppose that no rib is sewed together with more than one other rib and that there is no identification of vertices other than that called for by the pasting together of the ribs.

The point set \( \mathbb{R}^n \) thus obtained becomes a topological space \( \mathbb{R}^T \) in the natural way. It suffices for a neighborhood of each point \( A \) not involved in the identifications to take its previous neighborhood in \( \mathbb{R}_T \supseteq A \), and as a neighborhood of each point \( B \) obtained by identification to take the collection of points obtained by joining the collection of neighborhoods of the point \( B \) in all the \( \bar{R}_j \) to which the point \( B \) belonged before the pasting.

By hypothesis the \( \bar{R}_j \) have intrinsic metrics and are therefore connected. We suppose that under the pasting one may pass from any \( \bar{R}_h \) to any \( \bar{R}_h \) through manifolds \( \bar{R}_j \) that adjoin one another under the pasting. Then for any two points \( A, B \in \mathbb{R}^T \) we can construct a chain \( Z(A,B) \) which is a finite collection of points \( X_i \in \mathbb{R}^T \), the first of which is \( A \) and the last is \( B \), and each two successive \( X_i, X_{i+1} \) belong to one and the same original manifold \( \bar{R}_h \).

Define a function \( \rho \) in \( \mathbb{R}^T \) by putting

\[
\rho(A,B) = \inf_{Z(A,B)} \left[ \sum_i \rho_i(X_i, X_{i+1}) \right].
\]

It is not hard to verify that \( \rho(A,B) \) is an intrinsic metric. We shall
call the space $R$ with the metric $\rho$ thus obtained a space *pasted up* from the manifolds $\overline{R}_i$.

If all the sections of the boundary curves of the manifolds $\overline{R}_i$ undergo pasting, then $R$ becomes a two-dimensional manifold. In the contrary case it becomes a manifold with an edge or a manifold with an incomplete edge.

The simplest case of pasting consists of defining a polyhedral metric by evolutes of plane polygons.

In Chapters VI and IX we prove the following two theorems on pasting: 1) if the two-dimensional manifold $R$ is pasted up from manifolds $\overline{R}_i$ with edges, where the $\overline{R}_i$ are manifolds with intrinsic metrics excised from two-dimensional manifolds of bounded curvature, and if under the pasting the identification of boundaries took place with sections of equal length corresponding, then the space $R$ itself is a two-dimensional manifold of bounded curvature; 2) if the two-dimensional manifold $R$ is pasted up from manifolds $\overline{R}_i$ with boundaries, and if the $\overline{R}_i$ with their intrinsic metrics are excised from two-dimensional manifolds of bounded curvature, where the $\overline{R}_i$ were bounded by a finite number of simple curves with bounded variation of the rotation, and if the identification of the boundaries took place with correspondence of sections of equal length, then the space $R$ itself is also a two-dimensional manifold of bounded curvature.

The second theorem contains the first as a special case. In both cases each curve of the pasting conserves its rotation on the side of the corresponding $\overline{R}_i$, and at the vertices of the pasting the angles of the pasted sectors are preserved.

The operations of cutting and pasting are a convenient method of investigation. They make it possible, while preserving many properties to rebuild the enveloping space.

6. **Further investigations.** In this section we shall list further results from the geometry of two-dimensional manifolds of bounded curvature which do not lie within the scope of the present monograph. A more detailed survey will be found in [17, § 5–9].

17. **Linear element.** As Ju. G. Rešetnjak [61], [64], showed, on a piece of any two-dimensional manifold of bounded curvature homeomorphic to the disc with $\omega^+ < 2\pi$ it is possible to introduce an isometric coordinate net in which a metric linear element of the space will have
the form \( ds^2 = \lambda(u,v) \left( du^2 + dv^2 \right) \), where \( \lambda(u,v) \) is the difference of two subharmonic functions. Moreover, the possibility of introducing a net with those properties is not only a necessary but a sufficient test for a two-dimensional manifold of bounded curvature. Later this result was proved again by A. Huber [79]. Ju. G. Rešetnjak also cleared up the degree of arbitrariness with which the indicated net can be introduced. In the second of the papers just mentioned the author used the linear element for the study of two-dimensional manifolds of bounded curvature, and in particular for the analytic description of curves with bounded variation of rotation, in connection with the properties of preimages of these curves on the coordinate plane. These results establish the relationship of the theory in question with classical differential geometry and the theory of functions, and also give yet another general method of investigation. Apparently, similar results hold also for other types of nets. For Čebyšev nets a part of the investigation has already been carried out by I. Ja. Bakel’man.

The question as to what minimal properties of curvature guarantee the possibility of prescribing a metric by a regular linear element is an interesting one. In the case of convex surfaces, it was proved by A. D. Aleksandrov [5, Chapter XI] that for this purpose it suffices that there exist at each point a specific Gaussian curvature as the limit of the ratio of the curvature of an arbitrary region to the area of that region as the region contracts to a point.

18. Special questions of intrinsic geometry. The methods developed by A. D. Aleksandrov make it possible to pose and resolve many special questions of intrinsic geometry for two-dimensional manifolds of bounded curvature. We shall mention a number of them.

Extremal problems. The question of the maximum area of a region homeomorphic to the disc with a bounded positive part of the curvature \( \omega^+ \leq a < 2\pi \) and a bounded perimeter or diameter is a characteristic example of a problem in the calculus of variations, the solution of which is achieved on a nonregular surface. A. D. Aleksandrov solved this problem by approximating by polyhedra and by applying the method of cutting and pasting [2], [5], [17]. The papers of S. M. Lozinskiï, [52], Ju. G. Resetnjak [65], and A. Huber [78], touch on related questions. Here one must mention also the paper [68] of G. I. Rusiešvili. Moreover, A. D. Aleksandrov in [7], [17], [18] and V. V. Strel‘cov in [18], [71], [72] obtained interesting estimates of the length of a curve with a
bounded variation of rotation, lying in a region homeomorphic to the
disc with $\omega^r < 2\pi$ and having bounded perimeter or diameter. The
question of such estimates was already present in the paper of Cohn-
Vossen [50].

**Geodesics and quasigeodesics.** For nonregular convex surfaces A. D.
Aleksandrov introduced the concept of a quasigeodesic curve [5] as a
curve with a nonnegative right and left rotation on each section. A. V.
Pogorelov investigated these curves in detail [54]. They turn out to be
the natural closure of the class of geodesics. In order that a curve on
a convex surface should be quasigeodesic it is necessary and sufficient
that it be the limit of geodesic curves which lie on convex surfaces
converging to the given surface. A. V. Pogorelov established the existence
of three simple closed quasigeodesics on any closed convex surface, a
In a two-dimensional manifold of bounded curvature a quasigeodesic is
defined as a curve for which the sum of the variations of the right and
left rotation coincides with the absolute cuvature. In [9] and [10] A. D.
Aleksandrov also investigated quasigeodesics by the methods of intrinsic
gometry, and generalized the results of Cohn-Vossen on the behavior
in the large of geodesic curves (A. D. Aleksandrov [7], V. V. Strel'cov
[73], Ju. F. Borisov [30]).

**Circumference and disc.** On a convex surface a circumference with
a fixed center is a simple closed curve only for small values of the radius.
The length of the circumference and the area of the disc as a function
of the radius are functions of bounded variation: they admit simple
estimates in terms of the value of the radius and the curvature of the
disc. The properties of the circumference make it possible to establish
an almost isometric mapping of the neighborhood of a point onto the
tangent cone. These results, proved by V. A. Zalgaller [44] for convex
surfaces, admit a natural generalization for two-dimensional manifolds
of bounded cuvature, if the center of the circumference lies at a point
with a complete angle different from zero.

**Manifolds with an edge.** Ju. F. Borisov investigated in detail the
singularities of metrized two-dimensional manifolds with an edge [30],

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[^1]: Half the difference between the sum of the variation of the right and left curvature of a
curve and its absolute curvature is called the proper rotation or twist of the curve. Thus
quasigeodesics have a zero proper rotation.
Moreover, he investigated the structure of a semineighborhood of a curve and the variation of the length of a curve under one-sided displacement of the curve on a two-dimensional manifold of bounded curvature [32].

**Manifolds with bounded specific curvature.** It is possible to distinguish a class of two-dimensional manifolds by the property of being intermediate between spheres and pseudospheres of various radii. These are spaces in which the ratio of the curvature to the area for any region lies within the limits $K_1 \leq \omega/s \leq K_2$. One may obtain various estimates concerned with the comparison between figures in these manifolds and the corresponding figures on surfaces of constant curvature $K_1$ and $K_2$. See A. D. Aleksandrov [2], [3], [5, Chapter X], [13], [15], [17, §7], V. V. Strel’cov [74], [75], and G. I. Rusiešvili [68].

19. **Surfaces in a space.** The most studied among the nonregular surfaces are the general convex surfaces. For these the negative part of the intrinsic curvature $\omega^-=0$. In general, every two-dimensional manifold of bounded curvature with $\omega^-=0$ is locally isometric to a piece of a convex surface (see A. D. Aleksandrov [5], [6]). For convex surfaces the close connection of intrinsic and extrinsic geometric properties is preserved. First of all we have the generalized theorem of Gauss: the area of a spherical representation coincides with the intrinsic curvature of the same set (theorem of A. D. Aleksandrov [5]). Moreover, every shortest arc on a surface has at each of its points a right and left semitangent. In other words, it has directions in the space (theorem of I. M. Liberman [51]). Moreover, every curve with a bounded variation of rotation on a convex surface has a finite rotation in the space (theorem of V. A. Zalgaller [45]). The corresponding results are far from always holding for nonconvex surfaces. It suffices to note that the theorem of Liberman may fail to hold even for smooth surfaces isometric to the plane.

A series of papers have been devoted to the singling out of a class of nonconvex surfaces which with respect to their intrinsic geometry are two-dimensional manifolds of bounded curvature and to establishing connections between the extrinsic and intrinsic geometry of such surfaces.

A. D. Aleksandrov [8], [11] considered surfaces "representable as the difference of convex surfaces," i.e. admitting the representation $z = f(x,y)$, where $f$ is the difference of two convex functions. The paper of Ju. G. Rešetnjak on "generalized convex" surfaces [62] (i.e., surfaces
tangent at each point on a definite side to a sphere of fixed radius)\(^3\) deals with basically the same class of surfaces. All of these surfaces are two-dimensional manifolds of bounded curvature, and the generalized Gauss theorem and the Liberman theorem hold for them.

I. Ja. Bakel’man [21]–[25] considered smooth surfaces “with generalized second derivatives.” These are surfaces locally admitting the parametrization \( \mathbf{r}(u, v) \), where the vector-function \( \mathbf{r} \) has continuous first derivatives and square-summable generalized second derivatives. For these surfaces the fundamental relations of differential geometry are preserved in integral form. A. V. Pogorelov [57]–[59] investigated smooth surfaces of “bounded extrinsic curvature,” in other words, surfaces with finite area for the spherical representation. For all the surfaces mentioned in this part it has been proved that they are two-dimensional manifolds of bounded curvature and that the theorem of Gauss holds for them. The properties of curves on these surfaces have still not been studied.

The somewhat more general class of surfaces with bounded extrinsic curvature [25] considered by I. Ja. Bakel’man, include the surfaces enumerated above. For this class only one theorem has been proved: with respect to its intrinsic geometry, every surface in this class unquestionably is a two-dimensional manifold of bounded curvature.

For metrics of positive curvature there have been established the theorems of A. D. Aleksandrov [5] on their realization on convex surfaces, and also the theorems of A. V. Pogorelov on the uniqueness of the realization for a closed surface (or under prescribed boundary conditions) [56], and on the regularity of the realization for sufficiently regular metrics [55]. The situation is quite different in the case of general metrized manifolds. From the work of John Nash [53] and N. Kuiper [49] it follows that various realizations of Riemannian spaces \( \mathbb{R}^3 \) are possible in the form of smooth surfaces in \( \mathbb{E}^3 \). The realization of nonconvex polyhedral evolutes was treated in the papers of V. A. Zalgaller [48] and of Ju. D. Burago and V. A. Zalgaller [39]. The existence of a surface realizing the metric of any closed oriented two-dimensional manifold of bounded curvature was established by Ju. D. Burago [38]. However, there remains the important question of distinguishing a sufficiently general class of surfaces in which pieces of

\(^3\) For a more precise definition see [62].
any two-dimensional manifold of bounded curvature can be realized and for which results of the type of the Gauss theorem and the Liberman theorem would be valid.

20. Generalizations. Multidimensional polyhedral evolutes were used by Ju. A. Volkov [41]. The Gauss-Bonnet theorem for such evolutes was established by I. A. Brin [37]. But the general definition of two-dimensional manifolds of bounded curvature did not lead to any multidimensional analogue.

Independently of the number of dimensions, A. D. Aleksandrov considered metric spaces "with curvature less than $K$" [13], [15]–[17]. These are spaces in which the excess in any triangle with respect to the upper angles is not larger than the triangles with sides of the same length on a surface of constant Gaussian curvature $K$. Related questions were considered in the papers of V. A. Toponogov [76] and Ju. G. Rešetnjak [65], [67].

Nonregular smooth surfaces, possibly not satisfying the condition of bounded curvature (subsection 6 of Chapter I) but admitting the introduction of a parallel translation, were studied by Ju. F. Borisov [35], [36].

Some theorems were obtained by Ju. G. Rešetnjak [63] for metrized two-dimensional spaces with the single requirement that a tangent cone exists at each of their points.