CHAPTER IV

Metrics Admitting Approximation by Polyhedral Metrics

1. Convergent metrics and the limit metric.

1. Object of the investigation. The object of the present chapter is the investigation of two-dimensional manifolds which have intrinsic metrics admitting, at least locally, uniform approximation by polyhedral metrics for which the positive part of the curvature is uniformly bounded. In other words we suppose that the metric space under investigation is a two-dimensional manifold, has an intrinsic metric, and for each point $O$ there is a neighborhood $G$, homeomorphic to the disc, of that point, and also a sequence of manifolds $P_n$ with polyhedral metrics $\rho_n$ and homeomorphic mappings $\phi_n$ of the regions $G_n \subset P_n$ onto the region $G$. These mappings carry the metrics $\rho_n$ onto the region $G$. We suppose that the positive part of the curvature of the polyhedral metric $\rho_n$ in the region $G$ is bounded uniformly, and that the metrics $\rho_n$ converge uniformly to $\rho$ in the region $G$.

In the preceding chapter we have proved that two-dimensional manifolds with an intrinsic metric certainly admit such approximation. Moreover, it was proved that in a two-dimensional manifold with an intrinsic metric $\rho$ of bounded curvature each point has a small neighborhood in the form of a convex polygon homeomorphic to the disc. This polygon can play the role of the region $G$ indicated above, and the polyhedral metrics $\rho_n$ approximating $\rho$ may be chosen to have bounded absolute curvature and not just bounded positive curvature.

In this chapter we prove the converse assertion: each two-dimensional manifold with an intrinsic metric locally admitting uniform approximation by polyhedral metrics for which the positive part of the curvature is bounded uniformly has a metric of bounded curvature.

This makes it possible in what follows, depending on the needs of the situation, to begin either from the intrinsic properties of the metric itself, or from a consideration of the approximating polyhedral metrics. Thus, beginning from the approximation of the metric by polyhedral metrics, we establish the existence of an angle between arbitrary shortest arcs,
the boundedness of the angles of sectors and so forth. Moreover, we obtain a strengthening of the condition of boundedness of the curvature: it is proved that in a region \( G \) with a compact closure, for a finite system of arbitrary nonintersecting reduced triangles, and not just simple triangles, the sum of the absolute values of their excesses is a bounded number depending only on the choice of \( G \). On the other hand, for a space which is locally approximable by some sort of polyhedra, the fact that we have established the boundedness of the curvature of this space gives us the right to consider the particularly convenient approximations by polyhedra constructed in Chapter III with respect to the metric of the space itself.

The object of the next two subsections is to explain the concept of convergence of metrics.

2. **Possible peculiarities.** 1. A function \( \rho \) which is a limit of the metrics \( \rho_n \) is not in general necessarily a metric. (The condition that \( \rho(X,Y) = 0 \) only when \( X = Y \) may fail to hold.)

2. The metrics \( \rho_n \) may converge nonuniformly to a limit metric.

**Example.** Suppose that \( R \) is the unit plane disc with center at the origin of coordinates, \( \rho \) is its Euclidean metric, and \( \rho_n \) is the intrinsic metric of the surface obtained from \( R \) by removing a disc of radius \( 1/2n \) and center at the point \((1/n, 0)\) and replacement of the hole by a cone of height 1. We transfer \( \rho_n \) to \( R \) by simple projection. Evidently \( \rho_n \to \rho \), but the convergence is not uniform.

3. The topological space into which the set \( R \) is made by the metric \( \rho \) will be denoted in what follows by \( R(\rho) \). We shall say that the metrics \( \rho_n \) **define in** \( R \) **one and the same topology** if they define in \( R \) one and the same system of open sets. In other words, \( R(\rho_n) \) are not only homeomorphic but are identically mapped onto one another already by their homeomorphisms. The limiting metric \( \rho \) may, generally speaking, define a topology in \( R \) which is different from that defined in \( R \) by the metrics \( \rho_n \).

**Example.** Suppose that \( R \) is the set of points of the open arc \( AB \). As a metric \( \rho_n \) we take the distance measured in the Euclidean space containing the curve \( AB \). As the endpoint \( B \) of the curve approximates to an interior...
point $C$ (Figure 42) the metrics $\rho_*$ will uniformly converge to the metric $\rho$, converting $R$ into a different topological space. In fact, in the limiting position, for the points $X_i \subset R$, approaching the end point $B$, we have $X_i \to C$, while there is no such convergence in the metrics through which the limit is being taken.

4. We call a metric for a space complete if each sequence that is convergent to itself in that metric has a limit point. The limit of complete metrics may turn out, generally speaking, to be a noncomplete metric.

We give two examples.

Example 1. Suppose that $R$ is the Euclidean plane. To each point $X \in R$ lying at a distance $r$ from the origin of coordinates $O$ we assign a point $X'$ lying on the ray $OX$ and at a distance from $O$ equal to

$$r' = \begin{cases} \frac{2}{\pi} \arctg r & \text{for } r \leq n, \\ r \frac{2 \arctg n}{\pi n} & \text{for } r > n. \end{cases}$$

We define the metrics $\rho_*(X,Y)$ as the usual Euclidean distance between $X'$ and $Y'$. Evidently the metrics $\rho_*$ are complete. At the same time they converge to a metric, which in the large on the plane $R$ coincides with the incomplete metric of the open Euclidean disc on the usual plane.

Example 2. Suppose that $R$ is the segment $[0,1]$. We mark off the points $1/2$, $1/2^2$, $1/2^3$, $\cdots$, and we map this segment onto the segment $[0,1]$ according to the rule

$$x' = \begin{cases} x & \text{for } x \leq \frac{1}{2^k} \quad (k = n, n + 1, \cdots), \\ 0 & \text{for } x = \frac{1}{2^k}, \\ \frac{1}{2^{k-1}} & \text{for } x = \frac{1}{2^k} \quad (k = n + 1, n + 2, \cdots). \end{cases}$$

We define the metric $\rho_*(x_i, x_i)$ on $[0,1]$ as the usual distance between $x'_i$ and $x'_i$ on the segment $[0,1]$. Each of the metrics $\rho_*$ coincides with the usual metric of the closed segment $[0,1]$ and is complete. At the same time the metrics $\rho_*$ converge uniformly to the incomplete metric of the segment $(0,1]$.

In the first example the loss of completeness was connected with the nonuniform convergence of the metrics. In the second it was connected with the circumstance that the identical mappings of the spaces $R(\rho_*)$
onto one another were not homeomorphisms, although the metrics \( \rho_n \)
converted \( R \) into spaces homeomorphic to the disc.

5. A space is called **boundedly compact** if each set bounded in the sense of the metric of that space has a compact closure.

Generally speaking, even in a metrized manifold, the completeness of the metric does not imply bounded compactness. Suppose for example that \( \rho(X,Y) \) is the metric of the Euclidean plane. Consider the metric \( \rho^*(X,Y)=\min[\rho(X,Y),1] \). In the metric \( \rho^* \) the plane is complete but is not a boundedly compact manifold. This peculiarity is connected with the fact that the metric \( \rho^* \) is not intrinsic.

**Lemma 1.** In a locally compact space with an intrinsic metric, completeness is a necessary and sufficient condition for bounded compactness.

**Proof.** The completeness of the space always follows from the bounded compactness. Suppose now that the space is complete. Consider points \( X_n, n=1, 2, \ldots \), which lie at uniformly bounded distances \( r_n \) from the point \( O \). Keeping to the successive order of these points, we may suppose that \( r_n \to r \). Because of the intrinsic character of the metric there exist curves \( L_n=OX_n \) whose lengths \( L_n \to r \). The initial portion of \( L_n \) adjacent to \( O \) lies in a compact neighborhood \( V \) of the point \( O \). On these pieces the \( L_n \) may be assumed to be shortest arcs. Within the limits of \( V \) the pieces \( L_n \) converge to some curve \( L \). Suppose that \( X_n \) is its endpoint. The point \( X_0 \) in its turn has a compact neighborhood, in which also, by changing the curves somewhat and selecting a subsequence of numbers, we may regard the pieces of \( L_n \) as shortest arcs converging to an extension of the curve \( L \). Moreover, \( X_0 \) has moved off from \( O \). After a countable repetition of such shifts the points \( X_n \) form a sequence convergent in itself, which converges to a new point \( X_0 \) with respect to which we may continue the shift. Thus we extend the convergence to the entire extent of the curves \( L_n \) and establish that a subsequence of the points \( X_n \) converges to some point \( X \).

Lemma 1 makes it possible, in the two-dimensional manifolds of interest to us, to make no distinction between the properties of completeness and bounded compactness.

6. The limit of intrinsic metrics may turn out not to be an intrinsic metric.

**Example.** Let \( R \) be the plane square \( 0<x<1, \ 0\leq y \leq 1 \). We introduce a surface \( F_n \) as follows. On the part of \( R \) where \( x \equiv 1/n \), we construct
a "roof" with a total length of sloping sides equal to 3, and over the
remaining portion of the square we complete it by a slanted side, as
depicted in Figure 43. We define the metric $\rho_n$ as the distance on the
surface $F_n$ between the points lying over $X$ and $Y$. As $n \to \infty$ the slope
of the lateral side of the roof increases. The limiting metric $\rho$ of the
metrics $\rho_n$ may be represented as the result of measuring the distances
on the surface $F$ with a vertical side instead of a slanted side on the
roof. We note that the convergence of the metric $\rho_n \to \rho$ is uniform.

It is easy to see that the limit metric $\rho$ on the set $R$ is not intrinsic.
The points $X(1/4, 0), Y(1/4, 1)$, for example, cannot be joined, without
leaving $R$, in the metric $\rho$ by a curve close in length to
$$
\rho(X, Y) = \lim_{n \to \infty} \rho_n, \quad XY \leq 1.5.
$$

The loss of the intrinsic character of the metric is connected in this
case with the incompleteness of the metrics $\rho_n$.

3. **The properties of the converging metrics and limit metrics.** Evidently
the limit $\rho$ of the metrics $\rho_n$ will be a metric if and only if the equality
$\rho(X, Y) = 0$ holds only when $X = Y$. Indeed, the remaining conditions im-
posed on the metric are automatically preserved by the limit metric.

**Lemma 2.** *For the uniform convergence of the metrics $\rho_n \to \rho$ it is
necessary, and if $R(\rho)$ is compact it is also sufficient, that $\rho_n(X_n, Y_n)$
should converge to $\rho(X, Y)$ for any $X_n \to X$, $Y_n \to Y$ in $R(\rho)$.*

The necessity is proved by the following inequalities, valid for any $\varepsilon < 0$
and for sufficiently large $n$:

$$
|\rho_n(X_n, Y_n) - \rho(X, Y)| \leq |\rho_n(X_n, Y_n) - \rho(X_n, Y_n)| + |\rho(X_n, Y_n) - \rho(X, Y)|
\leq \varepsilon + \rho(X_n, X) + \rho(Y_n, Y) \leq 3\varepsilon.
$$

In order to establish sufficiency we suppose that the convergence $\rho_n \to \rho$
is not convergent and that there exist \( \varepsilon > 0 \), \( n \to \infty \), \( X_i, Y_i \in R \) such that
\[
|\rho_n(X_i, Y_i) - \rho(X_i, Y_i)| \geq \varepsilon.
\]

Since \( R(\rho) \) is compact we may suppose that \( X_i \to X \) and \( Y_i \to Y \) in \( R(\rho) \). Then for sufficiently large \( n \) we shall by hypothesis have
\[
|\rho_n(X_i, Y_i) - \rho(X, Y)| \leq \frac{\varepsilon}{2},
\]
so that along with the preceding inequality
\[
|\rho(X_i, Y_i) - \rho(X, Y)| \geq \frac{\varepsilon}{2},
\]
which contradicts the convergence of \( X_i \) to \( X \) and \( Y_i \) to \( Y \) in \( R(\rho) \).

**Lemma 3.** If the metrics \( \rho_n \) define in \( R \) one and the same topology and if the convergence \( \rho_n \to \rho \) is uniform, and the spaces \( R(\rho_n) \) are boundedly compact, then the limit metric \( \rho \) defines the same topology and the space \( R(\rho) \) is also boundedly compact.

First we shall prove the bounded compactness of \( R(\rho) \). Suppose that \( M \subset R \) is a set of points bounded in the metric \( \rho \). We need to show that one may select a sequence of points from \( M \) which converge to a point of \( R \). From the uniform convergence \( \rho_n \to \rho \) it follows that \( M \) is bounded also in the metric \( \rho_n \) for sufficiently large \( n \). In view of the bounded compactness of \( R(\rho_n) \) there exists a sequence \( X_i \in M \) converging to \( X \in R \) in the metric \( \rho_n \). In view of the identity of the topologies defined by the metrics \( \rho_n \) this sequence converges to \( X \) for all \( \rho_n \), and therefore, because of the uniform convergence, also in the metric \( \rho \).

We need to show that the limit metric \( \rho \) defines the same topology in \( R \) as the metrics \( \rho_n \). For this it suffices to verify that the convergence of \( X_i \) to \( X \) in any of the metrics \( \rho_n \) implies the convergence of \( X_i \) to \( X \) in the metric \( \rho \) and conversely. We have already verified the first of these assertions at the end of the preceding paragraph. It remains to verify the second assertion.

Suppose that \( X_i \to X \) in \( R(\rho) \). Then all the \( X_i \) lie in a region \( U \subset R \) which is bounded in the metric \( \rho \), and, from the uniform convergence of \( \rho_n \) to \( \rho \), also bounded in the metric \( \rho_n \) for large \( n \). Because of the bounded compactness of \( \rho_n \) there is a subsequence \( X_{i_n} \) converging to some point \( X_0 \in R(\rho_n) \) and therefore in all the \( R(\rho_n) \). But then, by the uniform convergence of the metric, \( X_{i_n} \to X_0 \) in \( R(\rho) \), from which we conclude that \( \rho(X, X_0) = 0 \). But since we have assumed that \( \rho \) is a metric, we have
$X = X_0$. Hence $X, \to X$ in $R(\rho_n)$ and in all the $R(\rho_i)$.

Lemma 3 is completely proved.

**Lemma 4.** Suppose that on the set $R$ the metrics $\rho_n$, defining one and the same topology, converge uniformly to the metric $\rho$. Suppose further that $U$ is a compact part of $R(\rho)$ and that $U$ contains an infinite sequence of curves $L_n$, with the lengths $s_n$ of the curves $L_n$ bounded uniformly in the corresponding metrics $\rho_n$. Then from the curves $L_n$ we may choose a subsequence converging in the metric $\rho$ to some curve $L$. The curve $L$ will be rectifiable in the metric $\rho$ and its length will satisfy $s \leq \lim \inf s_n$.

**Proof.** Because of the fact that they have common topologies, the $L_n$ are curves in any of the metrics. On each of the $L_n$ we select as the parameter $t$ (0 $\leq t \leq 1$) the relative length of the arc: $t = s'/s_n$, where $s'$ is the length of the arc measured from the beginning of the curve. The length for each curve $L_n$ is measured in its metric $\rho_n$. In the interval $[0,1]$ we choose a countable everywhere dense set of values $t_i$. Using the compactness of $U$ and a usual diagonal process, we choose a subsequence of the $L_n$ for which $X^{*}_{i} \to X_i$ in the metric $\rho$. Here $X^{*}_{i} = X^{*}(t_i)$ are points on the selected curves $L_n$ and the $X_i$ are certain limit points. Since from now on we shall use only the subsequence, we shall keep to the notation $L_n$.

Take an $\varepsilon > 0$. For each $t'$ we select a $t_i$ such that

$$|t' - t_i| < \frac{\varepsilon}{5S},$$

where $S = \sup s_n$. Then for all $n$

$$\rho_n(X^{*}(t'), X^{*}(t)) < \frac{\varepsilon}{5}.$$

We choose further $N$ so large that for $n,m > N$ the following inequality will hold for that value of $t_i$ and for any pair $X,Y$:

$$\rho(X^{*}(t_i), X^{*}(t_i)) < \frac{\varepsilon}{5}, \quad |\rho(X, Y) - \rho_n(X, Y)| < \frac{\varepsilon}{5}.$$

Then we have:

$$\rho(X_n(t'), X_n(t'))$$

$$\leq \rho(X^{*}(t'), X^{*}(t)) + \rho(X^{*}(t_i), X^{*}(t_i)) + \rho(X^{*}(t_i), X^{*}(t'))$$

$$< \rho_n(X^{*}(t'), X^{*}(t)) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \rho_n(X^{*}(t_i), X^{*}(t')) + \frac{\varepsilon}{5} < \varepsilon.$$
Thus the points \( X^*(t') \) converge to some limit \( X(t') \) for each \( t' \). The limit points \( X(t) \) form a continuous curve \( L \), since
\[
\rho_n(X^*(t_1), X^*(t_2)) \leq S |t_1 - t_2|,
\]
so that by Lemma 2
\[
\rho(X(t_1), X(t_2)) \leq S |t_1 - t_2|.
\]
From the last two inequalities it is easy to show that the points of the curves \( L_n \) converge to the corresponding points of the curve \( L \) in the metric \( \rho \) uniformly for all \( t \), i.e., \( L_n \to L \) in the metric \( \rho \). From the same inequality it follows that the limit curve \( L \) is rectifiable in the metric \( \rho \). Also for any system \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_{r+1} \leq 1 \)
\[
s_n \geq \sum_{i=1}^{r} \rho_n(X_{t_i}, X_{t_{i+1}}) \to \sum_{i=1}^{r} \rho(X_{t_i}, X_{t_{i+1}}),
\]
from which
\[
\lim_{n \to \infty} \inf_{r} s_n \geq \sum_{i=1}^{r} \rho(X_{t_i}, X_{t_{i+1}})
\]
and
\[
\lim_{n \to \infty} \inf_{r} s_n \geq s.
\]
Lemma 4 is proved.

**Lemma 5.** If the interior metrics \( \rho_n \) determine one and the same topology and converge uniformly to the metric \( \rho \), and the shortest arcs \( L_n \) in \( \rho_n \) converge in the metric \( \rho \) to the curve \( L \), then \( L \) is a shortest arc in the metric \( \rho \) and its length is equal to the distance \( \rho(X,Y) \) between its endpoints.

**Proof.** The equation \( s = \rho(X,Y) \) follows from the inequalities
\[
\rho(X, Y) \leq s \leq \lim_{n \to \infty} \inf_{r} s_n = \lim_{n \to \infty} \rho_n(X_n, Y_n) = \rho(X, Y).
\]
The fact that \( L \) is a shortest arc follows from \( s = \rho(X,Y) \).

**Lemma 6.** If the metrics \( \rho_n \) are intrinsic, the convergence \( \rho_n \to \rho \) uniform and all the metrics \( \rho_n \) determine one and the same topology, in terms of which the space \( R(\rho) \) is locally compact, then the limit metric \( \rho \) in the small is also intrinsic.

**Proof.** Choose a point \( O \in R \) and around it a compact neighborhood \( V \) of radius larger than \( 2r \) in the metric \( \rho \). Moreover, take a neighborhood \( U \) of radius \( r \). For a sufficiently large \( n \) every two points \( X, Y \in U \) may be joined in the metric \( \rho_n \) by shortest arcs \( L_n \) of length \( \rho_n(X,Y) \), and also \( L_n \subset V \). By Lemma 4, it is possible to select from the curves \( L_n \) a sequence
converging in the metric $\rho$ to some curve $L$. By Lemma 5, this length will be equal to $\rho(X,Y)$ and $L$ will be a shortest arc in the metric $\rho$. Thus every two points $X,Y$ of a sufficiently small neighborhood of an arbitrarily chosen point $O$ may be joined in the metric $\rho$ by a shortest curve of length $\rho(X,Y)$, so that the limit metric is intrinsic in the small.

**Lemma 7.** If the intrinsic metrics $\rho_n$, defining in $R$ one and the same topology, uniformly converge to a metric $\rho$ and the spaces $R(\rho_n)$ are boundedly compact, then $\rho$ is also an intrinsic metric and any two points in it can be joined by a shortest arc.

The proof of this lemma is analogous to the proof of Lemma 6 and will not be carried out.

**Remark.** Under the conditions of Lemmas 6 and 7 any two points in a sufficiently small neighborhood may be joined in the limit metric by a shortest arc which is a limit of some subsequence of shortest arcs in the converging metrics $\rho_n$. But we have no right to assert that every shortest arc in the limit metric $\rho$ can be considered as such a limit of shortest arcs from the metrics $\rho_n$. This compels us sometimes to resort to special constructions, making it possible to manage with shortest arcs that are easier to investigate and admit approximations of the indicated sort.

By joining one or another system of points by shortest arcs, we can always, as already mentioned in subsection 2 of Chapter III, avoid superfluous intersections of shortest arcs by requiring them to coincide on the part from the first to the last of their common points. If we proceed in this manner in joining a system of points by shortest arcs in the converging metrics, and then choose a subsequence for which the entire system of shortest arcs converges to some system of shortest arcs in the limit metric, we obtain a simultaneous joining of the points by the limiting shortest arcs. Such arcs cannot have essential intersections with one another (Figure 44a) but they may have various kinds of one-sided tangencies to one another, as depicted in Figure 44b.

2. **Two estimates for polyhedral metrics.**

4. **Variation of the angle $\gamma$.** Suppose that in a polyhedral metric, in
a region homeomorphic to a disc, there is a reduced triangle \( ABC \) without interior tails (see subsection 2 of Chapter III). From now on it will be assumed that this triangle is distinguished from the enclosing manifold. The measurement of distances and the laying off of shortest arcs will be carried out in the triangle itself.\(^1\)

We employ the following notations:

\[
(x)^+ = \begin{cases} 
  x & \text{if } x > 0, \\
  0 & \text{if } x \leq 0
\end{cases}, \quad (x)^- = \begin{cases} 
  0 & \text{if } x \geq 0, \\
  |x| & \text{if } x < 0
\end{cases}
\]

The positive part \( \omega^+ \) of the curvature of the triangle will be the sum \( \sum (2\pi - \theta_i)^+ \), where \( \theta_i \) is the complete angle around the vertices of the polyhedral metric lying inside the triangle, and the sum \( \sum \) is extended over all those vertices. The negative part \( \omega^- \) of the curvature is formed from the analogous sum \( \sum (2\pi - \theta_i)^- \), and also from the sum \( \sum (\phi_i - \pi) \) for all those interior points of the sides of the triangle at which the sector \( \phi_i \) from the side of the triangle is larger than \( \pi \). Moreover, in the presence of one or several exterior tails we add to \( \bar{\omega}^- \) the “portion of the negative curvature” which is concentrated at the branch point of the side, equal to the size of the sector from the side of the triangle. (Such a situation is illustrated in Figure 45 for the point \( D \). The portion of the negative curvature of the triangle concentrated there is \( \bar{\alpha} \).

Marking off points \( X, Y \) on the sides \( AB, AC \) of the distinguished triangle, we may as usual lay off distances \( AX = x, AY = y, XY = z \) which define a certain plane triangle \( T(AXY) \) with an angle \( \gamma_T(X,Y) \) at the vertex \( A \). (The subscript \( T \) recalls the fact that the measurement of the distance \( XY \) was carried out in the triangle \( T \) itself.) If the triangle \( T \) is convex, we have \( \gamma_T = \gamma \), where \( \gamma \) is the analogous angle obtained on measuring the distance \( XY \) in the original manifold. Under the conditions indicated above the following theorem holds for the quantity \( \gamma_T \).

**Theorem 1.** If the pairs of points \( X_i, Y_i, i = 1, \cdots, r + 1 \), form an in-

\(^1\) We may suppose that we are taking into account also triangles with interior tails, but along with distinguishing them from the enveloping manifold we regard them as having been “cut” along entrant tails.
creasing sequence on the sides $AB$, $AC$ of the triangle $ABC$, i.e., if $x_i \leq x_{i+1}$, $y_i \leq y_{i+1}$, then the sum of the positive increments of the quantity $\gamma_T$ does not exceed $\bar{\omega}^-$:

$$\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^+ \leq \bar{\omega}^-,$$

and the sum of the absolute values of the negative increments of $\gamma_T$ does not exceed $\omega^+$:

$$\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^\pm \leq \omega^+.$$

It follows from the assertion of the theorem that for a continuous monotone variation of the points $X$ and $Y$ the function $\gamma_T(X(t), Y(t))=\gamma_T(t)$ has bounded variation, the positive part of this variation not exceeding $\bar{\omega}^-$ and its negative part not exceeding $\omega^+$.

**Remark.** If the angle $\tilde{\alpha}$ of the sector at the vertex $A$ in the triangle $ABC$ exceeds $\pi$, then inequality (2) may be strengthened, being replaced by the following.

$$\pi - \tilde{\alpha}^- + \sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^+ \leq \omega^+.$$

Turning to the proof of the theorem, we may suppose that the side $BC$ is the unique shortest arc in the triangle.

Otherwise it may be replaced by one further left, i.e., passing closer to the vertex $A$, and the resulting two-gon may be discarded. This does not affect the value of $\gamma_T$ and does not increase $\bar{\omega}^-$ or $\omega^+$.

We begin the proof by marking off points $X, Y$ on the sides $AB$, $AC$ lying so close to $A$ that there are no points $X_i, Y_i$ between $X, Y$ and $A$, and that the triangle $AXY$ cut off from $ABC$ by the leftmost shortest arc $XY$ does not contain, either inside itself or on its sides $AX, AY$, vertices of the polyhedral metric, i.e., $AXY$ is isometric to a plane triangle. (If the angle $\tilde{\alpha}$ of the sector at the vertex $A$ in the triangle $ABC$ were less than $\pi$, then the triangle $AXY$ would be a nondegenerate plane triangle, and for $\tilde{\alpha} \geq \pi$ this triangle degenerates into a twice covered segment.) Thus we obtain a figure consisting of a plane triangle $T(AXY)$, of a two-gon $D(X,Y)$ bounded by a side $XY$ of the triangle $T(AXY)$ and the rightmost of the shortest arcs $XY$, and of the remaining portion $Q(XBCY)$.

We shall show that beginning with such a situation we may carry out a process of continuous deformation of the figure in question, in the course of which the following conditions will be observed.
a) The polyhedral character of the metric will be preserved. The lengths of the sides \( AB, AC, \) and \( BC \) will be preserved. The position of the point \( Y \) on the side \( AC \) will be preserved. The whole figure in the large remains a reduced triangle without interior tails, and in particular the sides \( AB, AC, \) and \( BC \) remain shortest arcs in the figure in question. Finally, the figure, as before, consists all the time of a certain plane triangle \( T(AXY) \), a two-gon \( D(XY) \), and the remainder \( Q(XBCY) \).

b) The point \( X \) moves continuously along \( AB \) toward the side of vertex \( B \), and the length of the side \( XY \) of the plane triangle \( T(AXY) \) in each position is equal to the distance \( XY \) for the corresponding points \( X,Y \) in the original triangle.

c) The piece \( Q(XBCY) \) in each position is isometric to the piece of the original triangle excised by the rightmost of the shortest arcs joining in it the corresponding points \( X \) and \( Y \). The form of the two-gon \( D(XY) \) may vary and does not necessarily coincide with that of the corresponding two-gon in the original triangle.

d) The total number of vertices of the polyhedral metric with curvatures distinct from zero does not increase. We are counting among these vertices also the point \( Y \), which we have fixed, and the moving point \( X \), whether or not these points have curvature distinct from zero.

e) The characteristics \( \omega^- \), \( \omega^+ \) of the triangle \( ABC \) do not increase in the course of the deformation.

f) The angle at the vertex \( A \) in the triangle \( T(AXY) \) is simply \( \gamma_{T}(X,Y) \). In passing from the original position of the point \( X \) to any of the positions following it this angle undergoes an increment \( \Delta \gamma_{T} \), which, if positive, does not exceed the decrease of \( \omega^- \), and if nonpositive does not exceed the decrease of \( \omega^+ \).

The existence of the process just described leads to the proof of Theorem 1. Indeed, we may in turn shift the point \( X \) in the indicated way, and then, changing the roles of \( X \) and \( Y \), move the point \( Y \) and make these points go through all the positions of the \( X, Y \). Thus, from the fact that conditions e) and f) are preserved, the validity of Theorem 1 will follow.

In order to prove the existence of the necessary process, we shall establish that, allowing an arbitrary position of the points \( X,Y \), we may push the point \( X \) to a further position while preserving all the conditions a)–f). Along with the passage to the limit this makes it possible to shift the point \( X \) to any point given in advance, in particular to the next point \( X \).
Thus we may suppose that the points \(X, Y\) have already been shifted into a certain position \(X_0, Y\) with conditions a)–f) preserved. In order to prove the existence of a further shift of the point \(X\) from the position \(X_0\), we consider five different possible cases.

**Case 1.** Suppose that for some point \(X' \in \{X_0, B\}\), and therefore for all the points \(X \in \{X_0, X'\}\), the leftmost shortest arc \(XY\) passes through \(X_0\) (Figure 46a).

![Figure 46.](image)

In this case we may continuously shift the point \(X\) from \(X_0\) to \(X'\), at the same time straightening out the quadrilateral \(AXX_0Y\) into the plane triangle \(AXY\) (Figure 46b). The distance from \(A\) to any point of the side \(XX_0Y\) does not decrease while this is being done, and thus the sides \(AB\) and \(AC\) remain shortest arcs.

If the triangle \(T(AX_0Y)\) degenerates into a segment, then the angle \(\gamma_T\) does not change. Otherwise, this angle increases. Suppose that \(\xi\) is the angle of the sector at the vertex \(X_0\) in the triangle \(T\), and \(\pi + \eta\) the angle at the vertex \(X_0\) in the remaining part \(D + Q\) (Figure 46a). On rectifying the quadrilateral the negative curvature at the point \(X_0\) will be lost. This is equal in absolute value to \(\xi + \eta\). The portion \(\eta\) goes into the new vertex \(X_0\), and the portion \(\xi\) is distributed between the increase in the absolute value of the negative curvature at the points \(X\) and \(Y\) and the change in \(\gamma_T\). Thus the latter completely coincides with the total decrease in the quantity \(\tilde{\omega}\).  

Evidently the remaining conditions a)–e) are preserved during the indicated process.

**Case 2.** Suppose that the conditions of Case 1 do not hold, i.e., that a leftmost shortest arc \(XY\) for \(X \in \{X_0, B\}\) does not go through \(X_0\). Suppose moreover that the two-gon \(D(X_0Y)\) merges into a unique shortest arc \(X_0Y\).
In this case the shortest arcs $XY$ converge to the side $X_0Y$ of the triangle $T$ as $X \rightarrow X_0$. Only vertices of the metric with negative curvature can lie on $X_0Y$. Therefore, for $X$ close to $X_0$ the shortest arcs $XY$ pass close to $X_0Y$ in a region not containing vertices with positive curvature, and therefore the shortest arcs $XY$ themselves are unique. It is easy to see that when the point $X$ moves from $X_0$ along $X_0B$ the shortest arc $XY$ originally changes continuously. Furthermore, it stretches along the shortest arc $YX_0$ on the portion from $Y$ to the last of the vertices with negative curvature lying on $YX_0$, and on the last portion $RX$ simply swings in a region isometric to the plane around the point $R$ (Figure 47a).

![Figure 47](image)

We shall move the point $X$ from $X_0$, simultaneously straightening out the planar pentagon $AX_0XRY$ into the plane triangle $AXY$ (Figure 47b). During this rectification the distance of points of the portion $XRY$ from the point $A$ does not decrease, so that in the entire figure in the large the sides $AB$, $AC$, remain shortest arcs. The decrease of the negative curvature at the points $X_0$ and $R$ is distributed into an increment of the negative curvature at the points $X$ and $Y$ and a decrease of the angle $\gamma_T$, which thus coincides with the total decrease of $\partial_{\gamma}$. The remaining conditions a) – e) are evidently satisfied.

Before turning to the further cases, we consider the possible structure of the two-gon $D(X_0Y)$, if it does degenerate into the unique shortest arc $X_0Y$, serving as a side of $T$.

Evidently, in general, all the shortest arcs $X_0Y$ moving in $D$ divide $D$ into a series of two-gons. The number of such pieces is finite, since in each of them there is at least one vertex with positive curvature. We consider that two-gon $O'O''$ (Figure 48) which entirely adjoins $X_0Y$, and if there are several such two-gons, the one which lies closest to $X_0$. The
two-gon $O'O''$ is cross-hatched in Figure 48.

There is at least one vertex $Q_i$ of positive curvature in $O'O''$. The point $Q_i$ can be joined to $A$ by a shortest arc which does not leave $T+O'O''$. If this shortest arc (see Figure 49) is not unique, then two

![Figure 48.](image1)

![Figure 49.](image2)

such shortest arcs form a two-gon inside which there is a new vertex $Q_i$ lying also inside $O'O''$. It may be joined to $A$ by a shortest arc which not only does not leave $T+O'O''$ but also does not issue from the two-gon $AQ_i$ in question. If that shortest arc is not unique, we turn to a new vertex $Q_i$ and so forth. Thus we arrive at some vertex $Q_i$, which is joined to $A$ by a shortest arc $AQ_i$ moving in $T+O'O''$, which is moreover unique among the the shortest arcs $AQ_i$ moving in a certain two-gon $AQ_{i-1}$ (or $T+O'O''$ if $i = 1$).

**Case 3.** Suppose that the shortest arc $AQ_i$ does not pass through any vertex with negative curvature. Suppose, moreover, that the triangle $T$ this time does not degenerate into a segment. In this case we extend the shortest arc $AQ_i$ beyond the point $Q_i$ to a segment $Q_iQ'$ which along with $AQ_i$ divides in two a sector around point $Q_i$. Any point $Q$ of the segment $Q_iQ'$ may be joined to $A$ by a shortest arc $AQ$ lying, as does the shortest arc $AQ_i$, in the two-gon $AQ_{i-1}$ (or in $T+O'O''$, if $i = 1$). Be-
cause of the uniqueness of $AQ_i$, as $Q \to Q_i$ the shortest arcs $AQ \to AQ_i$. Therefore, when $Q$ moves along a sufficiently small segment $(Q, Q')$, the shortest arc $AQ$ passes so close to $AQ_i$ that there are no other vertices between $AQ$ and $AQ_i$. In addition, there are exactly two shortest arcs $AQ$ and they encircle a single vertex $Q_i$ (Figure 50).

In each of the positions $Q \in (Q, Q')$ we may excise the indicated two-gon $AQ$ and paste its edges. In this process the distance $X_0Y$ decreases. Indeed, as is evident with the notations of Figure 50:

$$AG + GE \geq AE.$$  

Therefore, if one marks off a point $F$ so that $AGF = AE$, then $GE > GF$ and $X_0Y > X_0G + GF + EY$, i.e., after removing the triangle, when the points $E$ and $F$ coincide, there appears a shorter path $X_0G + GF + EY$ from $X_0$ to $Y$.

Suppose that when we excise the two-gon $AQ'$ the distance $X_0Y$ decreases by $2\varepsilon$. Then for any point $X \in (X_0, B]$ distant from $X_0$ by less than $\varepsilon$, the distance $XY$ after the excision of $AQ'$ turns out to be less than the original distance $XY$. Indeed

$$\overline{XY} \leq \overline{X_0Y} + X_0X = X_0Y - 2\varepsilon + XX_0 < X_0Y - XX_0 \leq XY.$$

In the original position there was no shortest arc $XY$ passing to the left of $Q = Q_i$, for otherwise the two-gon $O'O''$ would have been incorrectly chosen. Now after removing the two-gon $AQ'$ there appears such a shortest arc $\overline{XY}$, which is shorter than $XY$. Therefore, for some position $Q = Q(X)$ on the interval between $Q_i$ and $Q'$ the removal of the two-gon $AQ$ leads to the appearance of a shortest arc $\overline{XY}$ running to the left of $Q_i$, exactly equal in length to the original distance $XY$. This shortest arc $\overline{XY}$ begins at the point $X$, enters $O'O''$, and on the portion from $O''$ to $Y$ it may be regarded as coinciding with the side $X_0Y$ of the triangle $T$. As $X \to X_0$ the shortest arc $\overline{XY}$ converges to $X_0Y$, for otherwise we would have a new shortest arc $X_0Y$ going to the left of $Q_i$, which would contradict the choice of $O'O''$. By the same principle the shortest arcs $\overline{XY}$ are unique for $X$ close to $X_0$ and change continuously as $X$ shifts from $X_0$, beginning with the position $X_0Y$.

On excising the two-gon $AQ$ the sides $AB$ and $AC$ remain shortest arcs. The decrease of the angle at the vertex $A$ is equal to the decrease in $\omega^+$.

\footnote{For this inequality to be strict we have supposed that the plane triangle $AX_0Y$ does not degenerate into a segment, so that $X_0Y$ does not pass through $A$ and the points $E$ and $G$ do not coincide.}
arising from the replacement of the vertex \( Q_i \) by the vertex \( Q \). The shortest arc \( XY \) is obtained by a continuous shift from \( X_0Y \). If at the point \( X_0 \) or at any point \( R \) of the side \( X_0Y \) has negative curvature, then plane figure \( AX_0XRY \) would stop being a triangle. Then, simultaneously with the shift of \( X \) and the change of the excised triangle \( AQ \), we accomplish a rectification of the figure \( AX_0XRY \) into a triangle \( AXY \), as we already did in Case 2. The resulting decrease of the angle \( A \) will be exactly equal to the loss in \( \bar{\sigma}^\cdot \). Thus the change in the large of the figure will satisfy all the conditions a)–f).

**Case 4.** The shortest arc \( AQ_i \) passes through one or several vertices \( R \) with negative curvature. The triangle \( AX_0Y \) does not degenerate into a segment.

The shortest arc \( AQ_i \), passing through a vertex with negative curvature, leaves on at least one of the sides a sector angle larger than \( \pi \). Suppose \( R' \) is the vertex with negative curvature closest to \( Q_i \) for which to the right of \( AQ_i \) the sector is larger than \( \pi \), and that \( R'' \) is the closest of the vertices where the sector to the left of \( AQ_i \) is larger than \( \pi \). It is possible that \( R' = R'' \). The case when on one of the sides there is no vertex at all with a sector larger than \( \pi \) need not be considered; we may subsume it under the Case 3 already studied above. The difference is only that the two-gon \( AQ \) will have the form depicted in Figure 51. But, as in

![Figure 51](image1)

![Figure 52](image2)

Case 3, the removal of \( AQ \) leads to a shortening of the distance \( X_0Y \), making it possible to retain the discussion presented above. Thus in Case 4 we take only those situations for which the two-gons \( AQ \), for \( Q \) close to \( Q_i \), merge into a shortest arc within the limits of the triangle \( T \) and have their sides divergent only within the limits of the two-gon \( O'O'' \) (Figure 52).
In Case 4 we shall shift the vertex $Q$, moving away the two-gon $AQ$, until one of the following positions occurs:

1) A new shortest arc $O'O''$ occurs;

2) One of the vertices $R', R''$ stops being a vertex with negative curvature;

3) On $R'Q$ or $R''Q$ there appears at least one new vertex with negative curvature;

4) The vertex $Q$ merges with one of the vertices with negative curvature;

5) The curve $AQ$ (after removal of the two-gon) stops being the unique shortest arc $AQ$ in the region in question (in $AQ_{-1}$ or $T + O'O''$).

In each of these cases we turn to the consideration of the newly obtained figures. In view of the finiteness of the number of vertices of the polyhedral metric all these cases can appear only finitely many times, and we sooner or later arrive at Case 3, which allows us to continue the shift of the point $X$.

We note that in the process of removing the two-gon $AQ$ under the hypotheses of Case 4 the sides $AB, AC$ remained shortest arcs, the plane triangle $AX_0Y$ and its angle $\gamma_T$ remained invariant, the curvatures $\bar{\omega}^-, \omega^+$ did not increase, and all the conditions a)–f) remained fulfilled.

**Case 5.** Suppose finally that under the hypotheses of Case 3 or 4 the triangle $T$ degenerates into a segment.

Since Case 1 is excluded, degeneration of $T$ into the segment $AYX_0$ is impossible. If $T$ merges with the segment $AX_0Y$, then the shortest arc $AQ$, certainly passes through a vertex $Q'$ with negative curvature and the discussion may be carried out as in Case 4. It remains to consider the possibility of degeneration of $T$ into the segment $X_0AY$. If in addition the vertex $A$ lies outside the two-gon $O'O''$, the shortest arc $AQ$, passes through $O'$ or $O''$ and the discussion may be carried out as in Case 4. Suppose finally that the point $A$ belongs to the segment $O'O''$ (Figure 53). Then we shall distinguish two cases, depending on whether the angle $\bar{\alpha}$ of the sector at the vertex $A$ is larger than or equal to $\pi$.

If $\bar{\alpha} > \pi$, then removal of the two-gon $AQ$ for small shifts of $Q$ from $Q$, does not shorten $X_0Y$ and we are essentially back at Case 4.

If $\bar{\alpha} = \pi$, then removal of the two-gon $AQ$ is connected with the shortening of $X_0Y$ and we may proceed as in Case 3.

Thus it is proved that in any of the positions one may prolong the transformation of the figure $ABC$ while preserving the conditions a)–f).
At the same time, if the position of the point $X$ becomes unboundedly close to the position $X'$, one may pass to the limiting figure: the lengths of the sides of the triangle $T$ will converge to some quantities, the lengths of the sides of the limit triangle $T'$, and the piece $D+Q$ transforms by being monotonically shortened, at the expense of movement of the shortest arc $XY$ and the removal of two-gons $AQ$ with a finite number of displaced points $Q$. This piece also converges to some limit figure, dividing the rightmost of the shortest arcs $X'Y$ into pieces $D'+Q'$. In addition, the limit figure $T'+D'+Q'$ will satisfy all the conditions a)–f).

The possibility of prolonging the transformation from any position which may be reached and the possibility of passing to the limit enables us to shift the point $X$ in the indicated way and monotonically up to any position on the piece $X_0B$ of the side $AB$. As we have already noted, this proves Theorem 1.

Remarks. 1) Suppose that in the initial position the triangle $T$ reduces to a segment $XAY$, and the angle $\alpha$ of the sector adjacent to the vertex $A$ in the figure $ABC$ is larger than $\pi$. After rectifying $ABC$ into a plane triangle $T(ABC)$ this angle will be not greater than $\pi$. We may observe that the initial decrease of this angle to $\pi$ occurred in the process of deformation of the triangle $ABC$ by removing the two-gon $AQ$ (Case 5). In Case 5 the decrease of $\alpha$ is connected with the loss of the corresponding part of $\omega^+$. This makes it possible to strengthen the formulation of Theorem 1 and to pass from estimate (2) to estimate (3).

2) The proof of Theorem 1 can be somewhat simplified if the results of [65] are used. In fact, one may strike off the leftmost shortest arcs $X_iY_i$, and replace each of the regions homeomorphic to the disc into which these shortest arcs divide $T$ by a polygon on the cone. Then, to the resulting simpler polygons containing few vertices with $\omega^+ > 0$ and not containing any vertex with $\omega^- \neq 0$, we may apply the arguments presented above.

5. The oscillation of $\gamma$ farther from the vertex.

Theorem 2. Under the conditions of Theorem 1, if the positive part of the curvature of the triangle $ABC$ is concentrated near the vertex $A$,
then the decrease of $\gamma_T(X,Y)$ on pieces far from the vertex $A$ is small.

More precisely: for any $\Omega^+ \geq 0$, $R > 0$, $\varepsilon > 0$ given in advance, there exist $r > 0$ and $\delta > 0$ so small that in all cases when the positive part $\omega^+$ of the curvature of the triangle $ABC$ consists of the part $\omega^+_1 \leq \Omega^+$ concentrated in an $r$-neighborhood of the vertex $A$, and a remaining portion $\omega^+_2 \leq \delta$, the decrease of $\gamma_T(X,Y)$ on any segment outside $R$ of a neighborhood of the vertex $A$ does not exceed $\varepsilon$, i.e., for $R \leq AX_1 \leq AX_2$ and $R \leq AY_1 \leq AY_2$

$$(4) \quad \gamma_T(X_1, Y_1) - \gamma_T(X_2, Y_2) \leq \varepsilon.$$  

The idea of the proof of Theorem 2 is the following. While carrying out the excisions of the two-gons $AQ$ enclosing the vertices $Q$, of positive curvature, we may move the vertices $Q$ away from $A$. This makes the curvature $\omega^+_1$ decrease and become very small. At the same time all the changes in the curvature $\omega^+_2$ can be realized in a small neighborhood of the point $A$ and do not affect in any essential way the quantities $\gamma_T(X_1, Y_1)$ and $\gamma_T(X_2, Y_2)$. Then it remains to apply Theorem 1 to the estimation of $dT_T$ on the piece $X_1, Y_1; X_2, Y_2$ in the metric measured in the indicated way.

In order to carry out this plan, we prove two lemmas.

**Lemma 8.** If a vertex $Q$ lying at a distance $r$ from $A$ and having positive curvature $\omega$ is removed to a larger distance $\rho$ by excision of a two-gon $AQ$ enclosing $Q$ and containing no other vertex of the metric, then the curvature $\omega$ decreases not less than $r/\rho$ times, taking on a value

$$(5) \quad \omega' \leq \omega \frac{r}{\rho}.$$  

Indeed, in the general case the excised two-gon $AQ'$ has the form

*Figure 54.*
depicted on Figure 54a. On Figure 54b, we depict in the form of a
development on the plane one of the two triangles $AQQ'$ forming the
two-gon $AQ'$. Evidently, in the notations of Figure 54,

$$\omega' = \omega_1 + \omega_2, \quad r = r_2 + r_1, \quad \rho = \rho_2 + r_1, \quad r_2 < \rho_2.$$  

Using the decrease of the function $\sin x/x$ on the interval $[0, \pi/2]$ and the sine law, we obtain

$$\frac{\omega_1}{\omega} \leq \frac{\sin \omega_1}{\sin \omega} = \frac{r_2}{\rho_2} \leq \frac{r}{\rho}.$$  

Along with the analogous estimate for $\omega_1$ this yields inequality (5).

**Lemma 9.** If the shortest arc $XY$ passes close to $A$, then $\gamma_T(X,Y)$ is
close to $\pi$.

More precisely: if on a shortest arc joining the points $X$ and $Y$, which are
more distant from $A$ than from $R$, there is a point distant from $A$
by a distance $\rho \leq (R/2) \sin^2 (\varepsilon/6)$, then

$$\pi - \frac{\varepsilon}{3} \leq \gamma_T(X,Y) \leq \pi.$$  

Indeed, in the indicated case, for the distances $x = AX$, $y = AY$, $z = XY$,
we have the inequality

$$x + y - 2\rho \leq z \leq x + y.$$  

In a plane triangle with sides $x, y, z$ we have for the angle $\gamma_T$:

$$\cos \gamma_T = \frac{x^2 + y^2 - z^2}{2xy},$$  

so that

$$1 + \cos \gamma_T = \frac{(x + y - z)(x + y + z)}{2xy} \leq \frac{2\rho \cdot 2(x + y)}{2xy}$$  

or in other words

$$\sin^2 \frac{\pi - \gamma_T}{2} \leq \rho \left(\frac{1}{x} + \frac{1}{y}\right) \leq \frac{2\rho}{R} \leq \sin^2 \frac{\varepsilon}{6}.$$  

Taking into account the fact that $0 \leq (\pi - \gamma_T)/2 \leq \pi/2$, we therefore
obtain (6) and Lemma (9) is proved.

Now we proceed to the proof of Theorem 2.

Take $\delta = \varepsilon/3$, $\rho = (R/2) \sin^2 (\varepsilon/6)$, $\gamma = \rho (\varepsilon/3\Omega^+)$.
Suppose that in an $r$-neighborhood of the vertex $A$ there are vertices of the metric with the
total positive curvature $\omega'_1 \leq \Omega^+$. By excising two-gons $AQ'$ we may shift
them off to the distance \( \rho \). (If positive curvature exceeding \( \pi \) is concentrated at a vertex, then initially in the shift such a vertex approaches \( A \) and the curvature concentrated at it decreases. Then the curvature continues to decrease, and the vertex starts to move off from \( A \).) After this, since all these vertices have moved off to a distance \( \rho \), their total curvature, in accordance with Lemma 8, takes a value

\[
\omega_1^{+} \leq \omega_1^{+} \frac{r}{\rho} \leq \frac{\varepsilon}{3}.
\]

The shift of the vertices is connected with a change in the polyhedral metric, but none of these changes leaves the limits of the exterior boundary of the \( \rho \)-neighborhood of the vertex \( A \). The distances of all points up to \( A \) remain unchanged.

Suppose that \( \gamma_1 = \gamma_1(X_1, Y_1), \gamma_2 = \gamma_1(X_2, Y_2) \) in this initial metric, and that \( \gamma_1', \gamma_2' \) are the corresponding angles in the transformed metric.

We apply Theorem 1 to the altered metric. We have:

\[
\gamma_1 - \gamma_2 \leq \omega_1^{+} + \omega_2^{+} \leq \frac{2}{3} \varepsilon. \tag{7}
\]

Now we consider the following three possibilities:

1. The shortest arcs \( X_1Y_1, X_2Y_2 \) in the changed metric do not touch the \( \rho \)-neighborhood of the point \( A \). Then \( \gamma_1 = \gamma_1', \gamma_2 = \gamma_2' \) and the inequality \( \gamma_1 - \gamma_2 \leq \varepsilon \) mentioned in Theorem 2 follows from (7).

2. The shortest arc \( X_1Y_2 \) in the altered metric encounters the \( \rho \)-neighborhood of \( A \). Then the shortest arc \( X_1Y_1 \) also encounters that neighborhood, since that shortest arc may be considered to pass to the left of \( X_2Y_2 \).

From Lemma 9 we have:

\[
\pi - \frac{\varepsilon}{3} \leq \gamma_1' \leq \pi, \quad \pi - \frac{\varepsilon}{3} \leq \gamma_2' \leq \pi.
\]

But the shortest arcs \( X_1Y_1 \) and \( X_2Y_2 \) in the unaltered metric either also encountered the \( \rho \)-neighborhood of \( A \) or else passed through the unaltered zone, in which case they can only be longer than the newly appearing shortest arcs. Therefore

\[
\pi - \frac{\varepsilon}{3} \leq \gamma_1 \leq \pi, \quad \pi - \frac{\varepsilon}{3} \leq \gamma_2 \leq \pi,
\]

so that, further, for the absolute value of the difference in this case we have:

\[
|\gamma_1 - \gamma_2| < \varepsilon.
\]
3. In the altered metric the shortest arc $X_1Y_1$ touches, and the shortest arc $X_2Y_2$ does not touch the $\rho$-neighborhood of $A$. Then $\gamma_2 = \gamma'_2$, and with an argument analogous to that of the preceding section we establish that $|\gamma_1 - \gamma'_1| \leq \varepsilon/3$.

Thus

$$\gamma_1 - \gamma_2 \leq |\gamma_1 - \gamma'_1| + |\gamma'_1 - \gamma'_2| \leq \varepsilon.$$

Theorem 2 is proved.

**Theorem 3.** Under the hypotheses of Theorem 1, if the negative curvature of the triangle is concentrated near $A$, the increment of $\gamma_r(X,Y)$ on a segment distant from $A$ will be small.

More precisely: given any $\Omega^- \geq 0$, $R > 0$, $\varepsilon > 0$, there exists a pair $r > 0$, $\delta > 0$, so small that when the negative part of the curvature of the triangle $ABC$ consists of a part $\omega^- \leq \Omega^-$ and a remaining part $\omega^- \leq \delta$, the increment of $\gamma_r(X,Y)$ on any piece outside an $R$-neighborhood of the vertex $A$ does not exceed $\varepsilon$, i.e., for $A \leq AX_1 \leq AX_2$, $R \leq AY_1 \leq AY_2$

$$|\gamma_r(X_0, Y_2) - \gamma_r(X_1, Y_1)| \leq \varepsilon.$$

(8)

The proof of Theorem 3 is carried out on the same plan as that of Theorem 2. We need to remove vertices with negative curvature from the vicinity of $A$, to verify that their curvature decreases sufficiently fast, and that the altered metric does not essentially affect the difference $\gamma_2 - \gamma_1$, and then to apply Theorem 1.

Suppose that $P$ is a vertex with negative curvature of absolute value

![Figure 55.](image-url)
\( \omega \). Pass a shortest arc \( AP \) through \( A \) and prolong it by two segments \( PP' \) each forming with \( AP \) the angle \( \pi \), as in Figure 55a. Divide the sector \( \omega \) remaining between the two segments into \( n \) pieces by \( n-1 \) further segments \( PP' \). Then cut along the curve \( AP \) and along all the segments \( PP' \) and paste into the cut thus formed a plane figure as depicted in Figure 55b. In this pasting the vertex \( P \) disappears and \( n+1 \) new vertices \( P' \) appear in its place.

**Lemma 10.** If for the indicated pasting, there appear, in the place of the vertex \( P \) distant by \( r \) from \( A \) and having negative curvature of absolute value \( \omega \), \( n+1 \) vertices \( P' \), each of which is distant by a greater distance \( \rho \) equal to \( AP + PP' \), then the absolute value \( \omega' \) of the total curvatures of the vertices \( P' \) satisfies the inequality

\[
\omega' \leq \omega \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}.
\]

For the proof we consider one of the triangles \( APP' \), as in Figure 56: We have:

\[
\frac{\sin \frac{\omega - \omega'}{2n}}{\sin \frac{\omega}{2n}} = \frac{\rho - r}{\rho},
\]

so that

\[
\sin \frac{\omega}{2n} - \sin \frac{\omega - \omega'}{2n} = \frac{r}{\rho} \sin \frac{\omega}{2n},
\]

\[
2 \sin \frac{\omega}{4n} \cos \frac{2\omega - \omega'}{4n} = 2 \frac{r}{\rho} \sin \frac{\omega}{4n} \cos \frac{\omega}{4n},
\]

\[
\frac{\sin \frac{\omega'}{4n}}{\sin \frac{\omega}{4n}} = \frac{r}{\rho} \cdot \frac{\cos \frac{\omega}{4n}}{\cos \frac{2\omega - \omega'}{4n}} \leq \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}.
\]

But then

\[
\frac{\omega'}{\omega} \leq \frac{r}{\rho} \cdot \frac{1}{\cos \frac{\omega}{2n}}
\]

and Lemma 10 is proved.
We turn to the proof of Theorem 3. Suppose
\[ \delta = \frac{\epsilon}{3}, \quad \rho = \frac{R}{2} \sin^{2} \frac{\epsilon}{6}, \quad r = \rho \frac{\epsilon}{12 \Omega}. \]

First, by excising two-gons, we move all vertices \( Q \) with positive curvature to the exterior boundary of the \((\rho/2)\)-neighborhood of the point \( A \). Under this the distances to \( A \) do not change. Then we join each of the vertices with negative curvature and lying in the \( r \)-neighborhood of the point \( A \) to \( A \) by a shortest arc. These shortest arcs will be unique. They can only "flow together" in the direction towards the vertex \( A \), since they are traced out in a zone where there do not remain any vertices with positive curvature, so that the formation of two-gons is impossible. Each of the shortest arcs \( AP \) will be prolonged to a distance \( \rho/2 \) from \( A \) by a bundle of segments \( PP' \) as described above. All the curves \( PP' + PA \) remain shortest arcs and do not intersect, in view of the absence of vertices with positive curvature. The number \( n \), for each of the bundles thus drawn, will be chosen so large that for the given vertex \( P \) the condition
\[ \sin \frac{\omega}{4n} \leq \frac{1}{2} \leq \cos \frac{\omega}{2n} \]
will be satisfied.

Then we carry out the pasting described above for each of the vertices \( P \). Vertices \( P \) with total negative curvature of absolute value \( \omega^- \) are thus replaced by newly appearing vertices \( P' \) for which, according to Lemma 10, we will have the following estimate for the absolute value of their total negative curvature:
\[ \omega_i'^- \leq \omega^- \frac{2r}{\rho} \min \frac{1}{\omega} \cos \frac{\omega}{2n} \leq \Omega^- \frac{\epsilon}{6 \Omega} \leq \frac{\epsilon}{3}. \]

For the indicated pasting the distances of points to the vertex \( A \) does not decrease, so that \( AB \) and \( AC \) remain shortest arcs. Now the increase of any point \( M \) from \( A \) cannot be larger than \( \rho/2 \). Indeed, the shortest arc \( MA \) drawn before carrying out the pasting either did not touch the future cuts, in which case it has the same length also after the pasting, or else on some portion \( AM' \) it went along the curve of a cut (Figure 57a). In this case the shortest arc \( MA \), after pasting, is elongated by not more than \( 2x \sin(\omega'/4n) \), which does not exceed \( \rho/2 \), since \( x \leq \rho/2 \), and sin \( (\omega'/4n) \leq 1/2 \) by the choice of \( n \).
After the indicated shift of the vertices $Q$ and the vertices $P$ all the changes in the metric do not go beyond the exterior boundary of the $\rho$-neighborhood of the point $A$, either in the original or in the altered metric.

After all this the proof of Theorem 3 is completed in just the same way as that of Theorem 2. We shall not repeat the argument here.

Remark. Putting Theorems 2 and 3 together, we may assert that if the basic part of the absolute curvature of a polyhedral triangle $ABC$ is concentrated near its vertex $A$, the oscillation of the quantity $\tau(X,Y)$ far from $A$ turns out to be insignificant.

3. Existence of angles in the limit metric. In this section we prove the existence of an angle between any two shortest arcs in metrics admitting local approximation by polyhedral metrics for which the positive part of the curvature is uniformly bounded (see subsection 1). Thus we shall have established that in two-dimensional manifolds of bounded curvature any two shortest arcs which issue from the same point form a well defined angle. In subsection 13 of Chapter II this was proved only for shortest arcs forming a sector convex relative to the boundary.

6. Angle between shortest arcs. Suppose that the metric $\rho$ is the limit of uniformly converging polyhedral metrics $\rho_n$, all of which make $R$ into the same topological space, namely into a two-dimensional manifold. Let $G$ be a neighborhood homeomorphic to the closed disc of an arbitrary fixed point $O$. Suppose that in $G$ the positive parts $\omega^+_\rho$ of the curvatures of the metrics $\rho_n$ are bounded uniformly: $\omega^+_\rho(G) \leq C$. We shall consider $\omega^+_\rho$ to be a set function given on the Borel subsets of $G$.

It is well known that from an infinite system of completely additive functions, defined and uniformly bounded on the Borel sets of a com-
pact space it is possible to select a weakly converging sequence (see, for example, [1], Chapter III, p. 237). This makes it possible, retaining only some of the metrics $\rho_n$, to suppose that the functions $\omega^+_n$ weakly converge to some completely additive set function $\omega^+$.

Suppose that $U_r$ are neighborhoods of the point $O$ homeomorphic to the disc minus the central point $O$ and with radii not exceeding $r$. As $r \to 0$ the sets $U_r$ form a vanishing sequence. Therefore, in view of the complete additivity of the functions $\omega^+$ (see, for example, [1], Chapter II, p. 587),

$$\lim_{r \to 0} \omega^+(U_r) = 0.$$ 

We choose in $U_r$ a set $V_r$ homeomorphic to the circular annulus. It may be chosen so that the interior boundary of the ring is arbitrarily close to $O$ and the exterior to the outer boundary of $U_r$. Evidently $\omega^+(U_r) \geq \omega^+(V_r)$. Moreover, in view of the weak convergence $\omega^+_n \to \omega^+$ and the fact that $V_r$ is closed,

$$\omega^+(V_r) \geq \limsup_{n \to \infty} \omega^+_n(V_r)$$

(see, for example, [1], Chapter III, p. 235).

Suppose the $r$ is so small that $\omega^+(U_r) < \varepsilon$. Then for each fixed $V_r$ and for sufficiently large $n$ we have $\omega^+_n(V_r) < \varepsilon$.

From the above we may make the following assertion.

**Lemma 11.** There exists a subsequence of the metrics $\rho_n$ such that for any $\varepsilon > 0$ there exists an arbitrarily small $R$-neighborhood of the point $O$, and an arbitrarily small $r$-neighborhood of the point $O$, chosen after the choice of $R$, for which, beginning with sufficiently large $n$, all the metrics $\rho_n$ in the annulus between the $R$- and $r$-neighborhoods of the point $O$ have positive curvature less than $\varepsilon$.

Indeed, choose a sequence $\rho_n$ for which $\omega^+_n \wto \omega^+$. We take $R$ so small that $\omega^+(U_{2R} - O) < \varepsilon$. Then we take an arbitrarily small $r$ $(0 < r < R)$ and choose a ring-shaped region $v \subset U_{2R} - O$ whose interior boundary lies inside the $r$-neighborhood of the point $O$, and whose exterior boundary encloses the $R$-neighborhood of $O$ but does not go outside of $U_{2R}$. From what was said above, for sufficiently large $n$ we will have $\omega^+_n(V) < \varepsilon$.

Now it is easy to prove the following important assertion.

**Theorem 4.** In the limit metric any pair of shortest arcs issuing from one and the same point form a definite angle.
No additional hypotheses on the character of their relative positions is made.

In proving this theorem we shall not present precise \( \varepsilon \)-estimates, rather restricting ourselves to the essentials of the proof.

Suppose that the shortest arcs \( L \) and \( M \) issue from the point \( O \) and do not form a definite angle at that point. Then on these shortest arcs there exist two sequences of pairs of points \( X \) and \( Y \) converging to \( O \) for which the \( \gamma(X,Y) \) converge to limits differing by some \( \delta > 0 \).

Consider a very small \( R \)-neighborhood of the point \( O \). By the hypotheses, there exist on \( L \) and \( M \) pairs \( X_i, Y_i \) and \( X_2, Y_2 \), for which \( OX_i \leq OX_2, OY_i \leq OY_2 \) and

\[
\gamma(X_i, Y_i) - \gamma(X_2, Y_2) \geq \frac{\delta}{2}.
\]

Suppose further that \( \rho \) is a polyhedral metric with a very large index. In it we draw shortest arcs \( OX_i, OX_2, OY_i, OY_2, X_iX_2, Y_iY_2 \) without superfluous intersections, as for example in Figure 58. On the shortest arcs \( OX_i, OY_i \) we mark points \( X_i', Y_i' \) distant from \( O \) by distances equal to \( OX_i \) and \( OY_i \).

From Lemma 11 we may suppose that the curvature \( \omega^* \) of the polyhedral metric \( \rho \) in the entire region in question is insignificantly small except possibly for some portion not exceeding a constant \( C \) independent of \( n \) and concentrated in the \( r \)-neighborhood of the point \( O \), where \( r \) is chosen to be insignificantly small in comparison with the distances \( OX_i, OY_i \).

If we were to consider on the plane a triangle with the sides \( \rho(O,X_i), \rho(O,X_2), \rho(X_i,X_2) \), then its angle at the vertex \( O \) would be equal to zero. In view of the uniform convergence \( \rho \to \rho \) we may suppose that the angle is close to zero also in distances measured in \( \rho \).

By Theorem 2, this last angle cannot grow strongly on the passage to the triangle \( OX_i, OX_i', X_iX_i' \). Therefore \( X_iX_i' \) is very small in comparison with \( OX_i, OY_i \). By a similar principle \( Y_iY_i' \) is small in comparison with \( OX_i, OY_i \).

Now we have the following situation:

1) \( \gamma(X_i, Y_i) \approx \gamma(X_i, Y_i) \) because of the uniform convergence \( \rho \to \rho \);

2) \( \gamma(X_i, Y_i) \approx \gamma(X_i, Y_i) \) because of the smallness of \( X_iX_i', Y_iY_i' \);
3) \( \gamma(x, y) \) cannot greatly exceed \( \gamma(x, y) \).

If the shortest arc \( X'Y' \) lies in the triangle \( XOY \), then assertion (3) follows from Theorem 2. If now \( X'Y' \) goes outside this triangle, then on \( OX, OY \), between \( X' \) and \( X \) and between \( Y' \) and \( Y \) there exist points \( X'' \) and \( Y'' \) joined by shortest arcs passing both inside and outside that triangle, and we obtain the same result on applying Theorem 2 first to \( X'Y'' \) and \( X''Y'' \), and then to \( X''Y'' \) and \( X''Y'' \). Finally, \( \gamma(x, y) \approx \gamma(x, y) \) in view of the uniform convergence \( \rho \to \rho \).

From this last and relations 1)–3) enumerated above, it follows that \( \gamma \) cannot substantially exceed \( \gamma(x, y) \), in contradiction with hypothesis (9).

Thus Theorem 4 is proved.

4. Sector angle.

7. Sector angle. If two shortest arcs \( L_1, L_2 \) issue from the point \( O \) and on some initial segment have no essential intersections (not excluding the possibility that these shortest arcs overlap or touch), then it makes sense to say that these shortest arcs decompose a neighborhood of the point \( O \) into two sectors. Here the concept of sector is somewhat generalized; see subsection 11 of Chapter II.

A sector may in its turn be decomposed into pieces by a series of shortest arcs issuing from its vertex \( O \), going into the sector, and not having essential intersections. These shortest arcs, including the sides \( L_1 \) and \( L_2 \) of the sector itself, are ordered in their successive order around the point \( O \). The sector angle is the least upper bound of the sums of the angles between successive shortest arcs of each particular such decomposition of the sector, taken over all such decompositions. Below we shall verify that the sector angle is always finite in the spaces of interest to us. In the meantime we shall admit for it also infinite values.

The angle \( \alpha \) between the sides of a sector evidently does not exceed the angle \( \bar{\alpha} \) of the sector, since the sides of the sector themselves already form one of the possible decompositions referred to above.

**Lemma 12.** If the sector \( C(L_1, L_2) \) is divided into pieces by shortest arcs \( L_2, \ldots, L_{n-1} \), and if one of the subdividing shortest arcs forms a null angle with its successor, then such a shortest arc may be dropped from the curves of the subdivision, without changing the sum of the angles between successive shortest arcs.

If the shortest arcs \( L_1, L_2 \) and \( M \) issue from the point \( O \), and the shortest arc \( M \) has common points with \( L_1 \) and \( L_2 \) arbitrarily close to \( O \), then the shortest arcs \( L_1 \) and \( L_2 \) form a null angle.
Both assertions of Lemma 12 follow from the existence of angles and from the triangle inequality (Theorem 1 of Chapter II) for angles between three shortest arcs.

Lemma 12 makes it possible to supplement already existing subdivisions of a sector by any shortest arc lying in the sector. If in doing this the new shortest arc intersects, arbitrarily close to the vertex of the sector, one or several of the previous shortest arcs, then all of them along with the newly drawn shortest arc form pairwise null angles and may be replaced by a single new shortest arc without changing the sum of the angles between successive shortest arcs. But if the new shortest arc does not touch the other shortest arcs in a small neighborhood of \( O \), it simply supplements the decomposition, which can only increase the sum of the angles between successive shortest arcs of that decomposition.

The following important assertion therefore follows.

**Lemma 13.** In a two-dimensional manifold of bounded curvature, for sectors admitting decomposition into sectors convex relative to the boundary, the value of the sector angle in the sense of the definition of subsection 14 of Chapter II coincides with the value of that angle in the sense of the definition just given.

In fact, the new definition, extending the class of admissible decompositions, can only increase the value of the sector angle. But the possibility of supplementing any subdivision by the curves of the decomposition into sectors convex relative to the boundary shows that the increase of the value of the sector angle cannot take place.

The following theorem is proved analogously.

**Theorem 5.** Sector angles are additive.

More precisely, if the shortest arcs \( L_1, L_2, L_3 \) issue from the point \( O \) and do not have essential intersections near \( O \), then we may say that one of the sectors \( C_{13} \), bounded by the shortest arcs \( L_1, L_3 \), is made up of sectors \( C_{12} \) and \( C_{23} \). For the angles of these sectors \( \bar{a}_{13} = \bar{a}_{12} + \bar{a}_{23} \).

**Proof.** Each two decompositions of the sectors \( C_{12}, C_{23} \) forms a decompositions \( C_{13} \), so that \( \bar{a}_{13} \geq \bar{a}_{12} + \bar{a}_{23} \). Conversely, every decomposition \( C_{13} \) may be supplemented by the shortest arc \( L_2 \) without decreasing the sum of the angles between successive shortest arcs of the decomposition, so that \( \bar{a}_{13} \leq \bar{a}_{12} + \bar{a}_{23} \), whence \( \bar{a}_{13} = \bar{a}_{12} + \bar{a}_{23} \).

8. **Complete angle around a point.** The entire neighborhood of a point may be considered as a special sort of sector. We shall call its angle the
complete angle around the point and as a rule designate it by $\theta$. More precisely, $\theta$ is the upper bound of the sum $\sum_{i=1}^{n} \alpha_{i+1}$ of the angles between successive shortest arcs $L_1, L_2, \ldots, L_n, L_{n+1} = L_1$ of any decomposition of the neighborhood of this point into a finite number of sectors. Such decompositions exist. It suffices, for example, to take a single shortest arc issuing from $O$. The finiteness of $\theta$, which we shall verify later, is so far not assumed.

For points whose neighborhoods decompose into sectors convex relative to the boundary, the value of the complete angle in the sense of the definition of subsection 14 of Chapter II evidently coincides with the newly given definition.

It will be necessary for us to make an essential distinction between ordinary points, with $\theta > 0$, and singular points, with $\theta = 0$.

**Lemma 14.** At points where $\theta = 0$ any two shortest arcs form a null angle.

Indeed, if two shortest arcs intersect arbitrarily close to $O$, they form a null angle. But if they do not intersect near $O$, then they must also form in the present case a null angle. Otherwise a pair of these shortest arcs would give a decomposition of the neighborhood, reducing to a positive sum of angles, and we would have $\theta > 0$.

Evidently in this case, any sector at the vertex $O$ will have a null angle.

9. **Properties of the sector angle.**

**Theorem 6.** Suppose that from a point with complete angle $\theta$ there issue two shortest arcs, forming sectors $C_1, C_2$ with angles $\bar{\alpha}_1$ and $\bar{\alpha}_2$. Then the following assertions hold.

1) The angle between shortest arcs is equal to the smallest of the three numbers $\bar{\alpha}_1, \bar{\alpha}_2, \pi$:

$$\alpha = \min [\bar{\alpha}_1, \bar{\alpha}_2, \pi].$$

2) For a convex sector $\bar{\alpha}_1$

$$\alpha = \min [\bar{\alpha}_1, \pi].$$

3) $\theta = \bar{\alpha}_1 + \bar{\alpha}_2$.

4) If $\bar{\alpha}_1 < \bar{\alpha}_2$ (in particular if $\theta > 0$ and $\bar{\alpha}_1 < \theta/2$), then the sector $C_1$ is convex. If $\bar{\alpha}_1 = \bar{\alpha}_2 = \theta/2$ the sector $C_1$ may be either convex or not. If $\bar{\alpha}_1 > \bar{\alpha}_2 < \pi$ the sector $C_1$ is nonconvex.

5) If the supplementary sector $\bar{\alpha}_2 > \pi$, then the sector $C_1$ is convex relative
to the boundary; if $\alpha_2 < \pi$ the sector $C_1$ cannot be convex relative to the boundary; if $\theta < \pi$ there are in general no sectors at the vertex in question which are convex relative to the boundary; if $\alpha_2 = \pi$ the sector $C_1$ may or may not have the property of convexity relative to the boundary.

PROOF. 1. If close to the vertex $O$ the sides of the sector prolong one another and form a single shortest arc, then $\alpha = \pi$, $\alpha_1 \geq \pi$, $\alpha_2 \geq \pi$ and equation (10) holds. Otherwise shortest arcs $XY$ that are very close to $O$ and join points on the sides of the sector avoid $O$ and lie either in the sector $C_1$ or in the sector $C_2$. Suppose for $X, Y$ arbitrarily close to $O$ that there is such a shortest arc in the sector $C_1$. Then by Theorem 3 of Chapter II we have $\alpha = \alpha$. But always $\alpha \leq \pi$, $\alpha \leq \alpha_2$, i.e., again (10) is true.

2. Equation (11) is proved in the same way. In this case, $XY$ passes in $C_i$.

3. The equation $\theta = \alpha_1 + \alpha_2$ follows from the additivity of the sector angle.

4. If $\alpha_1 < \alpha_2$, then from (10) we have $\alpha < \alpha_2$ and the shortest arcs $XY$ close to $O$ cannot lie in $C_2$ since the sector $C_1$ is convex. If $\alpha_2 < \alpha_1$ and $\alpha_2 < \pi$, then, from (10), we have $\alpha < \alpha_1$ and the shortest arcs $XY$, if sufficiently close to $O$, cannot lie in $C_i$. Thus the sector $C_1$ is nonconvex. Examples for both convex and nonconvex sectors with $\alpha_1 = \alpha_2$ are easily constructed on surfaces close to the cone.

5. The proof of the last (fifth) assertion of Theorem 6 is given below in subsection 13.


THEOREM 7. A neighborhood of any point $O$ may be divided into sectors by a finite number of shortest arcs issuing from that point in such a way that they will successively form with one another angles less than any given $\varepsilon > 0$.

PROOF. In the case $\theta = 0$ only one shortest arc is needed. Suppose that $\theta > 0$. We choose a very small neighborhood $U$ of the point $O$ homeomorphic to the disc. From Lemma 11 one may assume that $U$ is so small, and the sequence of polyhedral metrics $\rho_n \rightarrow \rho$ is specialized in such a way, that for large $n$ the fundamental part $\omega^+_n(U)$ of the positive curvature in the metrics $\rho_n$ is concentrated in an insignificantly small neighborhood of the point $O$.

In $U$ we carry out the following construction. We encircle the point $O$ with a simple closed curve $\Gamma$. We subdivide this by points $X_n$ into
pieces so small that, first, the maximum $d$ of their diameters in the metric $\rho$ is very small with respect to the distance $r$ from the point $O$ to the curve $\Gamma$ and, second, any shortest arcs in the metric $\rho$, successively joining the endpoints of these segments, form a curve enclosing $O$. In view of the uniform convergence $\rho_n \to \rho$, these conditions will also be satisfied in the metrics $\rho_n$ with sufficiently large index.

In the polyhedral metric $\rho_n$ we successively join these division points by shortest arcs, avoiding superfluous intersections in the process. We obtain a closed broken curve enclosing the point $O$. If successive links of this polygonal curve have a common initial segment, we discard it. If the polygonal curve has self-intersections even after this, we reject the loops thus formed. We arrive at a simple closed polygonal curve enclosing $O$. In the process the total number of vertices does not increase and all the vertices will lie at a distance not less than $r - d - \delta(n)$, where $\delta(n) > 0$ is arbitrarily small for large $n$.

Now we join the vertices of the resulting polygonal curve to the point $O$. If the next shortest arc $AO$ leaves the limits of the broken polygonal curve and again intersects it at the point $M$, we reject the piece $AM$ of our curve, replacing it by the piece $AM$ of the shortest arc $AO$. After finishing this joining operation we obtain a finite system of reduced triangles $OAB$, of which there are no more than the number of points $X_i$ in the original subdivision of $\Gamma$. The sectors at the vertices $O$ of these triangles, successively adjoining one another, fill out a neighborhood of the point $O$. The sides $OA$, $OB$ of these triangles cannot turn out to be much smaller than the distance from $O$ to the curve $\Gamma$ in the original metric $\rho$, and the sides $AB$ will be very small in comparison with $OA$ and $OB$.

This construction may be repeated in each of the metrics $\rho_n$, and then a subsequence chosen for which the resulting curves of the subdivision of the neighborhood of $O$ converge to some subdivision of a neighborhood of the point $O$ by shortest arcs in the metric $\rho$.

We assert that under appropriate choice of the neighborhood $U$ and sufficient fineness of the subdivision of the curve $\Gamma$, the resulting subdivision of the neighborhood of $O$ by shortest arcs will satisfy the requirements of Theorem 7. In fact, if the angle $\alpha$ between successive shortest arcs were large ($\geq \epsilon$), then the angle $\gamma$ would also be large in certain positions of the points $X_i, Y_i$ on that shortest arc. But then the angle $\gamma^*$ for the corresponding points $X_i, Y_i$ on the sides converging to this shortest arc of the
triangle $OAB$ in some polyhedral metric $\rho_n$ would also be large. But then \textit{a fortiori} the angle $\gamma_\ast^a$ obtained by measuring the distance inside the triangle $OAB$ (in the metric $\rho_n$) would also be large. Finally, the quantity $\gamma_\ast^a(X_n, Y_n)$, in view of Theorem 2, cannot be noticeably decreased by shifting the points $X_n, Y_n$ into the position $A, B$, since the positive curvature $\omega_\ast^a$ of the polyhedral metric $\rho_n$ in the triangle $OAB$ is very small, with the exclusion possibly of a bounded quantity concentrated in a neighborhood of the point $O$ insignificantly small in comparison to $OX_n, OY_n$. This last may be accomplished by a sufficiently large choice of $n$. The quantity $\gamma_\ast^a(A, B)$ is small, so that the side $AB$ is very small in comparison with $OA$ and $OB$. Thus the quantity $\alpha$ is small and Theorem 7 is proved.

Remark. The required subdivision is based on a system of shortest arcs which are limiting for systems of arcs subdividing the neighborhood of the same point $O$ into sectors of reduced triangles in certain metrics $\rho_n$.

A number of corollaries follow from Theorem 7.

Theorem 8. If $\theta > 0$, the neighborhood of the point may be divided into arbitrarily small convex sectors.

In addition the subdivision may be accomplished by shortest arcs which are limits of shortest arcs in approximating polyhedral metrics.

Indeed, from Theorem 7, there exists a subdivision with angles between successive shortest arcs smaller than $\theta/2$. From item 4) of Theorem 6, these sectors are convex.

In this case it is not only true that in a sufficiently small neighborhood of the point $O$, for any pair of points $X, Y$, on the opposite sides of each sector of the subdivision, there exists a shortest arc $XY$ passing in that sector, \textit{but also that there does not exist any one shortest arc $XY$ which encloses $O$ from outside}. Indeed, if there were such shortest arcs arbitrarily close to $O$, then we would arrive at a contradiction with the condition $\bar{\alpha} < \theta/2$.

Remark. In the case $\theta = 0$ it may turn out in general to be impossible to subdivide a neighborhood of a point into convex sectors. One may obtain

\textbf{Figure 59.}
an example of this by an appropriate choice of the dimensions of the twice-covered plane figure depicted in Figure 59. Here the protuberances converge to the point $O$.

**Lemma 15.** If $\theta > 0$, then the equation $\theta = \sum \alpha_i$ holds for every subdivision by shortest arcs, successively forming the angles $\alpha_i \leq \min(\pi, \theta/2)$.

Indeed, in this case all the sectors of the decomposition are convex. By Theorem 6, $\alpha_i = \bar{\alpha}_i$ for each of them. But by the additivity of the sector angles we have $\sum \bar{\alpha}_i = \theta$, so that $\theta = \sum \alpha_i$.

**Theorem 9.** The complete angle $\theta$ around a point is always finite.

This follows from Lemma 15 and the existence of the necessary subdivision for $\theta > 0$.

**Corollary.** A sector angle is always finite.

5. **Boundedness of the absolute curvature of the approximating metric.**

In this section we prove that in the case of interest to us, the approximating polyhedral metrics have a local uniform bound not only for the positive parts of the curvature, but also for the negative parts and thus for the absolute curvatures.

11. **Polygonal neighborhood of a point.**

**Lemma 16.** Every point has a neighborhood $P$ of radius less than any given $\varepsilon > 0$, where $P$ is a polygon homeomorphic to a disc and with a perimeter less than $\varepsilon$. Moreover, for sufficiently large $n$ the point $O$ has analogous neighborhoods $P_n$ in the metrics $\rho_n$. The number of vertices of each of the polygons is bounded independently of $n$, and the point $O$ lies in all the $P_n$ and in $P$ along with some fixed neighborhood $U$.

We shall carry out the proof separately for the cases $\theta > 0$ and $\theta = 0$.

Suppose that $\theta \neq 0$. We decompose a neighborhood of the point $O$ by shortest arcs in the metric $\rho$ into sectors with angles $\alpha_i = \bar{\alpha}_i < \min(\pi, \theta/2)$. This is possible from Theorem 7. Moreover, we may suppose that the indicated shortest arcs are limits of sides of triangles $OA_iB_i$ enclosing the point $O$ in the metrics $\rho_n$. The number of these triangles is bounded by some number $N$ not depending on $n$.

The sector of this subdivision will be convex. On each of the shortest arcs of the subdivision we mark a point $X_i$ at a distance $\delta < \varepsilon/2N$ from $O$. Since $\delta$ is small and the sector convex, we may successively join these points by shortest arcs $X_iX_{i+1}$ passing through the corresponding sectors. We obtain a polygon $P$. 

It may occur that adjacent sides of $P$, issuing from the vertex $X_i$, coincide on some initial segment. In this case we move $X_i$ to the branch point of these shortest arcs. We obtain a polygon $P$ homeomorphic to the disc.

Note that, generally speaking the points $X_i$, $X_{i+1}$ can be joined by various shortest arcs, including some which leave the sector $X_iOX_{i+1}$, but for sufficiently small $\delta$ these shortest arcs, as has already been observed in the proof of Theorem 8, cannot enclose the vertex $O$ of the sector $X_iOX_{i+1}$, since that would contradict the condition $\alpha_i < \theta/2$. They cannot pass through $O$ either, since $\alpha_i < \pi$. Moreover, they cannot touch any very small (in comparison with $\delta$), neighborhood $U$ of the point $O$, since $\alpha_i$ is strictly less than $\pi$.

Suppose that $X_i^*$ are the points on the sides of the triangles $OAB_i$ converging to the points $X_i$. From what has just been said, beginning with some $n$ the shortest arcs $X_i^*X_{i+1}^*$ in the metrics $\rho_n$ cannot enclose the vertex $O$ or touch the closed neighborhood $\bar{U}$. Otherwise there would be a limiting shortest arc $X_iX_{i+1}$ with the same property. This makes it possible, for sufficiently large $n$, to draw shortest arcs $X_i^*X_{i+1}^*$ in the triangles $OAB_i$. The sectors of $O$ of these triangles for large $n$ are so to speak "convex away from $O."$ Laying off in this way the shortest arcs $X_i^*X_{i+1}^*$, we obtain in $\rho_n$ a polygon $P_n$. It will have a small perimeter, and a bounded number of vertices, and it will contain the point $O$ and the entire neighborhood $\bar{U}$. If adjacent sides of $P_n$, issuing from $X_i^*$, have a common origin, then we move $X_i^*$ to the branch point of these sides. Thus we may make all the $P_n$ homeomorphic to the disc.

Now suppose that $\theta = 0$. Encircle the point $O$ and a neighborhood of diameter $d < \varepsilon/2$ by a closed curve $\Gamma$. Since there does not exist any shortest arc through the point $O$, no shortest arc joining the points of the curve $\Gamma$ can either pass through $O$ or touch a certain neighborhood $\bar{U}$ of the point $O$. Therefore also in the metrics $\rho_n$, with $n$ sufficiently large, no shortest arc joining two points of the curve $\Gamma$ can encounter $\bar{U}$. This makes it possible both in the metric $\rho$ and in the metrics $\rho_n$ to construct, for larger $n$, two-gons which will lie in an $\varepsilon$-neighborhood of $O$, have perimeters less than $\varepsilon$, and enclose the point $O$ along with some neighborhood $U$ of $O$. These two-gons will play the roles of $P$ and $P_n$.

Remark. The polygons $P$, $P_n$ may be regarded as convex. It is sufficient to consider shortest loops, enclosing the originally chosen polygons $P$ and $P_n$, as was done in subsection 4 of Chapter III.
Theorem 10. In some neighborhood $U$ of the point $O$ both the positive and negative curvatures of the converging polyhedral metrics $\rho_n$ are bounded uniformly.

Proof. Suppose that $U$ is a neighborhood of the type indicated in Lemma 16. For each $n$ the neighborhood $U$ is contained in the polygon $P_n$. The number of sides $m$ of the polygon $P_n$ is bounded by a number $N$ not depending on $n$. Moreover, $\omega^+_n(P_n) \leq C$.

The polygon $P_n$ itself is homeomorphic to the disc and may be divided into plane triangles (Figure 60). As in subsection 9 of Chapter III, we find from the Euler theorem that

\[
\sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) + \left[ \sum \xi - (m - 2)\pi \right] = 0.
\]

Here the first sum is extended over all vertices of the polyhedral metric which are interior for $P_n$ and is equal to $\sum (\bar{\omega}_n^--\omega_n^+)$ for all these vertices. The second sum is extended over all the vertices of the polyhedral metric lying within the sides of $P_n$. It represents that portion of $\bar{\omega}_n^-(P_n)$ which was not taken into account in the previous sum.\(^3\) Finally, the third term is the difference between the sum of the angles at the vertices of $P_n$ and the sum of the angles of a plane $n$-gon.

From (13) we immediately obtain the estimate

\[
\bar{\omega}_n^-(P_n) = \omega_n^+(P_n) + (m - 2)\pi - \sum \xi \leq C + (N - 2)\pi,
\]

which proves Theorem 10.

6. Further information on angles.

12. Uniform closeness of $\gamma$ to $\alpha$. The boundedness of $\alpha^+_n$ made it possible for us to choose a subsequence of metrics $\rho_n$ for which the $\omega_n^+$, as set functions, converged weakly to some limit function. The boundedness of $\bar{\omega}_n^-$, established in Theorem 10, makes it possible to specialize this subsequence in such a way that the functions $\bar{\omega}_n^-$ will also converge weakly to some limit function.

In Lemma 11 we have proved the possibility of choosing a neighborhood

\(^3\)As in a triangle, we include in the quantity $\bar{\omega}^-(P)$ the absolute value of the negative rotations from the side of $P$ at the vertices of the polyhedral metric which lie on the sides of $P$ and do not coincide with the vertices of $P$. 

FURTHER INFORMATION ON ANGLES

of the point $O$ so small that for the selected metrics $\rho_s$ their positive curvature will be insignificant except possibly for a bounded part of the curvature localized in an insignificantly small (in comparison with the neighborhood chosen before) region of the point $O$. Now we may assert the same result also for the negative part of the curvature.

In the course of the proof of Theorem 4, in using Lemma 11 and Theorem 2, we established that for any point $O$ and any $\varepsilon > 0$ there exists an arbitrarily small neighborhood $G$ of that point such that within the limits of this neighborhood, for two shortest arcs issuing from $O$ and two pairs of points $X_i, Y_i$ and $X_2, Y_2$ on them, with $OX_1 \leq OX_2$ and $OY_1 \leq OY_2$, the decrease in the angle $\gamma$ on passing to the more distant pair of points does not exceed $\varepsilon$:

$$\gamma(X_i, Y_i) - \gamma(X_2, Y_2) \leq \varepsilon.$$ 

Arguing on the analogy of Lemma 11 and Theorem 3, we can now prove, for some neighborhood $G$ the corresponding estimate of the possible increment $\delta$ in $\gamma$

$$\gamma(X_2, Y_2) - \gamma(X_i, Y_i) \leq \varepsilon.$$

Putting these results together, we arrive at the following theorem.

**Theorem 11.** For any point $O$ and any $\varepsilon > 0$, there exists a neighborhood $G(\varepsilon, O)$ such that, for any pair of shortest arcs issuing from $O$ and any points $X, Y$ lying in $G$, on these shortest arcs the inequality $|\gamma(X, Y) - \alpha| \leq \varepsilon$ holds, where $\alpha$ is the angle between the shortest arcs.

We note a special consequence of Theorem 11.

** Lemma 17.** If at the point $O$ the complete angle $\theta = 0$ and if $L$ is a shortest arc issuing from $O$, then for any $\varepsilon > 0$ there exists an arbitrarily small neighborhood $G$ of the point $O$ such that through each point $X$ on $L$ in the region $G$ we may pass a loop enclosing $O$ whose length is less than $\varepsilon r$, where $r$ is the distance from $O$ to $X$.

**Proof.** We choose a neighborhood $G$ so small that in it $|\gamma - \alpha| < \varepsilon/2$ throughout. This we do by Theorem 11. Suppose that $X \in L \cap G$. Since $X$ lies on a shortest arc issuing from $O$ and which extends beyond $X$, $X$ is on the exterior boundary of the set of points distant from $O$ by a distance not larger than $r$. After choice of a very small $\delta > 0$, we may pass a simple closed curve $\Gamma$ enclosing $O$, all of whose points are distant from $O$ by a distance differing from $r$ by not more than $\delta$.

Joining $X$ to a point $Y$ running along the curve $\Gamma$, we find a two-gon
$D(X, Y)$ enclosing $O$. We join its vertices to $O$ by shortest arcs $L_1, L_2$ lying in $D$. Since $\theta = 0$, the angle between these shortest arcs is equal to zero as well, so that $\gamma(X, Y) < \epsilon/2$. But the distance $OX = r$, and the distance $OY$ differs from $r$ by less than $\delta$, so that the length of the side $XY$ of the two-gon $D$ is very small in comparison with $r$. The contour of the two-gon $D$, thus satisfies the requirements of Lemma 17.

13. **Angle from the side of the sector.** To each sector, besides the angle $\alpha$ between its sides and the angle $\hat{\alpha}$ of the sector itself, one may also attach the concept of the angle between sides *measured in the sector itself*. We understand by this the limit $\hat{\alpha}$ of the angles $\hat{\vartheta}(X, Y)$, constructed with respect to the distances $OX, OY$ and the distance $XY$ measured in the sector itself, i.e. with respect to the shortest of the curves $XY$ which lie in the sector.

**Theorem 12.** The angle $\hat{\alpha}$ between sides $M_1$ and $M_2$ of a sector, measured in the sector itself, always exists and is equal to

$$\hat{\alpha} = \min \{\pi, \bar{\alpha}\},$$

where $\bar{\alpha}$ is the sector angle.

**Proof.** 1. Suppose that $\theta > 0$ at the point in question, and that for the sector of interest to us $\bar{\alpha} < \min(\pi, \theta/2)$. Then, from Theorem 6, the sector is convex. In this case $\hat{\vartheta} = \gamma$, so that the limit $\hat{\alpha}$ exists and $\hat{\alpha} = \min \{\pi, \bar{\alpha}\}$.

2. Now suppose that $\theta > 0$ and $\bar{\alpha} \geq \min \{\pi, \theta/2\}$. We decompose a neighborhood of the point $O$ into fine sectors by shortest arcs whose successive angles between shortest arcs are larger than zero but less than $(1/2) \min \{\pi, \theta/2\}$. We supplement this decomposition with the sides $M_1$, $M_2$ of the original sector. If the latter intersects shortest arcs of the subdivision arbitrarily close to $O$, we drop the intersecting shortest arcs. We obtain a decomposition of our sector $M_1OM_2$ by a series of shortest arcs $L_i$ successively forming, including the sides of the sector, angles less than $\min(\pi, \theta/2)$. All the sectors of this decomposition are convex and the angles of these sectors are equal to the angles between their sides.

We now distinguish the whole sector $M_1OM_2$ and consider the intrinsic metric induced by this selection. Evidently, from the general theorem on upper angles, we have

$$\lim \sup \hat{\vartheta} \leq \sum \bar{\alpha}_i = \sum \alpha_i = \sum \bar{\alpha}_i = \bar{\alpha}.$$  

If the shortest arc $XY$ in the sector $M_1OM_2$ also goes through $O$ for some $X$ and $Y$, then for $X, Y$ close to $O$, we always have $\hat{\vartheta} = \pi$, so that
lim $\hat{\gamma}$ exists and is equal to $\pi$. Moreover, from (15) it follows that also $\bar{\alpha} \geq \pi$, so that equation (14) is satisfied in this case.

If now all the shortest arcs $XY$ avoid $O$, then in analogy with equation (4') in Chapter II we obtain

$$\liminf \hat{\gamma} \leq \sum \alpha_i = \bar{\alpha}. \quad (16)$$

Moreover, in this case $\limsup \hat{\gamma} \leq \pi$.

Along with inequality (15), inequality (16) obtained above indicates the existence of the limit $\lim \hat{\gamma} = \bar{\alpha}$, and again equation (14) is valid.

3. Suppose finally that $\theta = 0$. In this case $\bar{\alpha} = 0$. It remains for us to show that $\lim \hat{\gamma} = 0$.

Suppose that $X$ and $Y$ are points on the sides $M_1, M_2$ of the sector in question, very close to $O$, and suppose for definiteness that $OX \leq OY$. Using Lemma 17, through the point $X$ we draw a curve enclosing $O$ whose length is small in comparison with $OX$. On the shortest arc $M_2$ we then find a point $Y'$, joined to $X$ in the sector in question by a curve $XY'$ which is very short in comparison with the distances $OX$ and $OY'$.

If we were to develop the triangle $Y'OY$ on the plane, its angle at the vertex $O$ would be equal to zero. This angle changes very little if we replace the side $OY'$ by the side $OX$ close to it in length, and the side $Y'Y$ by a shortest arc in the sector of the curve $XY$, which cannot be longer than $YY' + Y'X$. Both changes in length are small in comparison with $OX$ and $OY'$. Therefore the angle $\hat{\gamma}(X,Y)$ will also be very small and $\lim \hat{\gamma} = 0$.

Theorem 12 is completely proved.

REMARKS. 1) From Theorem 12 it follows that on distinguishing a sector from the enclosing space the value of the sector angle in the induced intrinsic metric remains equal to the preceding.

2) The last of the assertions of Theorem 6 follows from Theorem 12. Suppose that $\bar{\alpha} > \pi$. If the sector $C_1$ were not convex relative to the boundary, then in $C_0$, for $X$ and $Y$ arbitrarily close to $O$, there would be a path $XY$ shorter than $XO + OY$. But then we would have $\bar{\alpha} \leq \pi$, which from (14) contradicts the hypothesis $\bar{\alpha} > \pi$. Suppose that $\bar{\alpha} < \pi$. Then $\bar{\alpha} < \pi$ and there is a path $XY$ in $C_1$ shorter than $XO + OY$. Hence $C_1$ cannot be convex relative to the boundary. Examples of $C_1$ which are both convex and nonconvex relative to the boundary may be constructed for surfaces close to conical surfaces when $\bar{\alpha} = \pi$. 
14. **Angle in the strong sense.**

**Theorem 13.** If the shortest arcs $OA$, $OB$ issuing from the same point $O$ lie along with all the shortest arcs joining their endpoints in a neighborhood of the point $O$, then between the segments $OA$ and $OB$ of these shortest arcs there exists an angle in the strong sense.\(^4\)

For the proof we have to consider sequences $\gamma(X_n, Y_n)$ for $X_n \to O$, $X_n \in [OA]$; $Y_n \to Y_0$, $Y_n \in [OB]$, $Y_0 \in [OB]$ under the condition that there exist shortest arcs $X_n Y_n$ converging to the piece $OY_0$ of the side $[OB]$.

The ordinary angle, i.e., $\lim \gamma(X_n, Y_n)$ as $X_n, Y_n \to O$, exists. Suppose that its value is equal to $\alpha$. It remains to be verified that the same limit will be obtained for any of the sequences indicated above with $Y_0 \approx O$.

We consider two cases.

1. Suppose that the shortest arcs $OA$ and $OB$ have points in common arbitrarily close to $O$. Then for $X_n$ sufficiently close to $O$ and $Y_n$ close to $Y_0$, the exact equation $\gamma(X_n, Y_n) = 0$ is valid and the limit of $\gamma$ evidently exists and is equal to zero. In this case $\alpha$ is also equal to zero.

2. Now suppose that the shortest arcs $OA$ and $OB$ divide the neighborhood of $O$ into two sectors.

![Figure 61.](image1)

![Figure 62.](image2)

On both sides of $OB$ and close to $O$ choose points $B'$ and $B''$, very close to a point $C$ of the shortest arc $OB$ distinct from $O$ (Figure 61). The points $B'$ and $B''$ are then joined to $O$ by shortest arcs $OB'$, $OB''$, without creating essential intersections with $OA$ and $OB$. By Theorem 11, these shortest arcs form very small angles with $OB$.

\(^4\) See subsection 5 of Chapter II.
Now suppose that \( n \) is so large that the shortest arc \( \overline{X_nY_n} \), passing close to \( OB \), intersects \( OB' \) or \( OB'' \) at some point \( Z \). We suppose that \( Z \approx O \), since if \( Z \) coincides with \( O \) the discussion is only simplified: in this case \( \gamma(X_n, Y_n) = \alpha = \pi \).

We develop on the plane the triangles \( OZX_n \) and \( OZY_n \), and we adjoin them to one another along the side \( OZ \) as in Figure 62. The angle \( \alpha_0 \) in the plane triangle \( OX_nZ \) is very close to the angle \( AOB' \), from Theorem 11. \( AOB' \) is, in view of the smallness of the angle \( BOB' \), close to the angle \( \alpha = \angle AOB \). Straightening out the side \( X_nZY_n \) into the plane quadrilateral \( OY_nZX_n \), we establish that

\[
\gamma(X_n, Y_n) \geq \alpha_0 \approx \alpha.
\]

Since the whole construction can be carried out in such a way that this last approximate equation is arbitrarily exact for large \( n \), we have

\[
\lim_{n \to \infty} \inf \gamma(X_n, Y_n) \geq \alpha.
\]

But from the general theorem on the upper angle (Theorem 4 of Chapter II) it follows that

\[
\lim_{n \to \infty} \sup \gamma(X_n, Y_n) = \bar{\alpha} = \alpha.
\]

Along with the foregoing this indicates the existence of the limit of the angle \( \gamma \) for the sequence \( X_n, Y_n \) in question, and the validity of the equation

\[
\lim_{n \to \infty} \gamma(X_n, Y_n) = \alpha.
\]

Theorem 13 is proved.

The coincidence of the upper and strong lower angles with the angle in the usual sense makes it possible in appropriate situations to apply Theorems 5 and 6 to the ordinary angle.

We note one further consequence of the density of shortest arcs (Theorem 7) and the existence of the angle in the strong sense.

**Theorem 14.** Let \( z(x) \) be the distance from the fixed point \( O \) to the point \( X(x) \) on the simple rectifiable curve \( L \). The parameter \( x \) is the length along \( L \). Suppose, moreover, that at the point \( X(x_0) \) the curve \( L \) has a definite direction to the right. Then at \( x = x_0 \) the right derivative exists and

\[
\left( \frac{\partial z}{\partial x} \right)_x = -\cos \alpha,
\]

where \( \alpha \) is the angle formed by the right branch of \( L \) and by that shortest
arc $OX(x_0)$ which is the limit of shortest arcs $OX(x)$ as $x \to x_0 + 0$.

This is easily established if the branch of $L$ is enclosed in an arbitrarily narrow sector between shortest arcs and the results of subsections 4 and 6 of Chapter II are employed.

15. **Preparatory estimates of the difference $2\pi - \theta$.** We again return to the consideration of the difference $2\pi - \theta$. This time we shall establish the estimates needed in §7 of this difference in terms of the characteristics of the converging polyhedral metrics.

**Lemma 18.** For every point $O$ with complete angle $\theta$ we have the estimate

$$
\theta - 2\pi \leq \liminf_{n \to \infty} \omega_n(U),
$$

where $U$ is an arbitrarily small neighborhood of the point $O$ and the $\omega_n$ are the negative parts of the curvatures of the converging polyhedral metrics.

The finiteness of the lower limit on the right follows from Theorem 10.

In the case $\theta \leq 2\pi$ the assertion of the lemma is trivial. Suppose that $\theta > 2\pi$. In accordance with Theorem 8 we decompose a neighborhood of the point $O$ into angles $\alpha_i < \pi < \theta/2$ by shortest arcs which are limits for the system of shortest arcs serving in the metrics $\rho_n$ as sides of reduced triangles $OAB_i$ $(i = 1, \ldots, m)$ surrounding the point $O$.

From Lemma 15 we will have $\sum_{i=1}^m \alpha_i = \theta$. Each $\alpha_i$ may be replaced by some $\gamma_i$, up to $\varepsilon/m$. The $\gamma_i$ in turn, for sufficiently large $n$, may be replaced by $\gamma^*_i$ with accuracy $\varepsilon/m$. The quantity $\gamma^*_i$ does not decrease on being replaced by $\gamma^*_i$. (We do not think it necessary to define the notation $\gamma^*_i, \gamma^*_i, \gamma^*_i$, which has occurred several times already.)

The quantity $\gamma^*_i$, from Theorem 3, if it decreases at all, decreases by no more than $\omega_n(OA,B)$ on replacement by $\alpha^*_i$.

The quantity $\alpha^*_i$ can only increase on replacement by $\alpha^*_i$. Finally, $\sum \alpha^*_i$ cannot exceed $2\pi$ by more than the negative part of the curvature at the point $O$ in the polyhedral metric $\rho_n$. Thus we have successively found:

$$
\theta = \sum \alpha_i,
\alpha_i \leq \gamma_i + \varepsilon/m,
\gamma_i \leq \gamma^*_i + m/\varepsilon,
\gamma^*_i \leq \gamma^*_i,
\gamma^*_i - \alpha^*_i \leq \omega_n(OA,B),
$$
\[ \alpha_i^* \preceq \bar{\alpha}_i^*, \quad \sum \bar{\alpha}_i^* - 2\pi \preceq \omega_n^-(O). \]

Adding these inequalities, we obtain
\[ \theta - 2\pi \preceq \bar{\omega}_n^-\omega^-_n(OA,B_i) + \omega_n^-(O) + 2\varepsilon \preceq \omega_n^-(U) + 2\varepsilon. \]

By increasing \( n \) we can make the quantity \( \varepsilon > 0 \) arbitrarily small and we thus arrive at inequality (18).

**Lemma 19.** For each point \( O \) with complete angle \( \theta \)
\begin{equation}
2\pi - \theta \leq \liminf_{n \to \infty} \omega_n^+(U),
\end{equation}
where \( U \) is an arbitrarily small neighborhood of the point \( O \).

For the proof we consider separately the cases \( \theta > 0 \) and \( \theta = 0 \).

Suppose \( \theta > 0 \). In this case the proof is carried out in just the same way as that of the preceding lemma, with some alterations of the corresponding inequalities. The first of these in this case take the form
\[ \gamma_i \preceq \alpha_i + \frac{\varepsilon}{m}, \]
\[ \gamma_i^* \preceq \gamma_i + \frac{\varepsilon}{m}, \]
\[ \gamma_i^* = \gamma_i. \]

The last equation holds in view of the fact that the sectors at the vertex \( O \) in the triangles \( OA,B_i \) are so to speak "convex away from \( O \)" (see the proof of Lemma 16 in subsection 11).

Further, from the estimate (3) we have from Theorem 1 that
\[ \bar{\alpha}_i^* - \gamma_i^* \preceq \omega_n^+(OA,B_i) \]
and finally
\[ 2\pi - \sum \bar{\alpha}_i^* \preceq \omega_n^+(O). \]

Adding all these inequalities, carrying \( \sum \alpha_i = \theta \) to the left and considering that all the triangles \( OA,B_i \) can be regarded as situated in the neighborhood \( U \), and that \( \varepsilon \) may be chosen arbitrarily small, we obtain
\[ 2\pi - \theta \leq \liminf_{n \to \infty} \omega_n^+(U). \]

Now suppose \( \theta = 0 \). As in the proof of Lemma 17, we may surround the point \( O \) in a very small neighborhood by a two-gon \( D \), which is limiting for two-gons \( D_n \) surrounding the point \( O \) in the polyhedral metrics \( \rho_n \).

In \( D_n \) we join the vertices \( X_n,Y_n \) by shortest arcs to the point \( O \), and the
vertices $X$ and $Y$ of the two-gon $D$ by the limiting shortest arcs $OX, OY$. The angle between $OX$ and $OY$ is equal to zero, since $\theta = 0$. The angle $\gamma(X, Y)$ is then very small (from Theorem 11). Because of the uniform convergence of the metrics, the angles $\gamma_n(X, Y)$ are also small for large $n$. But from Theorem 2 the angles $\alpha^*_n, \alpha^*_n$ at the vertex $O$ cannot exceed $\gamma_n(X, Y)$ in any of the triangles $OX, Y$ by more than the positive curvature of these triangles. Therefore we have

$$\alpha^*_n + \alpha^*_n = \theta^* \leq \omega^*_n(U - O) + \epsilon.$$ 

Moreover,

$$2\pi - \theta^* \leq \omega^*_n(O).$$

Adding these inequalities and using the smallness of $\epsilon$ for large $n$, we obtain

$$2\pi \leq \liminf_{n \to \infty} \omega^*_n(U).$$

which coincides with (19) for $\theta = 0$.

Lemma 19 may be strengthened as follows.

**Lemma 20.** If the neighborhood of the point $O$ is subdivided into convex sectors by successive shortest arcs, forming angles $\alpha_n$, then the sum of these angles, which in general may be less than $\theta$, satisfies the inequality

$$2\pi - \sum_{i=1}^n \alpha_i \leq \liminf_{n \to \infty} \omega^*_n(U),$$

where $U$ is any small neighborhood of the point $O$.

**Proof.** If in at least one of the sectors the shortest arc $XY$, lying in the sector and joining the points $X, Y$ on its sides, passes through $O$, then the corresponding angle $\alpha_i = \pi$, and the sum of the remaining angles $\sum_{j \neq i} \alpha_{ij} \geq \pi$. In this case $\sum \alpha_i \geq 2\pi$ and estimate (20) is trivial.

But if the shortest arcs $XY$ lie in each sector and do not touch the point $O$, then for each of the sectors $\alpha_i = \bar{\alpha}_i$. In this case $\sum \alpha_i = \sum \bar{\alpha}_i = \theta$ and equation (20) follows from (19). Lemma 20 is proved.

7. **Excesses of nonoverlapping triangles.** The fundamental problem of this section is the proof of the following theorem.

**Theorem 15.** Suppose that in a two-dimensional manifold with intrinsic metric $\rho$, the latter can in the neighborhood of each point be considered as the limit of a uniformly converging sequence of polyhedral metrics, whose positive curvatures are uniformly bounded. Suppose moreover that all the metrics $\rho_n$ convert $R$ into one and the same topological space. Then
for each region $G$ with compact closure the sum of the excesses of any finite system of nonoverlapping reduced triangles lying in $G$ is bounded in absolute value by some number depending only on the choice of the region $G$. This is valid both for the excesses with respect to the sides of the triangles and for the excesses with respect to the angles of the interior sectors of these triangles.

From the assertion of this theorem it follows in particular that in the metric $\rho$ the condition of boundedness of the curvature (subsection 6 of Chapter I) is satisfied. Moreover, it is satisfied in the following strengthened form:

1) not only locally (in some neighborhood of each point) but also in any compact region;
2) not only for simple (homeomorphic to the disc and convex relative to the boundary) triangles, but also for arbitrary reduced triangles;
3) not only the positive excesses, but also the negative excesses of the triangles are bounded;
4) the positive excesses, computed not only with respect to the angles of the triangles, but also with respect to the angles of their interior sectors, are bounded.

We divide up the proof of Theorem 15 into a series of steps. First we consider only the positive excesses of triangles homeomorphic to the disc. Here we are thinking of the excesses computed with respect to the sector angles. Then we consider the analogous negative excesses. Finally, we turn to other forms of reduced triangles and to negative excesses with respect to the angles between the sides of the triangles.

16. **Positive excesses of triangles homeomorphic to the disc.**

**Lemma 21.** There exists for each point a neighborhood $G$, within which for any system of nonoverlapping triangles homeomorphic to the disc the sum of the positive excesses of these triangles (computed with respect to the angles of the interior sectors) is bounded by some number not depending on the choice of the system of triangles.

**Proof.** Suppose that $G$ is a neighborhood of the point homeomorphic to the disc and that within $G$ the metric $\rho$ is the limit of a uniformly converging sequence of polyhedral metrics $\rho_n$ with bounded positive curvatures: $\omega^*(G) \leq C$. Suppose moreover that $\{T_i\}$ is any finite system of nonoverlapping triangles homeomorphic to the disc and lying in $G$. 
1. Passage to triangles consisting of segments of limiting shortest arcs. The sides of the triangles \( T \) may not be the limits of shortest arcs joining the same points in the metrics \( \rho_n \). This makes it difficult to use the condition of the approximability of \( \rho \) by the metrics \( \rho_n \). Therefore we shall alter the triangles \( T \) somewhat.

\[ \text{Figure 63.} \]

On the sides of any triangle \( T \) there may be points \( X_i \) which are vertices of other triangles \( T \) (Figure 63). On the boundary of each triangle we choose points \( A_i \), including all the vertices of \( T \), the points \( X_i \), and a sufficiently dense additional system of points. We successively join the points \( A_i \) by shortest arcs in the metrics \( \rho_n \) and choose a subsequence of the metrics for which these shortest arcs converge to some limiting shortest arcs. We thus obtain the following situation: 1) the new shortest arcs, successively joining in the metric \( \rho \) the points \( A_i \) lying on the boundary of \( T \), form a triangle \( P \) replacing \( T \) and having the same vertices; 2) the triangles \( P \) also do not overlap; 3) their sector angles are very close to the sector angles of \( T \); 4) for the selected subsequence \( \rho_n \), the shortest arcs joining the same points \( A_i \) in the metrics \( \rho_n \) form polygons \( P_n \) corresponding to \( P \).

In order to guarantee that assertions 1)–4) are satisfied, it suffices to choose the points \( A_i \) and the subsequence \( \rho_n \) in the following way. First
we mark off the points $X_i$, including the proper vertices of all the triangles $T$. Close to each of the points $X_i$, on all the sides of the triangles issuing from $X_i$, we mark off close points $A'_i$. We join the point $A_i$ to the points $A'_i$ surrounding it in the metrics $\rho_n$ by shortest arcs which have no superfluous intersections, and we choose a subsequence $\rho_n$ for which all of these shortest arcs converge to limiting shortest arcs in $\rho$. This whole construction for each point $X_i$ may be carried out in an arbitrarily small neighborhood of $X_i$, so as not to affect the sides of the triangles $T$ other than those issuing from $X_i$. Moreover, we may because of the closeness of all the $A'_i$ to the $X_i$ guarantee that the sector angles between the newly drawn shortest arcs differ from the corresponding angles of the sectors by arbitrarily small amounts.

Indeed, if the complete angle around a point $X_i$ is equal to zero, then both the original and new sectors at the point $X_i$ have a zero angle and therefore are equal to each other. But if the angle $\theta$ at the point $X_i$ is distinct from zero, then for $A'_i$ close to $X_i$, the old and new shortest arcs $A'_iX_i$, from Theorem 11, form a very small angle, so that the sector they form is also very small. Hence the angle of the sector between the new shortest arcs differs only by little from the angle of the sector between the original shortest arcs.

After the choice of the points $A'_i$, on each of the pieces of the side $T$ we choose additional points $A''_i$ sufficiently often that on successively joining them by shortest arcs we do not encounter curves drawn earlier other than the contiguous segments $A'_iX_i$. The points $A''_i$ are joined successively in the metrics $\rho_n$ by shortest arcs which do not have superfluous intersections with one another and with the shortest arcs drawn earlier. We then select a further subsequence $\rho_n$, for which these shortest arcs converge to some limiting shortest arc in the metric $\rho$.

The system of points $X_i$, $A'_i$, $A''_i$ forms in its totality the required system of points $A_i$, and the subsequence of $\rho_n$, beginning with a sufficiently large index, forms the required sequence of $\rho_n$.

The segments of the limiting shortest arcs, together replacing one of the sides of an original triangle $T$, have the same length as the side of $T$. Therefore they also constitute a single shortest arc and therefore in particular form a simple curve. The triangle $T$ is thus replaced by a triangle $P$. This last may now turn out not to be homeomorphic to the disc. Close to the vertices its contiguous sides may coincide, as depicted in Figure 64, i.e., $P$ might be only a reduced triangle.
The polygons $P_n$ corresponding to the triangle $P$ in the metrics $\rho_n$ may of course not be triangles. But they are nonoverlapping polygons. Close to each of their vertices adjacent sides may generally speaking coincide on some sections, depicted in Figure 65.

Because of the arbitrary nearness of the sector angles in the triangles $T$ and $P$, it remains for us to prove the boundedness of the excesses for the system of triangles $P$.

2. Limiting triangulation. The region $G$ in question has a compact closure $\bar{G}$. Each point $X \in \bar{G}$ may by Lemma 16 be enclosed in all the metrics $\rho_n$ by arbitrarily small absolutely convex polygons $Q_n$, where the number of their sides for each fixed point $X$ is bounded by one and the same number independently of the index $n$, and each polygon $Q_n$ covers not only the point $X$, but also some neighborhood $U$ of $X$.

From the neighborhoods $U$ we may select a finite covering of $\bar{G}$. The corresponding system of polygons $Q_n$ will cover $\bar{G}$ and a fortiori $G$, for each $n$. We may take as included in this collection arbitrarily small polygons $\hat{Q}_n$, isolatedly covering the neighborhoods of vertices of the polygons $P_n$, as depicted in Figure 66.

In each of the metrics $\rho_n$ the system of polygons $Q_n$ may be broken into a finite number of nonoverlapping polygons $Q'_n$ which are convex relative to the boundary, in a fashion similar to that of subsection 7 of Chapter III. Here we may suppose that the polygons $Q'_n$ enter unchanged into the system of polygons $Q_n$. In the process of the indicated decomposition one may not create superfluous intersections with the
earlier drawn shortest arcs, including the sides of the polygons \( P \).

Now we include the contours of the polygons \( P \) in the system of curves of the subdivision. Close to a vertex of \( P \), the sides of this polygon separate off pieces \( \tilde{Q}_1' \), which generally speaking are not convex polygons, from the corresponding polygons \( \tilde{Q}_1 \). These pieces are crosshatched in Figure 66. The same polygons \( \tilde{Q}_1 \) as are dissected by the various sides of \( P \), turn out to be decomposed into convex (relative to the boundary) polygons \( Q_1'' \). Thus each polygon \( P \) turns out to be decomposed into nonintersecting polygons \( Q_1'' \), which, with the exception possibly of the polygons \( \tilde{Q}_1'' \), are convex relative to the boundary.

We decompose each polygon \( \tilde{Q}_1'' \) into narrow triangles by shortest arcs issuing from the vertex \( P \). We decompose the remaining polygons \( Q_1'' \) by diagonals into triangles convex relative to the boundary. We obtain a triangulation of the polygons \( P \). All the triangles of this triangulation with the exception of those adjacent to the vertices \( P \) are \textit{a fortiori} convex.

In this way we may triangulate the \( P \) in each of the metrics \( \rho_n \). Since the number of initial polygons \( P, Q \), and their sides is uniformly bounded for all \( n \), then, as one may verify by considering the above description of the triangulation process, the number of elements of the triangulation (vertices, triangles, segments of adjacent sides) will be uniformly bounded for all \( n \).

Because of the uniform boundedness of the number of elements of the triangulation there exists a sequence of metrics \( \rho_n \) for which the triangulations of all the polygons \( P \) have the same topological structure, i.e., have the same number of triangles in the triangulation and the same rule for them to be adjacent.

From the above sequence of metrics \( \rho_n \) one may select a still narrower subsequence for which the corresponding vertices and edges converge to certain points and shortest arcs joining them in the metric \( \rho \). The limiting net of edges forms some triangulation of the triangles \( P \) in the metric \( \rho \).

We note that the limiting triangulation does not necessarily have the same structure as the triangulations converging to it. Certain groups of vertices of the triangulations \( P \) might converge to one point. This will be one vertex of the limiting triangulation of \( P \). The limiting triangulation consists in this case of a smaller number of elements than the converging triangulations. However, the sides of each triangle of the limiting trian-
gulations will be limits of sides of some definite triangle of the converging triangulations. Therefore, in particular, all the sectors of the triangles of the limiting triangulation adjacent to vertices not lying on the boundary of the triangles $P$ are convex sectors.

Finally, we may suppose that the polygons $Q_a$ and their pieces $Q_a''$ (decomposed into triangles) have been chosen so small that in the limiting triangulation of the triangle $P$ the angles of the triangles of the triangulation at a vertex $P$ are equal to the angles of the sectors of these triangles and in their sum constitute the angles of the interior sector of the triangle $P$.

3. Relations following from Euler's theorem. Consider one of the triangles $P$ and its limiting triangulation. We denote by $\alpha$ the angles of the triangles $t$ of the triangulation adjacent to the interior vertices of the triangulation and by $\beta$ the angles adjacent to points on the sides of $P$, and by $\xi$ angles adjacent to the vertices of $P$. From Euler's theorem, as in subsection 9 of Chapter III, we obtain for the sum of the excesses $\delta(t)$ of the triangles of the triangulation the following equation:

$$\sum \delta(t) = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) + (\sum \xi - \pi).$$

Here the sum on the left is extended over all triangles $t$ constituting the subdivision of the triangles $P$. Of the sums on the right, the first is extended over all interior vertices of the triangulation, and the sum $\sum \alpha$ appearing in it relates to the angles between the sides of the triangles encircling that vertex. The second sum is extended over all the vertices of the triangulation lying on the sides of $P$, and the sum $\sum \beta$ over the angles, between the sides of the triangles $t$, adjacent to the corresponding vertex. Finally $\sum \xi$ is the sum of the angles between the sides of triangles $t$ adjacent to the three vertices of $P$. Rewriting (21) in the form

$$\sum \xi - \pi = \sum (2\pi - \sum \alpha) + \sum (\pi - \sum \beta) + \sum \delta(t),$$

we note that the quantity $\sum \xi - \pi$ is simply the excess $\delta(P)$ of the triangle $P$, computed at the angles of its sectors. Therefore the sum of the positive excesses of all the triangles of $P$ is equal to the sum of the positive terms of the right side of equations of the type (22):

$$\sum \delta(P) = \sum (2\pi - \sum \alpha) + \sum (\pi - \sum \beta) + \sum \delta(t).$$

Here the sums are extended over all the triangles $P$ for which the excesses $\delta(P)$ are positive.

Because of the convexity of the sectors adjacent to the interior vertices of the triangulation, we find from Lemma 20 that
EXCESSES OF NONOVERLAPPING TRIANGLES

(24) \[ \sum (2\pi - \sum \alpha) \leq C, \]

where \( C \) is a common upper bound for the positive parts of the curvature of the approximating metrics \( \rho \). The terms \( \pi - \sum \beta \) are nonpositive, since the sum of the angles on one side of a shortest arc cannot be less than \( \pi \):

(25) \[ \sum (\pi - \sum \beta) \leq 0. \]

Thus it remains for us to estimate the excesses \( \delta(t) \) of the triangles of the triangulation.

Each angle \( \alpha \) of any of the triangles \( t \) may with arbitrary accuracy be replaced by the angle \( \gamma \) for some pair of points \( X, Y \) on the sides of that angle. In their turn the angles \( \gamma \) may be approximated, because of the uniform convergence \( \rho_\ast \rightarrow \rho \), by the quantities \( \gamma_\ast \) for the corresponding points \( X_\ast, Y_\ast \) on the sides of that triangle \( t_\ast \) of the triangulation \( P_\ast \) which converges to \( t \). Therefore

\[ \sum \delta(t) = \sum (\delta' + \delta'' + \delta''' - \pi) \leq \sum (\gamma' + \gamma'' + \gamma''' - \pi) + \varepsilon \leq \sum (\gamma_\ast' + \gamma_\ast'' + \gamma_\ast''' - \pi) + 2\varepsilon. \]

Each of the angles \( \gamma_\ast \) does not decrease if it is measured inside the triangle \( t_\ast \), so that

\[ \sum \delta(t) \leq \sum (\gamma_\ast' + \gamma_\ast'' + \gamma_\ast''' - \pi) + 2\varepsilon. \]

Finally, each angle \( \gamma_\ast \) decreases by no more than \( \omega_\ast(t_\ast) \) on swinging the triangle \( t_\ast \) onto the plane. But after doing this the excess of this triangle is equal to zero, so that

(26) \[ \sum \delta(t) \leq 3 \sum \omega_\ast(t_\ast) + 2\varepsilon, \]

and in view of the arbitrariness of \( \varepsilon \) and the estimate \( \sum \omega_\ast(t_\ast) \leq C \) we obtain

(27) \[ \sum \delta(t) \leq 3C. \]

From equation (23) and inequalities (24), (25), and (27), it follows that

(28) \[ \sum \delta(t) \leq 4C, \]

which proves Lemma 21.

Lemma 21 has a number of important consequences.

**Corollary 1.** The manifold in question is a manifold of bounded curvature in the sense of the definition of subsection 6 of Chapter I.

**Corollary 2.** By the theorem of subsection 17 of Chapter III we may consider the limit, not only locally, but in any region \( G \) with a compact
closure, of uniformly converging polyhedral metrics \((\rho_n)_n\), defined in a polygon \(P\) containing \(G\) and having absolute curvatures bounded uniformly, while close to the interior points of the region \(G\) these metrics will converge to the metric \(\rho\).

**Corollary 3.** Repeating the proof of Lemma 21 for a compact region \(G\) and the polyhedral metrics \((\rho_n)_n\) indicated above, we may verify that not only locally, but also for any compact region \(G\) and any system of non-overlapping triangles \(T_i \subset G\) homeomorphic to discs we have

\[
\sum_i \delta(T_i) \leq 4C(P).
\]

17. **Negative excesses of triangles homeomorphic to the disc.**

**Lemma 22.** For any finite system of nonoverlapping triangles homeomorphic to the disc, the sum of the negative excesses, computed at the angles of the interior sectors of these triangles, is bounded by one and the same number, depending only on the choice of the region \(G\):

\[
\sum \delta^-(T_i) \leq D,
\]

where \(D\) depends on the common estimate of the negative parts of the curvature of the polyhedral metrics by which one may uniformly approximate the metric \(\rho\) in a polygon \(G'\) enclosing \(G\).

For the proof, as in Lemma 21, we turn from the triangles \(T\) to somewhat altered triangles \(P\) and polygons \(P_n\) covering to them. Then we construct triangulations of \(P_n\) of the same type and a limiting triangulation of the triangles \(P\). Then we write down relation (21) for each of them, this time in the form

\[
\pi - \sum \xi = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) - \sum \delta(t).
\]

Summing this relation over all triangles in \(P\) having negative excesses, we obtain

\[
\sum \delta^-(P) = \sum (\sum \alpha - 2\pi) + \sum (\sum \beta - \pi) - \sum \delta(t),
\]

where the sums on the right are extended over the triangulations of all triangles \(P\) with negative excesses.

But for the first of the sums we have from Lemma 18:

\[
\sum (\sum \alpha - 2\pi) \leq \sum (\theta - 2\pi) \leq \sum \liminf \omega^-(U),
\]

where the \(U\) are arbitrarily small neighborhoods of the interior vertices of the triangulation.

For the second sum we have:

\[
\sum (\sum \beta - \pi) \leq \sum (\sum \beta - \pi) \leq \liminf \omega^-(U),
\]
where the $U$ are arbitrarily small neighborhoods of those vertices of the triangulation which lie on the sides of $P$.

It remains to estimate the sum of the negative excesses over the angles between the sides of the triangles of the triangulation themselves.

As in the preceding lemma, the angles $\alpha', \alpha'', \alpha'''$ of each triangle $t$ may be approximately replaced by the angles $\gamma', \gamma'', \gamma'''$ for any pair of points sufficiently close to the vertices on the sides of these triangles. These angles may be approximately replaced by angles $\gamma_n', \gamma_n'', \gamma_n'''$ for the corresponding points on the sides of the corresponding triangles $t_n$ in the metrics $\rho_n$. The basic difficulty remains, namely that of passing from the angles $\gamma_n', \gamma_n'', \gamma_n'''$ to the angles $\gamma_n', \gamma_n'', \gamma_n''$ obtained by measuring the distance between the same points inside the triangle $t_n$ itself.

It would seem that the angles may increase on this substitution, which would not suit us, since we would then be faced with obtaining an upper estimate for the expressions $\pi - (\gamma_n' + \gamma_n'' + \gamma_n''')$ from the corresponding estimate for $\pi - (\gamma_n' + \gamma_n'' + \gamma_n'''')$.

So let us consider the details of the transition from $\gamma_n$ to $\gamma_n$ for various angles of the triangles $t_n$.

1. Close to interior vertices of the triangulations of the polygons $P_n$, the sectors of the triangles $t_n$ are convex. Therefore for such angles $\gamma_n = \gamma_n$.

2. If the vertex $O$ of the corresponding triangle $t$ lies on the side of $P$, then the shortest arcs, for fixed (as to distance from the vertex) points $X_n, Y_n$ on the sides of $t_n$ for large $n$ cannot envelop the vertex $O$, since otherwise we would obtain in the limit a shortest arc $X_nY_n$ enveloping $O$. But then there exists a shortest arc $X_nY_n$ passing through $O$, and already that angle alone at the vertex $O$ in the triangle $t$ is equal to $\pi$. But we have been considering only triangles $t$ with negative excesses. Thus we may also assume that close to such vertices $\gamma_n = \gamma_n$.

3. If we are dealing with an angle of a triangle adjacent to a vertex of $P$ in which the complete angle $\theta > 0$, then, since the angle of the triangle $t$ was chosen less than $\pi$ and less than $\theta/2$, the sectors of the triangles $t_n$, as already noted in the proof of Lemma 16, are so to speak convex at some distance from the vertex. Therefore in this case as well we may suppose that $\gamma_n = \gamma_n$.

4. Finally, if the vertex $O$ of the triangle $t$ coincides with a vertex of $P$ and at that vertex $\theta = 0$, then on the basis of Lemma 17 we may encircle the point $O$ in the metrics $\rho_n$ with loops of two shortest arcs, converging to an analogous loop in the metric $\rho$. All of these loops lie close to $O$. 
and at the same time are arbitrarily many times smaller in perimeter than in distance to the point \( O \). We choose points \( X, Y \), limiting for points \( X_n, Y_n \) lying on the intersection of these loops with the sides of the triangle \( t_n \). Then we verify that the angles \( \gamma_n \) are insignificantly small and therefore they cannot exceed the \( \gamma_n \) by any more than an arbitrary \( \varepsilon > 0 \) given in advance.

Therefore for such vertices we may suppose that

\[ \gamma_n - \gamma_n \leq \varepsilon. \]

The number of such vertices is finite. Incidentally, it is easy to verify that we may assume \( \gamma_n \geq \gamma_n \) in general for not more than one of the triangles \( t_n \) adjacent to a vertex of \( P \).

All of this shows that for the estimation of \( \sum \delta_k(t) \) it is sufficient to estimate \( \sum (\pi - \gamma'_k - \gamma''_k - \gamma'''_k) \). But for these quantities we have the right to use the results of Theorems 1 and 3, which, as in Lemma 21, leads us to the required estimate.

18. **Excesses of arbitrary reduced triangles.**

**Lemma 23.** The assertions of Lemmas 21 and 22 are valid not only for reduced triangles homeomorphic to the disc, but also for arbitrary non-overlapping triangles.

1. For a triangle degenerating into a segment the angle of a sector on the side of the triangle for a vertex lying inside the segment is not defined. In this case both adjacent sectors are "exterior" for the triangle. For such triangles we consider the excess to be by definition equal to zero.

2. Consider positive excesses for other types of triangles. Exterior tails of triangles may be simply dropped. This does not increase the angles of the sectors on the side of the triangles. For the remaining triangles, homeomorphic to the disc or having interior tails, the same considerations as in the proof of Lemma 21 apply.

3. Consider negative excesses with respect to the angles of sectors for triangles of various types. If the triangle has exterior tails, then the rejection of these tails may somewhat increase the sum of the angles of the interior sectors of that triangle, by an amount equal to the angle of the interior sector \( \alpha \) at the newly appearing vertex \( D \) (see Figure 67). This last quantity is the contribution of this sector to the excess \( \theta - 2\pi \) at the point \( D \). But such an excess may be estimated, using Lemma 18, in terms of the quantities \( \omega_k(U) \) for small neighborhoods \( U \) of the point
D. These curvatures play no essential role in the other estimates of Lemma 22.\textsuperscript{5}

If the triangle consists only of three tails (Figure 67), it has a negative excess equal to $-\pi$. But no matter how many such nonoverlapping triangles are adjacent to the point $O'$, each of them gives a contribution not less than $\pi$ in the difference $\theta - 2\pi$ at the point $O'$. Therefore, also for such triangles, we obtain an estimate of the quantity $\delta^-(T)$ in terms of the quantity $\omega_{\pi}(U(O'))$.

To the remaining triangles homeomorphic to the disc and triangles with interior tails one may apply the same considerations as in Lemma 22.

Lemma 23 is thus proved.

**Lemma 24.** In a compact region $G$, for any finite system of nonoverlapping reduced triangles $T, \subset G$, not only the sum of the negative excesses with respect to the sector angles of these triangles, but also the sum of the negative excesses over the angles between the sides of these triangles, are uniformly bounded in absolute value.

**Proof.** In the cases when the angle $\alpha$ between the sides is equal to the angle $\bar{\alpha}$ of the interior sector of the triangle, we replace $\alpha$ by $\bar{\alpha}$. When at least one of the angles of the triangle $\alpha \geq \pi$, the excess of the triangle is nonnegative so that in general this triangle need not be taken into consideration.

There remain angles for which $\alpha < \pi$, $\alpha < \bar{\alpha}$. But always

$$\alpha = \min(\pi, \bar{\alpha}, \bar{\beta}),$$

\textsuperscript{5} In Lemma 22 there figured quantities $\omega_{\pi}(U)$ for neighborhoods of interior vertices of the triangulation. But the point $O$ cannot be such a vertex.
where $\tilde{\beta}$ is the angle of the exterior sector, so that in the indicated cases $\alpha = \tilde{\beta}$. Then we have:

$$\tilde{\alpha} - \alpha = \tilde{\alpha} - \tilde{\beta} = 2(\tilde{\alpha} - \pi) + (2\pi - \tilde{\alpha} - \beta) = 2(\tilde{\alpha} - \pi) + (2\pi - \theta),$$

where $\theta$ is the complete angle around the vertex $O$ of the angle $\alpha$.

But by Lemmas 21 and 19

$$\sum (\alpha - \pi) \leq \sum \delta(T_i) \leq 4C(P),$$
$$\sum (2\pi - \theta) \leq \sum \lim_{n \to \infty} \omega^i_n(U(O)) \leq C(P),$$

so that

$$\sum (\tilde{\alpha} - \alpha) \leq 9C(P).$$

Thus we finally obtain

$$\sum \delta^i(T_i) \leq \sum \delta^\alpha(T_i) + 9C(P) \leq D + 9C(P),$$

and Lemma 24 is proved.

The fundamental Theorem 15 follows from Lemmas 21–24.