CHAPTER IX

Curves with Rotation of Bounded Variation

1. Variation of the rotation. As in the preceding chapters, we will always be considering a space which is a two-dimensional manifold of bounded curvature.

1. Arc with rotation of bounded variation. An open simple arc $\mathcal{L}$ is a curve obtained from the simple arc $\mathcal{L}$ by excluding its endpoints.

Definition 1. We shall say that the open simple arc $\mathcal{L}$ has a rotation of bounded variation if the following three conditions are satisfied:

1) on the arc $\mathcal{L}$ there are no points with zero complete angle $\theta = 0$;
2) at each interior point the arc $\mathcal{L}$ has a definite direction of arrival at that point and departure from that point;
3) for all finite collections of points $A_i (i = 1, \cdots, n)$ following one another along $\mathcal{L}$ the sum

$$\sum_{i=1}^{n-1} |\tau_r(A_iA_{i+1})| + \sum_{i=1}^{n-1} |\tau_l(A_i)| \leq N(\mathcal{L})$$

remains bounded.

Here the right rotation $\tau_r(A_iA_{i+1})$ of each open piece $A_iA_{i+1}$ of the curve $\mathcal{L}$ certainly exists in view of condition 1 and Theorem 2 of Chapter VI. By the right rotation $\tau_r(A_i)$ at the point $A_i$ is meant the quantity $\pi - \tilde{\alpha}'$, where $\tilde{\alpha}'$ is the sector angle at the vertex $A_i$ formed by the branches of $\mathcal{L}$ and lying to the right of $\mathcal{L}$; this sector angle exists because of condition 2.

Definition 2. The least upper bound of the sums on the left side of inequality (1) with respect to all possible systems of points $\{A_i\}$ is called the variation of the right rotation of the open arc $\mathcal{L}$, and is denoted by $\sigma_r(\mathcal{L})$.

Analogously one defines the left rotation $\tau_l(\mathcal{L})$ and its variation $\sigma_l(\mathcal{L})$.

Since $\mathcal{L}$ is a piece of the compact set $\mathcal{L}$, the absolute curvature $\Omega(\mathcal{L})$ is finite. Moreover, for the left and right rotations of the pieces and points of the curve $\mathcal{L}$ we have from Theorem 6 of Chapter VI the relations

$$\tau_r(A_iA_{i+1}) + \tau_l(A_iA_{i+1}) = \omega(A_iA_{i+1}), \quad \tau_r(A_i) + \tau_l(A_i) = \omega(A_i).$$

From (2) follows

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Lemma 1. Under the conditions of Definition 1, not only the sums in the inequality (1) but also the analogous sums for the left and right rotations are bounded, and the following relation holds:

\[ |\sigma_1(\mathcal{S}) - \sigma_2(\mathcal{S})| \leq \Omega(\mathcal{S}). \]

Remark. A curve homeomorphic to the open disc generally speaking may fail to be a simple open arc. It may turn out that one cannot assign endpoints to it. On Figure 101 we depict such a geodesic on a regular surface, infinite on one side. In Figure 102 we show an open square with an infinite series of paste-ins in the form of twice-covered triangles. The segment $\mathcal{S}$ running in this square has no endpoints within the limits of the open square. In this case $\Omega(\mathcal{S}) = \infty$, and if one introduces the quantities $\sigma_1(\mathcal{S})$, $\sigma_2(\mathcal{S})$, only one of them will turn out to be finite.

2. Rotation, its positive and negative parts and its variation. Select on $\mathcal{S}$ a system $\{t\}$ of sets $t$, to which we refer all the closed portions, i.e., all partial arcs on $\mathcal{S}$ and all one-point sets on $\mathcal{S}$. Two sets $t_1, t_2 \subseteq \{t\}$ are taken to be nonoverlapping if they are either two arcs without common interior points or two noncoincident points, or an arc and a point not lying inside it. By $T$ we shall mean a finite system of pairwise nonoverlapping $t$.

On the sets $t$ there is defined the function $\tau_*(t_*)$, where by $t_*$ we mean $t$ itself if $t$ is a point and the interior portion of $t$ if $t$ is an arc. Therefore on $t$ we have defined the functions:

\[ \phi^+(t) = \max (0, \tau_*(t_*)), \]
\[ \phi^-(t) = \max (0, -\tau_*(t_*)), \]
\[ |\phi^*(t)| = |\tau_*(t_*)|. \]
Definition 3. By $G$ we denote open sets on the simple arc $\mathcal{A}$. We then have on sets $M \subset \mathcal{A}$ the nonnegative set functions

\begin{align}
\tau^+_\tau(M) &= \inf \sup \sum_{G \supset M} \phi^+(t), \\
\tau^-\tau(M) &= \inf \sup \sum_{G \supset M} \phi^-(t), \\
\sigma_\tau(M) &= \inf \sup \sum_{G \supset M} |\phi|(t)
\end{align}

and the function

\begin{equation}
\tau_\tau(M) = \tau^+_\tau(M) - \tau^-\tau(M).
\end{equation}

By the usual methods of measure theory we may prove the following two lemmas.

Lemma 2. Let $M$ be an open segment $AB$ on $\mathcal{A}$. Then the values of $\tau_\tau(M)$ and $\sigma_\tau(M)$ defined by (8) and (7) coincide respectively with the original value of the rotation, $\tau_\tau(AB)$, and the value $\sigma_\tau(AB)$ in the sense of Definitions 1 and 2. If $M$ is a one-point set $A$, then $\tau_\tau(M)$ is equal to the rotation $\tau_\tau(A)$, and

\begin{align}
\tau^+_\tau(M) &= \max \{0, \tau_\tau(A)\}, \\
\tau^-\tau(M) &= \max \{0, -\tau_\tau(A)\}, \\
\sigma_\tau(M) &= |\tau_\tau(A)|.
\end{align}

This lemma justifies the continuation in Definition 3 of the rotations $\tau_\tau$ and $\sigma_\tau$ required earlier. We keep for $\tau_\tau, \tau^+_\tau, \tau^-\tau, \sigma_\tau$ the expressions: right rotation of the curve, its positive and negative parts and the variation of the right rotation, understanding them in what follows as set functions on the curve.

Lemma 3.

\begin{equation}
\sigma_\tau(M) = \tau^+_\tau(M) + \tau^-\tau(M).
\end{equation}

Theorem 1. The functions $\tau^+_\tau, \tau^-\tau, \sigma_\tau$ are completely additive regular Carathéodory measures on the Borel sets of the arc $\mathcal{A}$.\footnote{On $\mathcal{A}$ we may introduce a strictly monotone parameter and take these functions to be given on sets of values of the parameter, i.e. in a metric space, and therefore speak of the Carathéodory measure.}

Proof. From Theorem 4, Chapter VI, on the additivity of the rotation, it follows that on decomposing the arc into pieces $t_i$, counting among the $t_i$ the successive intervals and division points, one always has the inequalities

\begin{align}
\tau^+_\tau(M) &= \max \{0, \tau_\tau(A)\}, \\
\tau^-\tau(M) &= \max \{0, -\tau_\tau(A)\}, \\
\sigma_\tau(M) &= |\tau_\tau(A)|.
\end{align}
\[
\sum \phi^+(t_i) \geq \phi^+(t), \\
\sum \phi^-(t_i) \geq \phi^-(t), \\
\sum |\phi|(t_i) \geq |\phi|(t).
\]

Therefore each of the functions \(\phi^+, \phi^-, |\phi|\) satisfies the conditions of Theorem 3 of Chapter IV, from which the validity of the present theorem follows.

**Corollary.** The function \(\tau_c(M)\) is also completely additive on the Borel sets \(M\) of the arc \(\mathcal{B}\).

3. Direction at the ends of an arc.

**Theorem 2.** An arc \(\mathcal{B}\) with rotation of bounded variation has at each of its endpoints a definite direction.

**Proof.** By definition of a curve with rotation of bounded variation, the complete angle \(\theta > 0\) at the endpoint \(A\) of the arc \(\mathcal{B}\). We shall show that for any shortest arc \(M\) issuing from \(A\) the curve \(\mathcal{B}\) close to \(A\) may, if it intersects \(M\) repeatedly, envelop \(A\) only a bounded number of times.

To this end we consider a polygonal neighborhood \(P\) of the point \(A\) so small that \(\Omega(P - A) < \theta/2\) and the shortest arc \(M\) leaves the point \(P\). Suppose that \(\mathcal{B}\) envelops the point \(A\) within \(P\) \(m\) times. In Figure

**Figure 103.**

103 we depict two circuits: \(A_{k-1}A_k\) and \(A_kA_{k+1}\). Choose \(m\) exemplars of the polygon \(P\). Cutting each of them along \(M\) and pasting the successive edges of the cuts together, we obtain an \(m\)-sheeted surface with branch
point $A$. Now consider the curve $A_1A_2$ on this multi-sheeted surface, taking the first circuit $A_1A_2$ to be accomplished on the first sheet of the surface, the second on the second and so forth, finally the last, $A_mA_{m+1}$ on the $m$th. Then the segments $AA_1$, $AA_{m+1}$ of the shortest arc $M$ and the piece $A_1A_{m+1}$ of the arc $\mathcal{R}$ excise from the multi-sheeted surface a region homeomorphic to the disc, as depicted on Figure 104. With the notations for the rotations, curvatures and sector angles indicated in Figure 104 we have, because of the connection between the curvature and the rotation of the boundary:

$$(\pi - \alpha) + \tau + (\pi - \beta) + \tau_1 + (\pi - m\theta) + \tau_2 = 2\pi - \omega.$$ 

Therefore noting that

$$\alpha, \beta, \tau, \tau_1, \tau_2 \leq 0, \quad \omega \leq m\Omega,$$

we obtain

$$m \leq \frac{\pi + \tau + \Omega}{\theta - \Omega} \leq \frac{\pi + \sigma + \theta/2}{\theta/2},$$

where $\sigma$ is the variation of the rotation of $\mathcal{R}$.

Thus the above assertion is proved.

Now suppose that $\mathcal{R}$ does not have a definite direction at $A$. Then on moving the point $X$ along $\mathcal{R}$ towards $A$ the direction of the shortest arc $AX$ (Lemma 1, Chapter VI) does not stabilize. Thus there are two shortest arcs $M_1$ and $M_2$ issuing from $A$, which form with one another an angle $\phi > 0$, and the arc $\mathcal{R}$ nevertheless intersects each of the shortest arcs $M_1$, $M_2$ an infinite number of times in an arbitrarily small neighborhood of the point $A$.

From what was proved above, near $A$ there are positions when $\mathcal{R}$ successively intersects one of the curves $M_1$, $M_2$, then the other and again the first, without in doing so enveloping the point $A$ (Figure 105). In this case, with the notations of Figure 105, $\alpha \geq \min \{\bar{\alpha}, \bar{\alpha}', \pi\} = \phi > 0$. We may suppose that the construction was carried out in a neighborhood of $A$ in which the rotations of $M_1$, $M_2$, $\mathcal{R}$ and the curvatures of the region, with the point $A$ removed, are very small in comparison with $\phi$. (The rotation of $\mathcal{R}$ near the endpoint is small because of the complete additivity of the rotation.)

Then from the consideration of the "two-gon" $BE$ (Figure 105) we conclude that the sector $\beta$ is small, and on considering the "two-gon" $CD$ that the sector $\delta$ is small and (taking account of the smallness of the absolute curvature at the point $C$) that the sector $\tau$ is close to $\pi$. Finally,
considering the “triangle” $ABC$, which has a small curvature and sides whose rotations are small, leads us to a contradiction with the fact that $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma}$ is appreciably (nearly by $\phi$) larger than $\pi$.

This proves Theorem 2.

Now consider any simple arc $\mathcal{L}$ with a rotation of bounded variation. According to Theorem 2, $\mathcal{L}$ has directions at the endpoints $A, B$. If we may pass shortest arcs from these endpoints which do not intersect $\mathcal{L}$ except as at the original points, then the rotation $\tau(AB)$ is defined for the entire arc $\mathcal{L}$. Earlier we had the rotation $\tau(A_nB_n)$ on each shortened piece of the arc $\mathcal{L}$ (Figure 106). Under these conditions we have:

**Lemma 4.**

$$\tau(AB) = \lim_{A_n \to A, B_n \to B} \tau(A_nB_n).$$

**Proof.** Because of the definition of rotation it suffices to show that for $A_n, B_n$ close to $A, B$ and polygons $L_n, L$ close to $\mathcal{L}(A, B), \mathcal{L}(A_n, B_n)$ (Figure 106), the values

$$\tau_n + \tilde{\alpha} + \tilde{\beta}, \quad \tau_n + \tilde{\alpha}_n + \tilde{\beta}_n,$$

are close. But we have
\[ \tau_\alpha + (4 \pi - \alpha - \alpha' - \beta^l - \beta) + \tau_\alpha' + \tau(\mathcal{A}_n) + \tau(\mathcal{B}_n) = 2 \pi - \omega(U), \]
\[ \tau_\alpha + \tau_\alpha' = \omega (L_0 - A - B), \]
\[ |\alpha + \alpha' - \pi| = |\tau(\mathcal{A}^\alpha)|, \]
\[ |\beta + \beta' - \pi| = |\tau(\mathcal{B}^\beta)|. \]

From these equations, taking account of the smallness of the quantities \( \omega(U), \omega(L_0 - A - B), \tau(\mathcal{A}_n), \tau(\mathcal{A}_n), \tau(\mathcal{B}_n) \), we find that the quantities \((12)\) are close to each other and thus that Lemma 4 is valid.

2. Approximation by polygonal curves.

4. Possibility of one-sided approximation by polygonal curves.

**Theorem 3.** If \( \mathcal{X} \) is a simple arc with rotation of bounded variation in the sense of Definition 1, then there exists a sequence of simple polygonal curves converging from the right to \( \mathcal{X} \) and with rotations whose variations have an upper limit not exceeding \( \sigma(\mathcal{X}) \). Further we may guarantee that these polygons have common endpoints with \( \mathcal{X} \) and form at the endpoints small angles with the curve \( \mathcal{X} \).

**Proof.** 1. Choose an arbitrary \( \varepsilon > 0 \) and draw a shortest arc \( \mathcal{L} \) joining the endpoints \( M \) and \( N \) of the arc \( \mathcal{X} \), passing to the right of \( \mathcal{X} \) and forming with \( \mathcal{X} \) at its endpoints angles less than \( \varepsilon \), and bounding along with \( \mathcal{X} \) a region which is homeomorphic to the closed plane disc and has an absolute curvature, on deletion of the points of the arc \( \mathcal{X} \) and its endpoints, also less than \( \varepsilon \). (All of this may be accomplished, since \( \mathcal{X} \) has a definite direction at its endpoints and at these endpoints \( \theta \approx 0 \).)

2. Consider moreover a sequence of other polygonal curves \( \mathcal{L}_n \), joining the points \( M \) and \( N \), passing between \( \mathcal{X} \) and \( \mathcal{L} \) and such that each successive polygon \( \mathcal{L}_n \) passes to the left of the preceding \( \mathcal{L}_{n-1} \) and the broken curves \( \mathcal{L}_n \to \mathcal{X} \) as \( k \to \infty \). The pair \( \mathcal{L}_n \) and \( \mathcal{L} \), for each \( k \), bounds a closed polygon \( P_k \) homeomorphic to the disc.

Draw in \( P_k \) the shortest curve \( I_k \) joining the points \( M \) and \( N \) and in addition the “rightmost”, i.e., the “closest” to \( \mathcal{L} \) of such curves if there are several. As is known, a shortest arc in a polygon will be a geodesic polygon each turning point of which coincides with one of the reentrant vertices of the polygon. In addition, if \( I_k \) has a turning point at the vertex \( Y \) of the polygon \( \mathcal{L}_n \), then the right rotation of \( I_k \) and moreover of \( L_n \) at

\( ^{2} \) These polygonal curves run in regions of infinitely decreasing absolute curvature and therefore it does not matter whether we are speaking of the variation of their left or right rotations.
that vertex is *a fortiori* nonpositive, and if the turning point takes place at a vertex \( X \) of the polygonal curve \( L \), then the left rotations of \( l_k \) and \( L \) at the vertex \( X \) are certainly nonpositive.

From the construction of the broken curves \( l_k \) it follows at once that each successive one of them has no points to the right of its predecessor. Therefore it follows that if at least one of the curves \( l_k \) fails to pass through some vertex \( X \) of the polygonal curve \( L \), then every following \( l_{k+p} \) also fails to pass through \( X \). Thus, one finds only a definite number of vertices \( M = X_1, X_2, \ldots, X_n, X_{n+1} = N \) of the polygon \( L \) through each of which there pass infinitely many polygons \( l_k \). Dropping a finite number of broken curves \( l_k \), we may suppose that all the \( l_k \) are stretched on the indicated vertices of \( L \) and not on the other vertices of \( L \).

3. The polygons \( l_k \) lie in a compact region \( D \) bounded by the curves \( \varnothing \) and \( L \). Moreover their lengths are bounded uniformly, since they do not increase, and the length of \( l_k \) does not exceed the finite length of \( L \). Therefore it is possible to find a sequence of \( k \) for which the \( l_k \) converge to some limit curve \( l \). Further we suppose that we are dealing only with this subsequence. From the minimal properties of the \( l_k \) it follows that \( l \) is a simple arc joining the points \( M \) and \( N \) and passing through the region \( D \). As with all the \( l_k \), the curve \( l \) will pass through \( X_1, \ldots, X_{n+1} \).

4. If on some portion \( X_n, X_{n+1} \) of the limit curve \( l \) there are no points in common with \( \varnothing \), then all the \( l_k \), not passing to the left of \( l \), are not spanned on the broken curves \( L_k \) after a certain \( k \), and are segments of geodesics on the portion \( X_nX_{n+1} \).

Now we consider a portion \( X_nX_{n+1} \) on which \( l \) has common points with \( \varnothing \). Among these common points there is a first \( Z_1 \) and a last \( Z_2 \), in the order of succession of points along \( \varnothing \) and \( l \). We do not exclude the possibility \( Z_1 = Z_2 \). We note that even if on the piece in question certain \( l_k \) are not spanned on \( L_k \), we may alter \( L_k \), moving back on the side of \( l_k \), in such a way that on the piece \( X_nX_{n+1} \) each polygon \( l_k \) is spanned at certain points on \( L_k \) and the first and last of these points, \( Y_1 \) and \( Y_2 \), converge respectively to \( Z_1 \) and \( Z_2 \). Everything we have said is sketched in Figure 107.

The pieces \( X_nY_1 \) and \( Y_2X_{n+1} \) are relative shortest arcs and therefore geodesics. By the same principle their limits \( X_nZ_1 \) and \( Z_2X_{n+1} \) are also geodesics. Therefore in particular the latter curves have definite directions at their endpoints. Moreover, from the convergence \( X_nY_1 \rightarrow X_nZ_1 \) and \( Y_2X_{n+1} \rightarrow Z_2X_{n+1} \) it follows that as \( k \rightarrow \infty \), the angles \( \angle Z_1X_nY_1 \) and \( \angle Z_2X_{n+1}Y_2 \) tend to zero.
5. We note further that the piece $Z_i Z_2$ of the limit curve $l$ also has definite directions at its endpoints, for example at $Z_2$. This follows from the fact that as the point $T \in \overline{Z_i Z_2}$ approaches $Z_2$ the direction of the shortest arc $T_iZ_2$, drawn so that it has no superfluous intersections with the earlier drawn shortest arcs $Z_i T_i$, may only swing monotonically to the right around $Z_2$. Indeed, if there were some shortest arc $T_iZ_i$ which passed to the left of a preceding shortest arc $T_iZ_2$, then as $k \to \infty$ the polygons $l_n$ converging from the right to $Z_i Z_2$, would have to intersect the shortest arc $T_iZ_2$ as they approach $T_2$. But then the polygon $l_n$ may either be shortened or replaced by one further left, which contradicts the choice of $l_n$.

6. Suppose finally that $k$ is so large that all the angles $\angle Z_i X_i Y_i < \varepsilon / 2n$ and $\angle Z_i X_{i+1} Y_i < \varepsilon / 2n$.

Denoting by $d_i^k$ the region bounded by the curves $X_i Z_i Z_2 X_{i+1}$ and the piece $X_i X_{i+1}$ of the curve $l_n$, with the exception of the curve $X_i Z_i Z_2 X_{i+1}$ itself. From the connection between the rotation of the boundary of this region and the curvature of the region it follows that

$$\tau_r(X_i Z_i Z_2 X_{i+1}) = \tau_r(X_i X_{i+1}) + \angle Z_i X_i Y_i + \angle Z_i X_{i+1} Y_i - \omega(d_i^k),$$

so that

$$\tau_r(X_i Z_i Z_2 X_{i+1}) < \tau_r(X_i X_{i+1}) + \frac{\varepsilon}{n} + \Omega(d_i^k).$$

Consider separately the rotation $\tau_r(X_i Z_i Z_2 X_{i+1})$ on the portions $X_i Z_i$, $Z_i$, $Z_i Z_2$, $Z_2 X_{i+1}$. On the open portions $X_i Z_i$ and $Z_2 X_{i+1}$ it is nonpositive and does not exceed $\Omega(X_i Z_i)$ and $\Omega(Z_2 X_{i+1})$ respectively in absolute value. On the open portion $Z_i Z_2$ the right rotation is connected with the right rotation of the curve $\mathfrak{K}$ itself on the corresponding portion $Z_i Z_2$ by the relation

$$\tau_r(Z_i Z_2) = \tau_r(Z_i Z_2) - \omega(g_i),$$

where the notations $\alpha$ and $\beta$ are given in Figure 107 and $g_i$ is the region bounded by $Z_i Z_2$ and $Z_i Z_2$ with the exception of the points of the curve $Z_i Z_2$ itself. Finally, at the points $Z_i Z_2$ the right rotation of the curve

\[\text{Figure 107.}\]
$X_iZ_iZ_{i+1}$ in question is connected with the right rotation $\tau_r(Z_i)$, $\tau_r(Z_{i+1})$ at the same points on the curve $\mathcal{C}$ by the inequalities

$$\begin{align*}
\tau_r(Z_i) &\geq \tau_r(Z_{i+1}) + \alpha, \\
\tau_r(Z_i) &\geq \tau_r(Z_{i+1}) + \beta,
\end{align*}$$

since $l$ passes to the right of $\mathcal{C}$.

Inequalities (13)—(15) allow us to conclude that

$$\tau_r([\overrightarrow{Z_iZ_{i+1}}]) < \tau_r(X_iX_{i+1}) + \frac{\varepsilon}{n} + \Omega(d_i^* + g_i + X_iZ_i + Z_iX_{i+1}),$$

where $[\overrightarrow{Z_iZ_{i+1}}]$ is a closed portion of the curve $\mathcal{C}$.

7. We have arrived at the following situation. The right rotations of $\mathcal{C}$ and $l_i$, because of the choice of the original curve $L$, differ by less than $3\varepsilon$. The variation $\sigma_r(l_i)$ is made up of the possible positive rotations at the $X_i$, $i = 2, \cdots, n$, and the sum of the negative rotations, which behave as follows: a) at the points $X_i$ they cannot exceed $\Omega(X_i)$, since the left rotations of the $l_i$ at these points are nonpositive; b) on the pieces $X_iX_{i+1}$, where $l$ was not spanned on $\mathcal{C}$, they cannot exceed $\Omega(X_iX_{i+1})$. On the other pieces, from (16), to the negative rotations $\tau_r(X_iX_{i+1})$ there correspond almost the same or still smaller rotations $\tau_r([\overrightarrow{Z_iZ_{i+1}}])$ on the pieces of $\mathcal{C}$. From all that has been said, recalling that the terms $\varepsilon/n$ cannot be repeated more than $n$ times, and that the absolute curvature $\Omega$ of all the regions, points, and curves in question cannot in sum exceed $\varepsilon$, it already follows, that

$$\sigma_r(\mathcal{C}) > \sigma(l_i) - 5\varepsilon.$$

Carrying out this construction as $\varepsilon_n \to 0$, we obtain the required sequence of polygons $l_i(m)$.

REMARKS. 1) As is clear from the construction, the vertices of the $l_i$ lie at certain vertices of $L$ and $L_k$. Therefore if desired, it may be arranged that the vertices of $l_i$ lie only at points where $\omega = 0$.

2) As we shall see further on (subsection 11) the variations $\sigma_r(l_i(m))$ of the curves $l_i(m)$ just constructed do in fact converge to $\sigma_r(\mathcal{C}_k)$ as $m \to \infty$.

5. Arc and chord.

LEMMA 5. If the shortest arc $l$ and the polygon $L$ bound a region $g$ homeomorphic to the disc, while $\omega^* + \tau^* < \pi$, where $\tau^*$ is the positive part of the rotation of the polygon $L$ from the side of $g$ and $\omega^*$ the positive part of the curvature of the open region $g$, then for the lengths of $L$ and $l$ we have the inequality
\[
L \cos \frac{\omega^+ + \tau^+}{2} \leq l.
\]

**Proof.** Excising the polygonal region \( g \) from the manifold in question and pasting a cylindrical piece to it, we may without loss of generality regard the region \( \tilde{g} \) as convex. Decompose \( \tilde{g} \) into reduced triangles by shortest arcs drawn from one common endpoint of \( l \) and \( L \) to all the remaining vertices of \( L \). Then construct plane triangles with sides of the same length and adjoin them to one another in the same order as in the region \( \tilde{g} \). Since the decrease of the angles of the triangles, from Theorem 11 of Chapter IV, does not in sum exceed \( \omega^+ \), the rotation from the side of \( l' \) of the plane polygon \( L' \) thus formed does not exceed \( \tau^+ + \omega^+ \). To \( L' \) and \( l' \) in the plane we apply estimate (23) from Chapter III, obtaining inequality (17).

**Lemma 6.** Suppose that inside the region \( G \), homeomorphic to the open disc and having \( \omega^+ \) as the positive part of its curvature, there lies a simple arc \( \mathcal{Q} \) with bounded variation \( \sigma \), of its rotation on the side of \( G \), and a shortest arc \( l \) joining its endpoints. We suppose that \( \sigma + \omega^+ < \pi \). Then the arc \( \mathcal{Q} \) is rectifiable and for the lengths of \( \mathcal{Q} \) and \( l \) we have the inequality

\[
\mathcal{Q} \cos \frac{\sigma + \omega^+}{2} \leq l.
\]

**Proof.** Take an \( \varepsilon > 0 \) such that \( \sigma + \omega^+ + \varepsilon < \pi \). Approximate \( \mathcal{Q} \) in accordance with Theorem 3 by a simple polygon \( L \) lying in \( G \) and having the same endpoints, with a variation of its rotation from each of the sides not exceeding \( \sigma + \varepsilon \). If \( L \) and \( l \) have no common points other than the endpoints, we may use estimate (17).

If \( L \) and \( l \) have other points of intersection, we assert that these points follow in the same order on each of the curves \( L \) and \( l \). Indeed, if this were not so, then on some portion of \( L \) and \( l \) there would be formed a two-gon of the type \( AB \) depicted in Figure 108. In it the sector angles \( \alpha \geq 0, \beta \geq \pi \), so that on comparing the rotation of the contour with the curvature of the region contained in it we arrive at a contradiction with the condition \( \sigma + \varepsilon + \omega^+ < \pi \).

*Figure 108.*
So, $L$ and $l$ can intersect, only forming successively a finite or countable number of nonoverlapping two-gons as depicted in Figure 109. Applying estimate (17) to the sides of $L_i, l_i$ of each of these two-gons, and taking account of the arbitrary smallness of $\varepsilon$ and the fact that the length of the limiting curve does not exceed the lower limit of the converging curves, we obtain the assertion of Lemma 6.

Remark. In the proof of Lemma 6 the condition $\sigma_i + \omega^+(G) < \pi$ was used only in clarification of the relative positions of $L$ and $l_i$. In the final estimate the curvature $\omega^+(G)$ may be replaced by $\omega^+(G - M - N)$, where $M$ and $N$ are the common endpoints of $G$ and $l_i$.

It follows from Lemma 6 that if $G$ and $l$ lie in a region $G$ with a small curvature $\omega^+(G)$ and if $\sigma(G)$ is small, then the ratio of the lengths of $G$ and $l$ is close to unity.

Theorem 4. Suppose that from the point $X_0$ at which $\rho = 0$, there issues an arc $G = X_i$ ($0 \leq t \leq 1$) with rotation of bounded variation. Then as $t \to 0$ the ratio of the lengths of the arc $X_0X_i$ to the chord $X_0X$, tends to unity.

Figure 109.

Proof. Make a cut along any shortest arc issuing from $X_0$. Pasting together the necessary number of copies of the same space, as was done in the proof of Theorem 2, we transfer our discussion to a space in which $\omega^+(X_0) < \pi$. Then we choose a neighborhood $G$ of the point $X_0$ homeomorphic to the disc and an initial segment of $G$, both so small that in $G$ the estimate $\sigma(G) = \omega^+(G) < \pi$ is valid and the shortest arc $X_0X_i$ does not leave $G$. Then from Lemma 6, taking account of the remark to that lemma, we will have

$$X_0X_i \cos \frac{\sigma_i(X_0X_i) + \omega^+(G - X_0)}{2} < X_0X_i,$$

which in view of the smallness of $\sigma(X_0X_i)$ and $\omega^+(G - X_0)$ for small $G$ and $t$ proves Theorem 4.

6. Singularities connected with points where $\theta = 0$.

Remarks. 1) In defining the simple arc $G$ with rotation of bounded variation we excluded the possibility of the existence on $G$ or at the endpoints of $G$ points where $\theta = 0$. If we consider an arc satisfying all the conditions of Definition 1 but having one or both endpoints at points where $\theta = 0$, then for such a curve Theorem 2 is true in an evident way. The direction at the endpoint of that curve where $\theta = 0$ exists because of the
trivial principle that any curve at such a point has a direction. In just the same way Lemma 4 also remains valid. Therefore to such a curve one may assign a definite rotation, as the limit of the rotations of extending arcs, even in the case when it is not possible to draw from the endpoints of $\mathcal{G}$ shortest arcs which do not intersect $\mathcal{G}$. Analogously we may consider simple arcs with any finite number of points where $\theta = 0$, with $\tau_0, \tau^+, \tau^-$, $\sigma$, retaining their meaning on Borel sets on such curves.

However, the following remarks show that if there appear points for which $\theta = 0$ there may occur essential singularities.

2) If at the endpoint $X_0$ of the curve $\mathcal{G} = X_t$ ($0 \leq t \leq 1$) the complete angle $\theta = 0$, then Theorem 4 generally speaking ceases to be valid. This may be observed in the example of a geodesic curve on a spire-shaped surface as depicted in Figure 110.

3) Such a curve $L$ (see Figure 110) in the large may fail to be rectifiable (for

![Figure 110.](image1)

![Figure 111.](image2)

the construction of an example see p.78) i.e., in an arbitrarily small neighborhood of the point $X_0$ its length will be infinite. Analogously one may construct a nonrectifiable simple arc with rotation of bounded variation, if on it (inside it) there lies a point $X_0$ with $\theta = 0$ (Figure 111).

Already using Figure 101 it has been shown that, in a bounded region with curvature $\omega^+ \geq 2\pi$, there can lie arbitrarily long curves with rotations of uniformly bounded variations. Figures 110 and 111 show that this can happen even for sequences of polygonal curves. We are thinking of polygons approximating from one side the curves $\mathcal{G}$ depicted there.

4) In defining the concept of rotation, we have used polygons lying on one side of the arc in question. The question arises: if the simple arc $\mathcal{G}$ has a rotation of bounded variation in the sense of Definition 1, then do polygons inscribed in it, provided they are sufficiently fine in the sense of subdivision with respect to the parameter $t$ on $\mathcal{G}$, necessarily have
rotations of uniformly bounded variations?

One may see from Figure 110 that this certainly may fail to be the case, if there is on $\mathcal{L}$ a point with angle $\theta = 0$. One need only take a polygon in Figure 110 starting with $A_1A_2$ and then follow along $\mathcal{L}$ to $A_3$, then $A_3A_4$, then again along $\mathcal{L}$ and so forth. Is this kind of singularity obviously connected with the presence of $\mathcal{L}$ of points where $\theta = 0$?

5) In the case of intrinsic geometry of convex surfaces, points where $\theta = 0$ do not exist and all the singularities mentioned in this subsection are missing.

7. Rectifiability of curves with rotation of bounded variation not passing through singular points.

**Lemma 7.** Suppose that $O$ is a fixed nonsingular point and that $U(r)$ is a convex neighborhood of the point $O$ homeomorphic to the disc, all of whose points are distant from $O$ by no more than $r$. Then the least upper bound $L$ of the lengths of simple curves $\mathcal{L}$ situated in $U(r)$ and having rotations $\alpha, (\mathcal{L}) \leq M$ of uniformly bounded variation tends to zero along with $r \to 0$.

**Proof.** Case 1. Suppose that $\omega(O) = \pi - 3\delta < \pi$. We take $r$ so small that already all the points distant from $O$ by a distance not greater than $2r$ lie in a neighborhood $V$ homeomorphic to the disc for which

$$\omega^+(V - O) \leq \delta.$$ We consider an arc $\mathcal{L}$ lying in $U(r)$ with variation of rotation $\alpha, (\mathcal{L}) \leq M$. The arc $\mathcal{L}$ may be decomposed into $2M/\delta$ pieces such that the variation of the rotation on each of them does not exceed $\delta$. Applying Lemma 6 to each of the pieces, we verify that $\mathcal{L}$ is rectifiable and that its length satisfies the inequality

$$\mathcal{L} \leq \frac{4Mr}{\delta \cos (\pi - \delta/2)},$$

from which it follows that $\mathcal{L} \to 0$ as $r \to 0$.

Case 2. Suppose that $\omega(O) \geq \pi$. By hypothesis, the complete angle $\theta > 0$ at $O$. We denote by $n \geq 2$ the integer part of the ratio $2\pi/\theta$. Cut the region $U(r)$ along one of its radii and paste together $n$ of its exemplars, identifying the vertices $O$ and successively pasting the edge radii. In the newly obtained metric space $\omega(O) = 2\pi - n\theta = \pi - 3\delta < \pi$.

If the radius $r$ is so small that in the original metric

$$\omega^+(V - O) \leq \frac{\delta}{n},$$
then consider the curve $\mathcal{L}$ transferred to the newly pasted metric space. We again have inequality (19), from which it follows that Lemma 7 is valid.

**Theorem 5.** If the simple arc $\mathcal{L}$ is the limit of the simple arcs $\mathcal{L}_n$, whose rotation is of uniformly bounded variation, $\omega(\mathcal{L}_n) \leq M$, and if there are no singular points on $\mathcal{L}$, then all the $\mathcal{L}_n$ are rectifiable, their lengths uniformly bounded, the arc $\mathcal{L}$ is rectifiable and the lengths of the $\mathcal{L}_n$ converge to the length of $\mathcal{L}$.

**Proof.** Taken an arbitrarily small $\varepsilon > 0$, and then a small number $0 < \delta < \pi/3$. On the curve $\mathcal{L}$ there are only finitely many points with curvature exceeding $\delta$, say $m$, such points. From Lemma 7, there exists an $r > 0$ so small that the length of the curves $\mathcal{L}_n$ lying entirely in a neighborhood $U(r)$ of one of these points is less than $\varepsilon/m$.

Around each point $X \in \mathcal{L}$ we choose a neighborhood $U$ of radius less than $r$ and such that even a neighborhood of a radius twice as large is contained in a neighborhood $V$ homeomorphic to the disc and with curvature $\omega^+(V - X) < \delta$. Each region $U$ covers on $\mathcal{L}$ some open interval around a point $X$. From these intervals, according to the Borel lemma, one may select a finite covering of the arc $\mathcal{L}$ formed by regions $U_i$ $(i = 1, 2, \ldots, m_1 + m_2)$ with centers $X_i$ enumerated in the order of succession on $\mathcal{L}$. Among these points there are a fortiori $m_1$ of the earlier mentioned points.

Correspondingly $\mathcal{L}$ may be divided into $m_1 + m_2$ pieces so that each of them along with some $\rho$-neighborhood is covered by one of the regions $U_i$.

We choose a curve $\mathcal{L}_n$ with an index so large that each of its points is distant by less than $\rho > 0$ from the corresponding (with respect to the parameter on the curve) point of $\mathcal{L}$. The curve $\mathcal{L}_n$ may be divided into $m_0 \leq 2M/\delta$ pieces with the variation of the rotation less than $\delta$. Supplementing with appropriate points of division, we divide it into $m_0 + m_1 + m_2$ pieces lying in the corresponding regions $U_i$ and having rotations less than $\delta$.

Pieces falling in the neighborhood of the distinguished $m_1$ points, because of the choice of $r$, are small in total length, and the lengths of the remaining pieces, from Lemma 6, do not exceed

$$\frac{2r}{\cos(3\delta/2)} (m_0 + m_1 + m_2).$$

Thus all the $\mathcal{L}_n$ are rectifiable, their lengths bounded uniformly by
some number $N$, and the limit curve $\mathcal{L}$ also rectifiable.

In order to verify that the lengths of the $\mathcal{L}_n$ converge to that of $\mathcal{L}$, we repeat our construction with the following additional requirements:

1) we suppose that $\delta$ was chosen so small that

$$1 - \frac{1}{\cos(3/2)\delta} \leq \frac{\varepsilon}{N};$$

2) suppose that $m_1$ is the number of links of a polygon inscribed in $\mathcal{L}$ with a length different from that of $\mathcal{L}$ by less than $\varepsilon$. We suppose that the quantity $\rho$ chosen earlier satisfied one more requirement:

$$\rho \leq \frac{\varepsilon}{2(m_1 + m_2 + m_3 + m_4 + 1)}.$$

We assert that then if $\mathcal{L}_n$ and $\mathcal{L}$ are closer than $\rho$ we have the following inequality for the lengths:

$$|\mathcal{L} - \mathcal{L}_n| < 4\varepsilon.$$

Indeed, suppose that $\mathcal{L}_n$ is subdivided as indicated above into $m_1 + m_2 + m_3 + m_4$ pieces. Those which fell into the $m_1$ particular regions $U$, do not, because of the choice of $\rho$, exceed $\varepsilon$ in total length. The lengths of the limit curve $\mathcal{L}$ in these regions also do not exceed $\varepsilon$. Thus we cannot here accumulate a difference in length larger than $\varepsilon$. On the remaining pieces, by condition (1) and Lemma 6, the total length of $\mathcal{L}_n$ differs from the sum of the lengths of the shortest arcs spanning the endpoints of these pieces by less than $\varepsilon$. In view of condition (2) these shortest arcs altogether differ less than $\varepsilon$ from the length of the corresponding links of the polygon inscribed in $\mathcal{L}$. This last, by the choice of the pieces $m_1$, is in its totality close in length to the remaining piece of $\mathcal{L}$. Therefore we obtain inequality (20) and along with it the convergence of the lengths.

Theorem 5 is proved.

**Corollary 1.** Every simple arc $\mathcal{L}$ with rotation of bounded variation and not having on the interior or at the endpoints singular points with $\theta = 0$ is rectifiable. Simple polygons converging to it on one side and having rotations of uniformly bounded variation, as constructed in Theorem 3, converge to $\mathcal{L}$ along with the lengths.

**Corollary 2.** Every simple arc $\mathcal{L}$ with rotation of bounded variation admits one-sided approximation by simple arcs, whose lengths converge to the length of $\mathcal{L}$.
3. **Second theorem on pasting.**

8. *Pasting of a manifold from pieces bounded by curves with rotation of bounded variation.* Suppose that $\overline{C}_i$, ($i = 1, \cdots, m$) are compact regions with edge, distinguished from certain manifolds $R_i$ of bounded curvature and having a boundary in the form of a finite number of simple closed curves. Each curve is supposed to be subdivided into a finite number of curvilinear links. All the links are supposed to be curves with rotations of bounded variation in the original spaces $R_i$. The links and their endpoints, by hypothesis, do not contain singular points with $\theta = 0$, so that all the links are certainly rectifiable.

Suppose further that the distinguished regions $\overline{C}_i$ are pasted along each
tended by admitting also curves $\mathcal{L}$ with a finite number of simple pieces entirely passing along curves of the pasting.

This follows immediately from Corollary 2 of Theorem 5.

**Lemma 9.** For any pair of points $X,Y$, the value of $\rho(X,Y)$ is finite.

This follows from Lemma 1 and the connectedness of the net of curves of the pasting.

Moreover, the two following assertions are true.

**Lemma 10.** The function $\rho(X,Y)$ is a metric.

**Lemma 11.** The metric $\rho$ defines in the pasted manifold the same topology as was defined by the process of pasting itself.

Now we may formulate the fundamental assertion.

**Theorem 6 (Second theorem on pasting).** The manifold $R$ with metric $\rho(X,Y)$, pasted according to the rules described above, and with $\rho$ given by (21), is a two-dimensional manifold of bounded curvature.

**Corollary.** In a sufficiently small neighborhood of any point which is interior for one of the pasted regions $G_i$, the metric $\rho$ coincides with the original metric $\rho_i$ of the space $R_i$. Therefore within the region $G_i$ the curvatures of sets are preserved. In the pasted manifold, by the same principle, one-sided rotations of curves of the pasting are preserved, and also, at vertices the adjacent sector angles $\hat{\theta}_i$. The curvatures $\omega_1, \omega_2$ of the curves of the pasting and of the vertices of the pasting are defined by the indicated rotations and angles according to the usual rule

$$\omega_1 = \tau_r + \tau_i, \quad \omega_2 = 2\pi - \sum \hat{\theta}_i.$$

9. **Plan of proof of Theorem 6.** In order to establish Theorem 6, we may attempt to verify directly that the axioms of boundedness of curvature are satisfied in $R$ by a method similar to that used in the proof of the first theorem on pasting, Theorem 7 of Chapter VI. But it will be easier to show that the metric $\rho$ may be obtained as the limit of some uniformly converging sequence of metrics $\rho_n$ whose curvatures are uniformly bounded. From the results of Chapter IV it follows that Theorem 6 will then be valid. We shall hold our exposition of the proof to such a plan.

**Approximating metrics.** In order to construct the approximating metric $\rho_n$, we carry out in each of the $R_i$, where $G_i$ lies, the following construction.

1. Each vertex of $\tilde{G}_i$ is encircled in a convex polygon $Q$ of small ($\ll 1/n^3$) radius and small perimeter. We suppose further that $Q$ lies in a
neighborhood with small curvature (aside from the vertex itself).

2. Each link of the boundary $\overline{G}_i$ is approximated from within $G_i$ in such a way that the variation of its rotation be little ($< 1/n^3$) different from the variation of the rotation of the link itself, that the points of these curves are distant by less than $1/n^3$ from the points on the approximated piece of the side (link of the boundary) of $\overline{G}_i$ which correspond to them with respect to the parameter (relative length), that the length of the polygon differ by little ($< 1/n^3$) from the length of the piece of the boundary being approximated, that the endpoints of each polygon lie on the approximated links at the same distance $\varepsilon_n > 0$ from its endpoints, and finally that the resulting polygons do not touch one another and do not form superfluous intersections with the polygons $Q$. All of this is schematically depicted in Figure 112.³

3. Now the region $\overline{G}_i$ decomposes into regions of three types: a polygonal interior region $\overline{G}_i'$ heavily outlined in Figure 112, narrow regions $\overline{G}_i''$ adjacent to the sides of $\overline{G}_i$, and regions $\overline{G}_i'''$ adjacent to the vertices of $\overline{G}_i$. The last are cross-hatched in Figure 112.

4. We select only the regions $\overline{G}_i'$. If earlier along a link of length $l + 2\varepsilon_n$, the region $\overline{G}_i$ was pasted with another region, then for $G_i'$ the corresponding piece of the boundary does not have entirely the same length $l'$. We paste to $\overline{G}_i'$ along $l'$ a planar trapezoidal strip of width $1/n$ with bases

³ We suppose the $\varepsilon_n$ so small that the endpoints of the polygons lie within the polygons $Q$. 
l' and l. Only after this we paste the regions \( \bar{G}^j_i \), along the sides, in a manner analogous to the pasting of the regions \( \bar{G}_i \) in \( R \). Here we obtain along each “side” between the regions \( \bar{G}_i \) so to speak a “building” in the form of a pair of narrow, nearly right-angled trapezoids (Figure 113).

5. In the neighborhood of the vertices of \( \bar{G}_i \), there remain “holes”. Into each of them we paste a right circular cylinder with a base of the appropriate perimeter and a height equal to half that perimeter.

6. According to the first theorem on pasting the resulting space \( R_s \) will have an intrinsic metric \( \rho_s \) of bounded curvature. It is easy to verify that the absolute curvatures of these metrics are uniformly bounded.

Transfer of the metrics \( \rho_s \) to a single region of representation. We compare (topologically) the region \( R_s \) in which \( \rho_s \) is given with the basic manifold \( R \). To this end we repeat in \( R \) all the subdivisions of \( \bar{G}_i \) into parts \( \bar{G}'_i, \bar{G}''_i, \bar{G}'''_i \). We map each piece \( \bar{G}''_i \) onto the trapezoidal strip replacing it and map onto one of the bases of the trapezoid the curve of division of \( \bar{G}'''_i \) and \( \bar{G}'_i \), with exact correspondence in length, and we map the other base of the trapezoid onto the piece of the region \( \bar{G}_i \) adjacent to \( \bar{G}''_i \).

Finally, we map the pieces of the pasted cylinders cross-hatched in Figure 114 topologically onto the regions \( \bar{G}'''_i \), making the mapping consistent with the mapping of the pieces of the boundary already established.

Further we shall show what exactly is the correspondence of the trapezoidal regions and the regions \( \bar{G}''_i \). On the side of \( \bar{G}_i \) we lay off points dividing the side into pieces of length of order \( 1/n^2 \). From each such point \( A \) we draw in \( R_i \) a shortest arc \( AM \) up to the polygon \( l \) ap-
proximating the side, as in Figure 115. Some piece $BM$ of this curve divides $\tilde{G}^{i''}$. Inasmuch as the "width" of $\tilde{G}^{i''}$ is of order $1/n^2$, and the distance of the points $A$ of order $1/n^3$, these divisions occur independently of one another. The entire region $\tilde{G}^{i''}$ is subdivided into "cells". Correspondingly (in terms of the parameters along the bases) we divide up the plane strip and topologically match $\tilde{G}^{i''}$ and the strip along cells, adjusting only the relations on the junctions of the cells and the edges of the region and the strip.

After all these relations have been established, the metrics $\rho_n$ may be considered to be given on $R$.

**Estimate of $\rho$ from below.** Suppose that $X$ and $Y$ are two fixed points in $R$, not lying on the curves of the pasting. For any $\varepsilon > 0$, from the definition of $\rho(X, Y)$, there exists a curve $\mathcal{C} \in K$ joining $X$ and $Y$ with the length of $\mathcal{C}$ satisfying

$$\rho(X, G) > \mathcal{C} - \varepsilon.$$

**Figure 116.**

For sufficiently large $n$ the endpoints and basic portions (with respect to length) of the segments of $\mathcal{C}$ will lie inside the regions $\tilde{G}^{i''}$, as in Figure 116. These pieces, directly in the structure of $\tilde{G}^{i''}$, carry over into $R_n$, as in Figure 117. There they may be augmented to a continuous curve $\mathcal{C}'$ with a small additional expenditure of length, i.e., for $n \geq N(\varepsilon)$

$$\rho > \mathcal{C} - \varepsilon > \mathcal{C}' - 2\varepsilon \geq \rho_n - 2\varepsilon,$$

so that

$$\rho(X, Y) \geq \limsup_{n \to \infty} \rho_n(X, Y).$$
**Estimate of \( \rho \) from above.** Write \( A = \liminf_{s \to \infty} \rho_{s}(X, Y) \). It follows from (22) that \( A \) is finite. In what follows we shall preserve only those \( n \) for which \( \rho_{n}(X, Y) \to A \).

Suppose that \( n \) is so large that \( X \) and \( Y \) lie inside the regions \( G_{n}^{i} \) and \( \rho_{n}(X, Y) < A + 1 \). Join \( X \) to \( Y \) in \( R_{n} \) by a shortest arc \( L_{n} \). This shortest arc certainly does not touch the cylinders pasted up in the construction. If it touches a pair of strips pasted up in the construction, then it already certainly intersects them. The number of such passages through the strips is not greater than \( (A + 1)n/2 \), since on each intersection the length decreases by at least \( 2/n \).

We construct a curve \( \mathcal{G} \) joining \( X \) and \( Y \) in \( R \) as follows. First we transfer to \( R \) those pieces of \( \mathcal{G}_{n} \) which lay in the regions \( G_{n}^{i} \). These pieces still do not form a continuous curve, since they are broken off on passing through the regions \( G_{n}^{j} \). To each intersection \( MN \) of the shortest arc with \( \mathcal{G}_{n} \) a plane strip we assign a piece \( M'N' \) on the curve of the pasting of the pair of strips in \( R_{n} \) (Figure 118) and the piece \( M''N'' \) corresponding to it on the curve of the pasting in \( R \). We note that in length \( M''N'' = M'N' \leq MN \). Finally, we join \( MM'' \) and \( N''N \) by sufficiently short curves in \( R \).

![Figure 118](image)

The supplementary pieces \( MM'' \) and \( N''N \) will have lengths of order \( 1/n^2 \), while their total number does not exceed \( (A + 1)n \). Therefore for large \( n \) these pieces will be sufficiently small in total, and we may verify that

\[
\mathcal{G} \leq \rho_{n} + \varepsilon,
\]

from which follows the validity of the inequality \( \rho(X, Y) \leq A \).
Uniformity of the convergence $\rho_n \to \rho$. From (21) and (22) it follows that for each fixed pair of points $X, Y \in R$, not lying on the curves of the pasting, $\rho_n(X, Y) \to \rho(X, Y)$. It remains to prove that this convergence holds (uniformly!) for any pair of points $X, Y \in R$.

We formulate an auxiliary assertion.

**Lemma 12.** For any $\varepsilon \geq 0$ there exists an $N$ and a finite system of points $A_i$, not lying on the curves of the pasting, such that for $n \geq N$ all the $A_i$ lie inside the regions $G_i$ and have the property that any point $X \in R$ may be put into correspondence with one of the points $A_i$, this being done in such a way that for all $n \geq N$ we will have the inequalities

$$\rho(X, A_i) < \varepsilon, \quad \rho_n(X, A_i) < \varepsilon.$$

Uniform convergence of $\rho_n$ to $\rho$ on the entire manifold $R$ follows from Lemma 12 because of the following inequalities:

$$| \rho_n(X, Y) - \rho(X, Y) | \leq \rho_n(X, A) + \rho(X, A) + \rho_n(Y, B) + \rho(Y, B) + | \rho_n(A, B) - \rho(A, B) | < 5\varepsilon,$$

where $A$ and $B$ are points of the system $A_i$ corresponding to the points $X$ and $Y$. The first four terms of the right side are small by the construction of the net of points $A_i$, and the last term is small for large $n$ since $\rho_n \to \rho$ for each pair of fixed points $A_i = A, A_i = B$.

**Plan of the proof of the lemma.** We shall outline the construction of the net of points $A_i$ and the assignment of points $X \in R$ to points of this net.

1. For some $\delta > 0$ we choose in the regions $G_i$ in the original spaces $R_i$ some $\delta$-net. We preserve from it only those points which lie strictly inside $\mathcal{G}_i$. This will be the “first net”. On each side of each region $\mathcal{G}_i$ we choose points $C$ forming a $\delta$-net in the sense of pieces along the length of the side. We shift these inside $G_i$ by less than $\delta$. This will be the “second net”. Together, for sufficiently small $\delta$, they constitute the net of the $A_i$.

2. We turn to the “attachment” of the points $X \in R$ to the points $A_i$. First we consider all points $X \in G_i$ distant by more than $3\delta$ from the boundary of $\mathcal{G}_i$. Each such point may be joined in $R_i$ by a shortest arc to the closest point $A$ of the first net. We attach $X$ to this point. The entire shortest arc $XA$ lies inside $G_i$ at a distance of at least $2\delta$ from the boundary of $G_i$.

To each point $X$ of the $3\delta$-edge of the region $G$, we join some point
of the second net by a curve consisting of the following pieces: a) a shortest arc $XB$ from $X$ to the boundary of $\bar{G}_i$; b) a path $BC$ along the boundary to the closest point $C$ (a still unshifted point of the second net) in a direction fixed in advance; c) a path $CA$ along which the point $C$ was shifted inside $\bar{G}_i$. The length of the entire path satisfies $XB + BC + CA \leq 3\delta + \delta + \delta$. We attach each point $X$ to the point $A$ obtained along such a path.

If $5\delta < \varepsilon$, the first of the requirements (23) is satisfied.

3. Now we choose $n$ so large that the following conditions are satisfied: a) all the points $A$, chosen above lie inside the regions $G'_i$; b) each region $G'_i$ contains all the points of the region $G_i$ distant by more than $\delta$ from the boundary of $G_i$; c) the boundaries of the $G'_i$ will be close to the boundaries of the $G_i$ in the sense of $\delta$-closeness of the points corresponding with respect to the parameter, with respect to relative length; d) the mappings of the regions $G'_i$ onto plane strips and the regions $G''_i$ onto cylinders preserves the $\delta$-closeness of points of the regions to the boundary points with similar values of the parameter on the boundary.

4. It remains to observe that for sufficiently small $\delta$ and sufficiently large $n$ the second of the inequalities (23) is satisfied.

For points $X \in G_i$ distant by more than $3\delta$ from the boundary of $G_n$, this is certainly satisfied, since the entire path $XA$ lies in $G'_i$ and therefore preserves its length in the metric $\rho_n$.

Consider a point $X$ in the $3\delta$-edge of the region $G_i$. In the metric $\rho$ the point $X$ is joined to $A$ by the path $XBCA$, as in Figure 119. In the metric $\rho_n$ there remains unchanged some initial piece $XB'_i$ of the shortest arc $XB$. The point $B'_i$ on the boundary of $\bar{G}'_i$ corresponds, in the parameter along the boundary, to some point $B''_n$ on the boundary of $\bar{G}_n$. Analogously the curve $CA$ has a piece $C'_iA$ in $\bar{G}'_i$.

In the metric $\rho_n$ the point $X$ may be joined to $A$ by the path $XB'_iB'_iBCC'_iC'_iA$.

For sufficiently small $\delta$ and large $n$ this path will be smaller than $\varepsilon$.

The cases $X \in G''_i$ or $X \in G'''_i$ introduce only nonessential changes in the above construction.

In such a way the validity of Lemma 12 is established, and along with it the uniform convergence of metrics $\rho_n \to \rho$, and from this Theorem 6.
10. Lemmas on sequences of polygons.

Lemma 13. Suppose that the endpoints \( Y_n \) of the simple polygons \( L_n \) converge to a point \( O \) with a complete angle \( \theta \approx 0 \), and the origins \( X_n \) of these polygons remain all the time outside some fixed neighborhood of the point \( O \). Then for sufficiently large \( n \) one may in an arbitrarily small neighborhood of the point \( O \) replace the terminal portion of each polygon \( L_n \) by a shortest arc going into \( O \), while the polygons remain simple and the variation of their rotations does not increase by more than some \( \varepsilon > 0 \) given in advance.

Proof. Consider the neighborhood \( U \) of the point \( O \) where \( r < \varepsilon \) and \( U \) lies along with the shortest arcs joining its points in an absolutely convex neighborhood of the origin \( V \) with absolute curvature \( \Omega(V-O) = u \) so small that

\[
\begin{align*}
u < \frac{\varepsilon}{3}, & \quad u < \theta, \quad \left(\frac{3}{2\pi} + \theta\right)\frac{u}{\theta - u} < \frac{\pi}{4}
\end{align*}
\]

and all the \( X_n \) lie outside \( V \).

We choose a polygon \( L_n \) with index so high that

\[
\rho(O, Y_n) < re^{-\frac{2\pi}{\tan(\theta/2)}}.
\]

We take \( Y_n \) to be the closest point to \( O \) on \( L_n \) (otherwise the endpoints may be dropped).

Suppose that \( M \) is the first and \( N \) the last of the points of \( L_n \) distant from \( O \) by exactly \( r \) (\( M \) and \( N \) may be the same). If the variation of the rotation \( \sigma(NY_n) > (\pi/2) - \varepsilon \), then we replace \( MY_n \) by the shortest arc \( MO \). From the choice of \( MO \) it follows that \( MO \) forms with the preceding part of the polygon on both sides angles not less than \( \pi/2 \), from which it follows that the variation of the polygon increases if at all by less than \( \varepsilon \).

Now suppose that

\[
\sigma(NY_n) \leq \frac{\pi}{2} - \varepsilon.
\]

We join the point \( O \) with all the vertices \( N = A_0, A_1, \ldots, A_n = Y_n \) of the polygon \( NY_n \). This may be done in such a way that one obtains reduced triangles \( OA_1A_2, OA_2A_3, \ldots, OA_{n-1}A_n \).

Case 1. Suppose that the sectors of these triangles adjacent to \( O \) follow
in a strictly monotone order around the point \( O \). Then we develop these
triangles on the plane and distribute them in the same sequence around
the point \( O' \). Suppose that \( \phi_i \) are the angles of the sectors adjacent to \( O \)
\((i = 1, \cdots, m)\), \( \phi'_i \) the corresponding angles of the plane triangles, \( \alpha_j \) the
sector angles of the triangles at the vertices \( A_j \) \((j = 1, \cdots, 2m)\), \( \omega_i \) the
curvatures of the triangles including the interior rotations of their sides.

We shall prove that \( \sum \phi'_i < 2\pi \).

We have:

\[
(27) \quad \sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} \phi_i + \sum_{j=1}^{2m} \alpha_j - \pi m.
\]

Moreover

\[
(28) \quad \tau(A_2 + \cdots + A_{m-1}) = (m - 2)\pi - \sum_{j=2}^{2m-1} \alpha_j \leq \sigma(NY_\alpha) < \frac{\pi}{2}.
\]

From (27) and (28) it follows that

\[
\sum_{i=1}^{m} \phi_i \leq \sum_{i=1}^{m} \omega_i + \pi m - \sum_{j=2}^{2m-1} \alpha_j < ku + \frac{3}{2}\pi
\]

where \( k \) is the largest number of coverings of one point by the triangles
in question. But evidently \( \sum \phi_i \leq (k - 1)\theta \), so that

\[
(k - 1)\theta \leq \frac{3}{2}\pi + ku,
\]

from which

\[
ku \leq \left( \frac{3}{2}\pi + \theta \right) \frac{u}{\theta - u} < \frac{\pi}{4}.
\]

Finally,

\[
\sum \phi'_i \leq \sum \phi_i + ku \leq 2ku + \frac{3}{2}\pi < 2\pi.
\]

Now we show that at least one ray \( O'A'_i \) forms with the segment \( A'_i A'_{i+1} \)
an angle less than \( \varepsilon/2 \). In fact otherwise it would be possible to pass
through the point \( N \) a logarithmic spiral

\[
\rho = re^{-\frac{4}{\tan(\varepsilon/2)}},
\]

with center \( O' \), as in Figure 120. This spiral forms with the radii the
angle \( \varepsilon/2 \) and will pass closer to \( O' \) than our polygon. But this spiral,
even after a complete revolution, approaches \( O' \) only at the distance

\[
re^{-\frac{2\pi}{\tan(\varepsilon/2)}}
\]
which contradicts condition (25) on the smallness of $O'Y'$. Thus, for some point $\angle O'A_iA_{i+1} < \varepsilon/2$. Replacing $A_iY_n$ by the shortest arc $A_iO$, we obtain the required change in the polygon.

*Case 2.* It remains to consider the case when the order of succession of the sectors of the triangles adjacent to $O$ is violated. Suppose it is first violated by the triangle $OA_iA_{i+1}$.

![Figure 120.](image1)

![Figure 121.](image2)

We first suppose that the link $A_iA_{i+1}$ goes outside the sector of the preceding triangle (Figure 121). Then we consider the piece from $A_i$ to the following intersection with $OA_i$ or to the endpoint $Y_n$. We mark off on it the point $T$ closest to $O$ and consider the rotation of the contour $OA_iTO$ enclosing the region $g$.

From the choice of $T$ and the fundamental properties of angles and rotations, we have in the notation indicated in Figure 121:

$$\beta \geq \frac{\pi}{2}, \quad \phi \geq 0, \quad \gamma \leq \pi + \omega^+(OA_iA_{i+1}),$$

$$\alpha = 2\pi - \gamma - \omega(A_i), \quad \tau(OT) \leq 0, \quad \tau(OA_i) \leq 0,$$

$$\omega(g) + \pi - \phi + \pi - \alpha + \pi - \beta + \tau(OT) + \tau(OA_i) + \tau(A_iT) = 2\pi,$$

which gives

$$\tau(A_iT) \geq \frac{\pi}{2} - \omega^+(OA_iA_{i+1}) - \omega(A_i) - \omega(g) \geq \frac{\pi}{2} - \varepsilon,$$

in contradiction to inequality (26).

This means that $A_iA_{i+1}$ passes through the sector $A_i$ of the triangle
$OA_{i-1}A_i$, as in Figure 122. In this case we replace $A_iY_i$ by $OA_i$. We have
\[
\gamma \leq \pi + \omega^+(OA_{i-1}A_i),
\]
\[
\pi - \alpha = \omega(A_i) + \gamma - \pi \leq \omega(A_i) + \omega^+(OA_{i-1}A_i).
\]

Now we compare the rotations $\tau^\circ$ and $\tau^\prime$ of the original and changed polygons at the vertex $A_i$. For $\alpha \geq \pi$ we have
\[
|\tau^\circ(A_i)| = \alpha + \beta - \pi \geq \alpha - \pi = |\tau^\prime(A_i)|,
\]
and for $\alpha < \pi$
\[
|\tau^\circ(A_i)| = \pi - \alpha \leq \omega(A_i) + \omega^+(OA_{i-1}A_i) < \frac{\varepsilon}{3},
\]
i.e., the rotation of the new polygon at the vertex $A_i$ is less than the previous rotation, or is in general small. We note further, that for the shortest arc $OA_i$ the variation of the rotation is certainly less than $\varepsilon/3$.

Lemma 13 is proved.

**Lemma 14.** If the simple arc $\mathcal{S}$ is the limit of simple polygons $L_n$ which have rotations of uniformly bounded right rotation, i.e.
\[
(29) \quad \sigma(L_n) \leq S,
\]
then $\mathcal{S}$ and each of its points has a definite direction to right and left.

Figure 122.

It suffices to carry out the proof for the endpoint $A$ of the arc $\mathcal{S}$. If the complete angle $\theta$ at $A$ is equal to zero, then $\mathcal{S}$ has a direction as does any curve issuing from $A$. We shall suppose that $\theta(A)\equiv0$. Then the proof of Lemma 14 simply follows the proof of Theorem 2. If in formula (10) of the present chapter, with the curve $L_n$ in the place of $\mathcal{S}$, one replaces $\sigma$ by $S$, then formula (10) remains true. In the proof of Theorem 2 the situation depicted for a piece of $\mathcal{S}$ in Figure 105, led to a contradiction. This time the same situation for a piece of $L_n$ leads to the conclusion that on all $L_n$ for sufficiently large $n$ in the vicinity of the piece $EB$ there exists a charge $\sigma$, comparable with $\alpha$ (see Figure 105). Since this may be repeated on pieces ever closer to $A$, for sufficiently large $n$ there are so many pieces with $\sigma \approx \alpha$ that this leads to a contradiction with inequality (29).

This proves the validity of Lemma 14.

**Lemma 15.** Suppose that a sequence of simple polygons $L_n$ with endpoints
$A_n, B_n$ converges to a simple arc $\mathcal{L}$, the variation of the right rotations of these broken curves being bounded uniformly:

$$\sigma_r(L_n) \leq S.$$  

(30)

Suppose further that the complete angle $\theta \equiv 0$ at some point $C$ on $\mathcal{L}$. Then beginning with sufficiently large $n$ one may replace $L_n$ by a polygon $L'_n$, by rejecting some piece $M_n N_n$ of $L_n$ and replacing it by a pair of shortest arcs $M_n C$ and $C N_n$. In addition one may adjoin the following conditions:

1) the variation takes place in an arbitrarily small neighborhood of the point $C$ given in advance;

2) the shortest arcs $CM_n$ and $CN_n$ have no common points other than $C$ or an initial segment adjacent to $C$;

3) the pieces $A_n M_n C$ and $C N_n B_n$ of the altered polygon $L'_n$ are each separately simple arcs;

4) the right rotations of $L'_n$ at the points $M_n$ and $N_n$ are less in absolute value than some $\varepsilon > 0$ given in advance;

5) the shortest arcs $M_n C$ and $C N_n$ form at the point $C$ angles less than $\varepsilon$ with the branches of $\mathcal{L}$;

6) if $C$ is not a cusp on the limit curve $\mathcal{L}$, (i.e., the branches of $\mathcal{L}$ do not form a zero angle at it), then $L'_n$ remains a simple polygon in the large;

7) if $C$ is a cusp, then the shortest arc $M_n C$ essentially intersects the piece $N_n B_n$ not more than a finite number of times, and similarly for the shortest arc $C N_n$ and the piece $A_n M_n$;

8) after altering $L_n$ to $L'_n$ the variation of the right rotation of $L'_n$ on the open portions $A_n C$ and $C B_n$ and the absolute value of the right rotation at the point $C$ on the limit curve $\mathcal{L}$ do not in sum significantly exceed $\sigma_r(L_n)$, or more precisely

$$\alpha'_r(A_n C) + \sigma_r(C) + \alpha'_r(C B_n) \leq \sigma_r(L_n) + 2\Omega(C) + \varepsilon,$$

(31) 

where the primes denote the variations of the rotations on the altered polygons.

PROOF. 1. Suppose that $C_n$ are points on $L_n$ corresponding in parameter on the converging curves $L_n \to \mathcal{L}$ to the point $C$ on $\mathcal{L}$. In accordance with Lemma 13 we will replace certain endpieces $M'_n C_n$ of the pieces $A_n C_n$ by shortest arcs $M'_n C$. We carry out this construction in ever smaller neighborhoods of the point $C$, i.e., such that $M'_n \to C$ as $n \to \infty$. Moreover, we ensure that $A_n M'_n C$ remain simple arcs and that the variations of the right rotations increase by less than $\varepsilon/2$. If for an infinite number of altered polygons there is formed an absolute rotation at the point $M'_n$.
which exceeds $\varepsilon$, i.e., if condition 4 of Lemma 15 under proof is not satisfied, this means that on the substituted pieces $\sigma_1(M^*_1C_n) \geq \varepsilon/2$. Then we reject the pieces $M^*_1C_n$ and apply Lemma 13 to the shortened polygons $A_nM^*_1$, replacing the endpieces $M^*_2M^*_3$ by the shortest arcs $M^*_2C$. If again for an infinite number of altered polygons $\sigma_1(M^*_3) \geq \varepsilon$, then for them $\sigma_1(M^*_2M^*_3) \geq \varepsilon/2$. Repeating this contraction more than $2S\varepsilon^{-1}$ times, we arrive at a contradiction with the estimate (30). This means that condition 4 of Lemma 15 may be satisfied.

2. Condition 1 of Lemma 15 may be considered as satisfied, since the construction may be carried out in a neighborhood of the point $C$ agreed on in advance. Moreover, the shortest arcs $M_nC$ and $CN_n$ may each time be drawn so that conditions 2 and 7 of Lemma 15 are satisfied.

3. From Lemma 14, curve $\mathfrak{I}$ has at each point, including $C$, definite directions of its branches. We select on $\mathfrak{I}$ a point $X_0$ such that the piece $X_0C$ of the curve $\mathfrak{I}$ lies in an absolutely convex neighborhood $U$ with an absolute curvature $\Omega(U - C) < \delta$ and such that for all $X \in X_0C$ the shortest arcs $XC$ form with $\mathfrak{I}$ at the point $C$ angles less than $\delta$.

Suppose further that $X_1, X_2, \ldots$ is a sequence of points on $X_0C$ converging to $C$. Beginning with some $n \geq N$, one may select pieces $Y_0Y_i$ corresponding in parameter which are so close to $X_0X_i$ that for each point $Y \in Y_0Y_i$ in the plane triangle with the sides of the triangle $XYC$ the angle at the vertex $C$ will be less than $\delta$. For such $Y \in Y_0Y_i$ the angle between any shortest arc $YC$ and the direction of $\mathfrak{I}$ at the point $C$ will be less than $3\delta$. Rejecting in advance from the polygons $A_nC_n$ their ends beyond the points $Y_i$ for $n > N$ and requiring that the points $M_n$ should be chosen on pieces beyond the point $Y_0$, we thus satisfy condition 5 of Lemma 15 if $\delta < \varepsilon/3$.

The validity of condition 6 of Lemma 15 already follows from conditions 5 and 1.

4. Suppose that $C$ is not a cusp on $\mathfrak{I}$. Then on the piece $M_nN_n$ we

![Figure 123.](image-url)
have two simple polygons, before and after the alteration. On Figure 123
the new pieces are depicted by broken curves. In order to compare the
right rotations for the original and altered polygons $L_n, L'_n$, we choose on
them points $M, N$ between which there lies the piece $M_nN_n$ undergoing
alteration.

We may suppose that the points $M, N$ lie in a neighborhood $U$ and that
the alteration of $L_n$ is carried out in a neighborhood of the point $C$ small
in comparison with the distances $CM, CN$.

We join $M$ with $N$ by a new polygon $l_1$ going to the right of $L_n$ and
$L'_n$ (Figure 123). Applying the Gauss-Bonnet theorem to the region bounded
by $l_1 + L_n$ and to the region bounded by $l_1 + L'_n$, we verify that the right
rotations on the closed segment $[M_nN_n]$ for the polygons $L_n$ and $L'_n$ differ
by no more than $\Omega(U) < \Omega(C) + \delta$.

But the right rotation of $L'_n$ at each of the points $M_n, N_n$ and on each
of the open segments $M_nC, CN_n$ is less than $\delta$ in variation. Therefore
\[
\sigma'(\{M_nN_n\}) \leq |\tau'(C)| + 4\delta < |\tau'(\{M_nN_n\})| + 8\delta
\leq |\tau,(\{M_nN_n\})| + \Omega(U) + 8\delta \leq \sigma,(\{M_nN_n\}) + \Omega(C) + 10\delta,
\]
where the primes denote the rotations on $L'_n$.

Since at the point $C$ the absolute values of the right rotations of $L'_n$
and $\omega$ differ, by the construction, by no less than $2\delta$, we have
\[
\sigma'(A_nC) + \sigma,(C)\omega + \sigma'(CB_n) \leq \sigma,(L_n) + \Omega(C) + 12\delta,
\]
which for $\delta < \epsilon/12$ guarantees that inequalities (31) hold.

5. It remains for us to establish inequalities (31) for the more complicated
case when $C$ is a cusp on $\omega$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure124.png}
\caption{Figure 124.}
\end{figure}
CONVERGING CURVES

In this case the new pieces $MM_aC$ and $CN_aN$ may intersect one another and the right rotation at $C$ for the curve $L'_1$ may be close to $\pi$ or $\theta - \pi$ independently of which of these two numbers is equal to the right rotation of $\mathcal{Q}$ at the point $C$. This possibility is illustrated geometrically in Figures 124 a and b.

We choose as earlier points $M,N$ and a polygon $l_i$. Moreover we draw two polygons, $l_i$ from $M$ to $C$ and $l_i$ from $C$ to $N$, going to the left of $L'_1$ and forming with $MM_aC$ and $CN_aN$ at the endpoints $C$ sectors smaller than $3\delta$ and at the endpoints $M$ and $N$ sectors smaller than $\delta$. This construction may be realized, since $CM_a$ and $CN_a$ issue from $C$ in a sector of width $2\delta$ around the common direction of the branches of $\mathcal{Q}$, and the sector $2\delta$ may be considered as small in comparison with the complete angle $\theta \equiv 0$ at the point $C$.

Now we will have

$$
\sigma_r([M_aN_a]) \geq |\tau_r([M_aN_a])| = |\tau_r(C)_{\mathcal{Q}} + (\tau_r(MN) - \tau_r(l_i + C + l_i)) + (\tau_r(l_i) - \tau'_r(MC)) + (\tau_r(l_i) - \tau'_r(CN)) + \tau'_r(M_aC) + \tau'_r(CN_a) + \tau'_r(N_a)|
$$

(32)

$$
\geq |\tau_r(C)_{\mathcal{Q}}| - \Omega(U) - 5\delta - 5\delta - \delta - \delta - \delta
\geq |\tau_r(C)_{\mathcal{Q}}| - \Omega(C) - 15\delta.
$$

We note further that

$$
|\tau_r(C)_{\mathcal{Q}}| \geq \min \{\pi - 6\delta; \theta - \pi\},
$$

$$
\sigma_r(C)_{\mathcal{Q}} = |\tau_r(C)_{\mathcal{Q}}| \leq \max \{\pi; \theta - \pi\}.
$$

Therefore

(33)

$$
|\tau_r(C)_{\mathcal{Q}}| \geq \sigma_r(C)_{\mathcal{Q}} - \Omega(C) - 6\delta,
$$

where $\Omega(C) = |\theta - 2\pi|$. From inequalities (32) and (33) it follows that

$$
\sigma'_r(A_aC) + \sigma_r(C)_{\mathcal{Q}} + \sigma'_r(CB_a) \leq \sigma_r(L_a) + 2\Omega(C) + 21\delta.
$$

If $21\delta < \varepsilon$ this guarantees that inequalities (31) hold.

Lemma 15 is completely proved.

11. Variation of the rotation of a simple arc and of arcs converging to it from one side.

**Theorem 7.** If the simple arcs $\mathcal{Q}_n$ converge on the right to the simple arc $\mathcal{Q} = AB$, there being on $\mathcal{Q}$ no singular points at which $\theta = 0$, and the variations of the right rotations $\sigma_r(\mathcal{Q}_n)$ are bounded uniformly, then $\mathcal{Q}$ is an arc with rotation of bounded variation and

(34)

$$
\sigma_r(\mathcal{Q}) \leq \liminf_{n \to \infty} \sigma_r(\mathcal{Q}_n).
$$
PROOF. 1. From Theorem 3, it is sufficient to establish Theorem 7 for the case when all the $\mathcal{L}_n$ are polygons. By Lemma 14, the arc $\mathcal{L}$ will have at each point definite directions to the right and left, and therefore also definite rotations on each open arc and at each point.  

2. Divide $\mathcal{L}$ by points $C_i$ ($i = 0, 1, \cdots, m + 1$), $C_0 = A$, $C_{m+1} = B$ into a finite number of pieces. From Lemma 15, for any $i$ and beginning with sufficiently large $n$, one may in arbitrarily small neighborhoods of the points vary the polygons $\mathcal{L}_n$ so that the new polygons $\mathcal{L}_n'$ will pass through the points $A_i, C_i, \cdots, C_m B$ and have at these points directions close to $l$. On the pieces $AC_i, C_iC_{i+1}, \cdots, C_mB$ these polygons remain simple, and the sum of the absolute values of their right rotations and of the right rotations of $\mathcal{L}$ at the points $C_i$ is bounded by the quantity

\[
\sum_{i=1}^{m} |\tau_i(C_iC_{i+1})|_\mathcal{L} + \sum_{i=1}^{m} |\tau_i(C_i)|_\mathcal{L} \leq \sigma_i(\mathcal{L}_n) + \sum_{i=1}^{m} \Omega(C_i) + \varepsilon.
\]

Recalling that on the pieces $C_iC_{i+1}$ the curves $\mathcal{L}_n'$ mostly (excluding a small neighborhood of the endpoints) go to the right of $\mathcal{L}$ and have directions at the ends which are close to $\mathcal{L}$, we may, by drawing a third curve and making use of the Gauss-Bonnet theorem and of the smallness of the curvature of the region between these curves and the third curve, conclude that for sufficiently large $n$ we will have the inequality

\[
\sum_{i=1}^{m} |\tau_i(C_iC_{i+1})|_\mathcal{L} + \sum_{i=1}^{m} |\tau_i(C_i)|_\mathcal{L} \leq \sigma_i(\mathcal{L}_n) + \sum_{i=1}^{m} \Omega(C_i) + 2\varepsilon.
\]

We note that it was on the transition from (35) to (36) that we made use of the fact that the $\mathcal{L}_n$ lay to the right of $\mathcal{L}$.

Passing in (36) on the right to the lower limit with respect to $n$ and taking account of the arbitrary smallness of $\varepsilon$, we obtain

\[
\sum_{i=1}^{m} |\tau_i(C_iC_{i+1})|_\mathcal{L} + \sum_{i=1}^{m} |\tau_i(C_i)|_\mathcal{L} \leq \mu + \sum_{i=1}^{m} |\tau_i(C_i)|_\mathcal{L} \leq \mu + \sum_{i=1}^{m} \Omega(C_i) \leq \mu + \Omega(\mathcal{L}),
\]

where for short we have denoted by the letter $\mu$ the right side of inequality (34). Because of the arbitrary choice of the points $C_i$ this shows that $\mathcal{L}$ has rotation of bounded variation.

3. On a curve with rotation of bounded variation the variation is a completely additive function of the arc. Moreover on a simple arc points $C$ with zero curvature are everywhere dense. Therefore the variation of the rotation $\sigma(\mathcal{L})$ may be defined as
\[
\sup_{\{C_i\}} \left\{ \sum_{i=0}^{m} |\tau_\sigma(C_iC_{i+1})| + \sum_{i=1}^{m} |\tau_\sigma(C_i)| \right\},
\]

restricting ourselves to choices of the points \(\{C_i\}\) for which \(\Omega(C_i) = 0\). Therefore it follows from (37) that

\[\sigma_\sigma(\mathcal{I}) \leq \mu,\]

which proves inequality (34) and Theorem 7 in its entirety.

**Corollary 1.** In Theorem 3 of this chapter it was proved that each simple arc \(\mathcal{I}\) with rotation of bounded variation may be approximated by polygons \(L_n\) converging to it from the right for which

\[\sigma_\sigma(\mathcal{I}) \geq \limsup_{n \to \infty} \sigma_\sigma(L_n).\]

It follows from Theorem 7 that for such a sequence

\[\sigma_\sigma(\mathcal{I}) = \lim_{n \to \infty} \sigma_\sigma(L_n).\]

**Remark.** Under the conditions of Theorem 7 the characteristic \(\tau_\sigma\), and under the conditions of Corollary 1 also \(\sigma_\sigma^+, \tau_\sigma^-, \sigma_\sigma\), as set functions in the domain of variation of the parameter on converging curves, converge weakly to the corresponding functions for the limit curve.

**Corollary 2.** It follows from Theorems 3 and 7 that for an arc with rotation of bounded variation

\[\sigma_\sigma(\mathcal{I}) = \inf \liminf_{n \to \infty} \sigma_\sigma(\mathcal{I}_n),\]

where the greatest lower bound is taken over all possible sequences of simple curves \(\mathcal{I}_n\) converging to \(\mathcal{I}\) from the right.

12. Variations of the rotation of a simple arc and of arcs converging to it.

**Theorem 8.** If the simple arcs \(\mathcal{I}_n\) with right rotations of uniformly bounded variations \(\sigma_\sigma(\mathcal{I}_n)\) converge to the simple arc \(\mathcal{I}\), on which there are no points with \(\theta = 0\), then \(\mathcal{I}\) is an arc with rotation of bounded variation and

\[\sigma_\sigma(\mathcal{I}) \leq \liminf_{n \to \infty} \sigma_\sigma(\mathcal{I}_n) + \Omega(\mathcal{I}).\]

In distinction from Theorem 7, we do not here suppose that \(\mathcal{I}_n\) converges to \(\mathcal{I}\) from one side, in fact from the right. As a result (34) is replaced by a quite different inequality (39). The necessity of such a change is easily observed in the simplest examples. It is sufficient, for instance, to
consider on the surface of a cube, as $\mathcal{Q}$, two ribs with a common vertex and to approximate $\mathcal{Q}$ by polygons $\mathcal{Q}_n$ depicted with broken curves on Figure 125.

The proof of Theorem 8 may be carried out same way as in the case of Theorem 7. Only in passing from (35) to (36) we may write in place of (36) just the inequality

$$\sum_{i=0}^m |\tau_i(C_{i+1})| + \sum_{i=0}^m |\tau_i(C_i)| \leq \sigma_0(\mathcal{Q}_n) + \sum \Omega(C_i) + \Omega(\mathcal{Q}) + 2s,$$

since in the region between the pieces $C_{i+1}$ of the curves $\mathcal{Q}, \mathcal{Q}_n$ and an auxiliary third curve there may fall essential pieces of the curve $\mathcal{Q}$, and not just insignificant pieces of small curvature adjacent to $C_i$ and $C_{i+1}$ as was the case in the proof of Theorem 7.

Correspondingly, instead of inequality (34) we obtain inequality (39), which proves Theorem 8.

**Remark.** An arc $\mathcal{Q}$ of bounded variation may be characterized in the following way, not depending on the choice of a concrete side of the curve. The arc $\mathcal{Q}$ is decomposed into a finite number of open arcs and points dividing them. For each of these elements one chooses the smaller of the right and left variations. These numbers are added, and one considers the number

$$\sigma(\mathcal{Q}) = \inf \sum \min(\sigma_+, \sigma_-),$$

where the greatest lower bound is taken over all possible finite subdivisions.

For the characteristics $\sigma(\mathcal{Q})$, apparently, it is always true that under the conditions of Theorem 8 the condition

$$\sigma(\mathcal{Q}) \leq \lim \inf_{n \to \infty} \sigma(\mathcal{Q}_n)$$

will be satisfied, and also the relation

$$\sigma(\mathcal{Q}) = \inf \lim \inf_{\mathcal{Q}_n \to \mathcal{Q}} \sigma(\mathcal{Q}_n),$$

where the greatest lower bound is taken over all possible sequences of simple arcs $\mathcal{Q}_n \to \mathcal{Q}$. 
5. Possible extensions of the class of curves with rotations of bounded variation.

13. Simple arcs with singular points. According to Definition 1, a curve with rotation of bounded variation can be only a simple arc interior to which and at the endpoints of which there are no singular points where $\theta = 0$. We may extend the class of curves in the following way.

Definition 4. We include in the class of curves with rotations of bounded variation every simple arc $\mathcal{L}$ which has definite directions at each of its interior points provided that for any decomposition into a finite number of arcs and open pieces the sum of the absolute values of the right rotations, extended only over those points and pieces whose closures do not contain singular points, remain bounded by some number $N(\mathcal{L})$ not depending on the subdivision.

In subsection 6, it was shown by simple examples that for such curves generally speaking there are many results of this chapter which are not valid. In particular, such curves may be nonrectifiable, so that this class decomposes into rectifiable and nonrectifiable curves.

Theorem 6 on pasting apparently remains valid if one pastes regions bounded by rectifiable curves satisfying Definition 4. It follows from this assertion, for example, that if from the singular point $A$ there issue two nonintersecting curves satisfying Definition 4, and one of them is rectifiable close to $A$, then also the other is rectifiable close to $A$. (If we cut the space along the first curve and paste into the cut a plane sector with nonzero angle, then $A$ ceases to be a singular point and the second curve will be rectifiable close to $A$ as a curve satisfying Definition 1.

For a curve $\mathcal{L}$ satisfying Definition 4, many of the constructions carried out above for ordinary arcs with rotation of bounded variation may not be realizable. In particular, polygons approximating $\mathcal{L}$ on one side as constructed in Theorem 3 may not always be made to have common endpoints with $\mathcal{L}$. However, for these curves there will always exist a one-sided rotation on each open piece, if this is defined in the sense of Lemma 4. There will also exist rotation at separate points, and variation of the rotation.

It is easy to show that if the curves $\mathcal{L}_n$ of this class converge to a simple arc $\mathcal{L}$ and one-sided variations of $\mathcal{L}_n$ are uniformly bounded, then also $\mathcal{L}$ satisfies the conditions of Definition 4. For the proof it suffices to note that there can only be a finite number of singular points on $\mathcal{L}$, so that the alteration of the polygons approximating $\mathcal{L}$ close to these
points leads to a bounded change in the variations of their rotations, and near the other points one may carry out the same constructions as in the proofs of Theorems 6 and 7.

We note once again that points where $\theta = 0$ and the singularities connected with them do not appear on convex surfaces.

14. Curves made up from simple arcs. It is natural to extend further the class of curves with rotations of bounded variation in the following way.

**Definition 5 (and 6).** We include in the class of curves with rotation of bounded variation every curve $\mathcal{C}$ which may be decomposed into a finite number of simple pieces each of which is an arc satisfying the conditions of Definition 1 (or 4, respectively).

Depending on whether it is Definition 1 or Definition 4 which we are generalizing, we obtain Definition 5, under which the curves are certainly rectifiable, or Definition 6, under which we admit also certain nonrectifiable curves.

**Singularity connected with cusp points.** For separate simple pieces of a curve satisfying Definition 5 or 6, it makes sense to speak of the one-sided rotation and its variation. The distinction between the right and left sides may be extended in a natural way along the curve from one simple piece to the next. If, moreover, one defines the right and left rotations at the junction points of successive pieces, then the one-sided rotations and their variations extend by additivity to arbitrary pieces along the whole curve. However in the case when the junction point of two simple pieces is a cusp point for the curve in the large, then the two branches issuing from this point may have infinitely many intersections close to the point in question, as in Figure 126c. At such a point $A$ the direction of the branches themselves form two completely defined sector angles $\theta\theta$ and $\theta(A)$. However in the case of Figure 126c it becomes indefinite which of the two rotations $\pi, \pi - \theta$ should be considered the right and which the left rotation of the curve at the point $A$.

In the presence of such singularities of the one-sided rotations and their variations on the curve in the large turn out to not be fully defined numerically. It is natural in this case to speak of the upper and lower
rotations \( \tau_r, \tau_{\leq r} \) and of the upper and lower variations \( \bar{\sigma}_r, \sigma_{\leq r} \), choosing at each such point correspondingly the largest or smallest of the two possible values of the rotation or variation.

This kind of singularity may also arise on a convex surface.\(^4\)

Consideration of Definitions 4, 5 and 6 is interesting because the limiting passage from curves satisfying Definition 1 may be carried to more complicated curves.

The following assertions are apparently true.

1. If the curves \( \mathfrak{L}_n \) converge to the curve \( \mathfrak{L} \), with the \( \mathfrak{L}_n \) satisfying Definition 6, and if \( \sigma_{\leq r}(\mathfrak{L}_n) \leq 0 \), then also \( \mathfrak{L} \) belongs to this same class of curves and

\[
\sigma_r(\mathfrak{L}) \leq \liminf_{n \to \infty} \sigma_{\leq r}(\mathfrak{L}_n) + \Omega(\mathfrak{L}),
\]

where \( \Omega(L) \) is defined taking account of the multiplicity of points of \( \mathfrak{L} \).

2. If moreover, there are no points on \( \mathfrak{L} \) where \( \theta = 0 \), then, beginning with some \( n \), all the \( \mathfrak{L}_n \) are rectifiable and their lengths are uniformly bounded.

Finally we note that we may also speak of the right and left rotations not of arcs but of closed curves under the condition that along the curve the distinction between the right and left rotations is preserved. This last does not hold for example for the closed center line of the Möbius strip.

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\(^4\) This was not mentioned in the book [42].