CHAPTER VII

Converging Metrics

1. Convergence of metrics. In the study of figures in metric spaces it is useful to approximate them by other figures lying in the same space or in spaces approximating the given space. The approximation of figures and spaces may be given various meanings. In this section we give some possible definitions, enumerating them for convenience. The exposition refers to two-dimensional manifolds of bounded curvature.

1. Convergence of figures in metric space.

Definition 1 (Convergence of curves). The arcs $L_n$ in the space $R$ with the metric $\rho$ converge in this space to the arc $L$ if there exist parametrizations $L_n = X_n(t)$, $L = X(t)$, $0 \leq t \leq 1$, of these arcs, under which for any $\varepsilon > 0$ for $n \geq N(\varepsilon)$ and for all $t \in [0,1]$ the inequality

$$\rho(X_n(t), X(t)) < \varepsilon$$

is satisfied.

Definition 2 (Convergence of polygons). Suppose that in a metrized two-dimensional manifold there are situated polygons $P, P_*$ ($n = 1, 2, \cdots$), i.e., compact connected two-dimensional manifolds with edges, bounded by a finite number of simple closed polygonal curves. We shall say that the polygons $P_*$ converge to $P$ along with vertices and sides if beginning with some $n$ the following conditions are satisfied:

1) The polygons $P_*$ and $P$ have the same number of vertices $A_*, A'$, and these may be so associated that the vertices $A'_n$ converge to the corresponding vertices $A'$, the sides of $P_*$ join corresponding pairs of vertices, and the sides of $P_*$ converge in the sense of Definition 1 to the corresponding sides of $P$.

2) There exists at least one interior point of the polygon $P$ which is also interior for all $P_*$. Under the conditions of Definition 2 it is not difficult to show that every compact set $K \subset P_*$ lies inside all the $P_*$ for $n \geq n(K)$.

For nonconnected polygons one may speak of the corresponding convergence of their separate components.

Definition 2 may also be used in the case when the "sides" of $P_*, P$ are
not shortest arcs, but rather arbitrary simple arcs.

Below we shall have in view the following definition of a sector. Suppose that from the point \( O \) in a two-dimensional manifold of bounded curvature there issue two simple curves \( L \) and \( M \) not having common points other than \( O \). Suppose that \( U \) is a neighborhood of \( O \) homeomorphic to the closed disc and sufficiently small so that \( L \) and \( M \) depart from the limits of \( U \). The pieces of the curves \( L \) and \( M \) up to the first intersection with the boundary of \( U \) divide the neighborhood \( U \) into two pieces \( U', U'' \), which we shall call bounded sectors. The pieces of the curves \( L \) and \( M \) themselves are counted in both bounded sectors. Bounded sectors \( U'_1, U'_2 \) obtained for different neighborhoods \( U_1, U_2 \) are regarded as equivalent if there exists a bounded sector \( U'_1 \subset U'_1 \), \( U'_2 \subset U'_2 \). The equivalence of bounded sectors is reflexive, symmetric, and transitive. An equivalence class of bounded sectors will be called a sector, the point \( O \) its vertex, and \( L \) and \( M \) its sides.

If the curves \( L \) and \( M \) coincide on the portion adjacent to \( O \), and then have no further common points, we also say that they form two sectors, one of which is degenerate. The bounded sectors for it are the pieces \( L = M \), and for the complementary one nondegenerate sectors. Both sectors cannot be degenerate at the same time. In this definition, in distinction from subsection 7 of Chapter IV, we exclude the case of multiple contact and divergence of the sides of the sector.

A concrete sector may be indicated by giving a bounded sector. One may indicate a definite one of two nondegenerate sectors by giving the orientation of a neighborhood of the point \( O \) and distinguishing in the order of circuit the sectors from \( L \) to \( M \) and from \( M \) to \( L \). A nondegenerate sector may be characterized by "a curve passing in it" (i.e., a curve issuing from \( O \) and passing, on its initial portion, in bounded sectors of one class) or a sequence of points tending to \( O \) "in this sector".

**Definition 3 (Convergence of sectors).** Suppose that in a two-dimensional manifold of bounded curvature there are given a sector \( V \) with vertex \( O \) and sides \( L, M \) and a sequence of sectors \( V_n \) with vertices \( O_n \) and sides \( L_n, M_n \). We shall say that the sectors \( V_n \) converge to the sector \( V \) if the following conditions are satisfied: 1) \( O_n \to O \); 2) there exists a neighborhood \( U \) of the point \( O \) homeomorphic to the disc such that if we restrict consideration to the pieces of the curves from \( O, O_n \) to the boundary of \( U \), then \( L_n \to L, M_n \to M \) in the sense of Definition 1; 3) if
the sector $V$ is nondegenerate, then there exists at least one point $A$ lying inside a bounded sector $U'$ belonging to $V$ which for $n \geq N(U, A)$ lies inside all the $U'_{\epsilon} \subset V_{\epsilon}$. But if the sector $V$ is degenerate the analogous requirement is imposed on the supplementary sectors $U''_\epsilon, U'''_\epsilon$.

Under the conditions of Definition 3, for a nondegenerate $V$ not only the separate point $A$, but also each compact $K \subset U'$, for $n \geq N(U', K)$, lies inside all the $U'_{\epsilon}$.

Most of all we shall be dealing with sectors bounded by shortest arcs.

2. Convergence of spaces.

Definition 4 (Uniform convergence of metrics). The following definition differs from the definition of subsection 8 of Chapter I in that the mappings $\phi_n$ are now required to be homeomorphisms. We shall say that the metric spaces $R_n$ uniformly converge to the metric space $R$, or that the metrics $\rho_n$ uniformly converge to the metric $\rho$, if there exist homeomorphic mappings $\phi_n$ of the spaces $R_n$ onto $R$ such that for any $\epsilon > 0$ for $n \geq N(\epsilon)$ and any $X, Y \in R$

\[(1) \quad |\rho(X, Y) - \rho(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \epsilon.\]

As a rule we shall use Definition 4 in considering closed two-dimensional manifolds of bounded curvature and compact manifolds with edge, in particular in the majority of local arguments. For noncompact manifolds in the large we introduce a somewhat different definition.

Definition 5 (Local uniform convergence of metrics). Suppose that $R_n$, $R$ are metrized two-dimensional manifolds. We shall say that $R_n$ converges locally uniformly to $R$ if there is in $R$ a sequence of open sets $G'_n$ with compact closures $\bar{G}'_n$, where $\bar{G}'_n$ are closed manifolds or compact manifolds with an edge while

\[G'_1 \subset \bar{G}'_1 \subset G'_2 \subset \bar{G}'_2 \subset \cdots \subset \bar{G}'_n \subset G'_{n+1} \subset \cdots \subset R = \bigcup_{n=1}^{\infty} G_n,\]

and there are given homeomorphic mappings $\phi_n$ of the pieces $\bar{G}'_n \subset R_n$ onto $\bar{G}'_n \subset R$ which satisfy the following requirements:

1) for any $\epsilon > 0$ and any compact $K \subset R$, $K \subset G'_n$ is satisfied for $n \geq N(\epsilon, K)$, and (1) holds for all $X, Y \in K$;

2) for any $\epsilon, A > 0$ and $X_0 \in R$, for $n \geq N(X_0, A, \epsilon)$ and for all $X \in R_n$ satisfying the condition $\rho_n(X, \phi_n^{-1}(X_0)) \leq A$, the inequality $\rho_n(X, G_n) < \epsilon$ is satisfied.

Definition 4 may be considered as a special case of Definition 5 for $G_1 = G_2 = \cdots = R_n$. 
3. Convergence of figures in converging spaces. Suppose that $R_*$ and $R$ are metrized manifolds and $R_* \to R$ in the sense of Definition 5. We introduce the following definitions.

1'. The arcs $L_n$ lying in $R_*$ converge to the arc $L$ lying in $R$ if for $n \geq N$ the arcs $L_n$ lie in the regions of definition of the $\phi_n$ and their images $L'_n = \phi_n(L_n)$ converge in $L$ in the sense of Definition 1.

2'. The polygons $P_n \subset R_*$ converge along with their vertices and sides to the polygon $P \subset R$, if for $n \geq N$ all the $P_n$ lie in the regions of definition of the $\phi_n$ and the figures $P'_n = \phi_n(P_n)$ converge in $R$ to $P$ in the sense of Definition 2, taking of course account of the fact that the sides of $P'_n$ may not be shortest arcs in $R$.

3'. The sectors $V_n$ of $R_*$ converge to the sector $V$ in $R$ if for $n \geq N$ their vertices $O_n$ lie inside the regions of definition of the $\phi_n$ and the images $V'_n$ of the sectors $V_n$ which are sectors in $R$, converge in $R$ to $V$ in the sense of Definition 3.

2. Converging curves and polygons.

4. Lengths of converging curves.

Theorem 1. Suppose that $R_*$, $R$ are metric spaces, with the $R_*$ converging uniformly to $R$ (or else $R_*$, $R$ are metrized two-dimensional manifolds and the $R_*$ locally uniformly converge to $R$), and suppose that the rectifiable curves $L_n \subset R_*$ converge to the curve $L \subset R$ in the sense of Definition 1'. Then

\[ s(L) \leq \lim \inf_{n \to \infty} d(L_n), \]

where the length $s(L)$ is measured in $R$ and the lengths $s(L_n)$ in $R_*$.

Proof. Suppose that $\rho_*$ and $\rho$ are the metrics of the spaces $R_*$ and $R$, and that $\phi_n$ are homeomorphisms of regions of the spaces $R_*$ containing the $L_n$ onto a region $R$ containing $L$, under which in some compact neighborhood $G$ of the curve $L$ one has uniform convergence $\rho_* \to \rho$ and convergence $L'_n = \phi_n(L_n) \to L$. The points of the curves $L_n, L'_n, L$ corresponding with respect to the parameter under this convergence will be denoted by $X_n, Y_n, X$.

We select on $L$ an arbitrary sequence of points $X^i (i = 0, 1, \cdots, n)$. Then we have:

\[ \sum_{i=1}^m \rho(X^{i-1}, X^i) \leq \sum_{i=1}^m [\rho(X^{i-1}, Y_n^{i-1}) + \rho(Y_n^{i-1}, Y_n^i) + \rho(Y_n^i, X^i)]. \]

Because of the convergence $\rho_* \to \rho$ for $n \geq N_i(\varepsilon)$ this sum
\[ \leq \sum_{i=1}^{m} \rho(Y_{n_i}^{-1}, Y_{n_i}) + 2 \epsilon. \]

Because of the uniform convergence \( \rho_n \to \rho \) this sum
\[ \leq \sum_{i=1}^{m} \rho_n(X_{n_i}^{-1}, X_{n_i}) + 3 \epsilon \leq s(L_n) + 3 \epsilon. \]

Because of the arbitrariness of the original system of points \( X \) it therefore follows that for \( n \geq N(\epsilon) \)
\[ s(L) \leq s(L_n) + 3 \epsilon. \]

Passing to the lower limit with respect to \( n \) and using the arbitrary smallness of \( \epsilon > 0 \), we therefore obtain inequality (2).

**Theorem 2.** Suppose that \( R_n \) and \( R \) are metric spaces, while \( R_n \) uniformly converges to \( R \) (or \( R_n \) and \( R \) are metrized two-dimensional manifolds and \( R_n \) locally uniformly converges to \( R \)), and suppose that the lengths of certain rectifiable curves \( L_n \subset R_n \) are bounded uniformly by the number \( S \). Suppose moreover that the \( L_n \) lie in regions \( G_n \subset R_n \), mapped into \( R \) by those homeomorphisms \( \phi_n \) under which the convergence of \( R_n \) to \( R \) takes place, while \( \phi_n(G_n) \subset K \), where \( K \) is a compact region in \( R \).

Then one may select a subsequence from the system of curves \( L_n \) which converges under the mappings \( \phi_n \) to some curve \( L \subset R \).

**Proof.** 1. We select from all \( L_n = X_n(t) \) as a parameter \( 0 \leq t \leq 1 \) the relative arc length, and we choose a subsequence of curves for which the points \( Y_n(t) = \phi_n(X_n(t)) \) converge in \( R \) as \( n \to \infty \) to some point \( X(t) \) for each rational \( t \in [0, 1] \). This is possible in view of the countability of rational \( t \) and the compactness of \( K \). In what follows we shall consider only the \( L_n \) from this subsequence of curves.

2. Taking an arbitrary \( \epsilon > 0 \), we choose rational \( t_i = i/m \), where \( i = 1, 2, \ldots, m \) and \( m \geq S/2 \epsilon \).

Now suppose that \( t \) is any value of the parameter from \([0, 1]\). For some one of the \( t_i \) chosen above we will have \(|t_i - t| \leq \epsilon/3 \). Then because of the uniform convergence of the metrics \( \rho_n \to \rho \) and the choice of the parameters, we will have for sufficiently large \( n \):
\[ \rho(Y_{n_i}(t), Y_{n_i}(t)) \leq \rho_n(X_n(t_i), X_n(t)) + \epsilon \leq |t_i - t| S + \epsilon < 2 \epsilon. \]

Therefore for \( n, m \geq N(\epsilon) \), taking account of the convergence of the points \( Y_{n_i}(t_i) \), we obtain
\[ \rho(Y_n(t), Y_m(t)) \leq \rho(Y_n(t), Y_n(t_i)) + \rho(Y_n(t_i), Y_m(t_i)) + \rho(Y_m(t_i), Y_m(t)) \]
\[ < 2 \epsilon + \rho(Y_n(t_i), Y_m(t_i)) + 2 \epsilon < 5 \epsilon. \]
Therefore it follows that $Y_n(t)$ converges to some point $X(t)$ and this convergence is uniform for all $t$.

3. For the limiting points $X(t)$, for $n \geq N(\varepsilon, t_1, t_2)$ we have
\[
\rho(X(t_1), X(t_2)) \leq \rho(Y_n(t_1), Y_n(t_2)) + 2\varepsilon \\
\leq \rho_n(X_n(t_1), X_n(t_2)) + 3\varepsilon \\
\leq |t_1 - t_2| S + 3\varepsilon,
\]
and thus
\[
\rho(X(t_1), X(t_2)) \leq |t_1 - t_2| S,
\]
so that the points $X(t)$ depend continuously on $t$ in $R$ and form some curve $L$.

Because of the uniform convergence $Y_n(t) \to X(t)$ proved above we have $L_n \to L$. Theorem 2 is proved.

5. **Convergence of polygons.**

**Theorem 3.** Suppose that $R_n$ and $R$ are two-dimensional manifolds of bounded curvature with metrics $\rho_n$, $\rho$, and $P_n$, $P$ polygons in them, with $R_n$ converging uniformly to $R$ and $P_n$ converging to $P$ along with vertices and sides (in the sense of Definition 2'). Suppose that $\phi_n$ are homeomorphisms under which the uniform convergence of metrics $\rho_n \to \rho$ and the convergence of polygons hold. Then beginning with some $n$ the homeomorphisms $\phi_n$ can be replaced by homeomorphisms $\phi_n$ defined in the same regions under which we will also have uniform convergence of the metrics $\rho_n \to \rho$, and the polygons $P_n$ will be exactly mapped onto the polygon $P$ with preservation of the relations of vertices and sides (more precisely, with the preservation of that relation between the boundaries of $P_n$ and $P$ under which the uniform convergence of the boundaries $\phi_n(P)$ to the boundary of $P$ took place).

**Proof.** For definiteness we restrict ourselves to the case when $R_n$ and $R$ are homeomorphic to the open disc and the polygons $P_n$, $P$ are each bounded by a simple closed polygon $\gamma_n$, $\gamma$. Moreover, for simplicity we shall suppose that the mapping $\phi$ is given on the entire space and that the uniform convergence of metrics holds also on the entire space. We give the construction of the homeomorphisms $\phi_n$ in the interior regions of $P_n$ and $P$; the continuation of these homeomorphisms to the portions of $R_n$ and $R$ exterior to $P_n$ and $P$ is constructed analogously.

1. Take an $\varepsilon_1 > 0$. We may suppose that $2\varepsilon_1$ is significantly smaller than the length of the links of $\gamma$. Suppose that $9\varepsilon$ is the smallest of the distances
from a link of \( \gamma \), shortened at the ends by \( \varepsilon_i \), to the family of the remaining links of \( \gamma \). (Of course \( 9\varepsilon \leq \varepsilon_i \).) We shall take \( n \) so large that the metrics \( \rho_n, \rho \) for corresponding pairs of points in \( R_n, R \) differ by less than \( \varepsilon \), and the points of \( \gamma_n' = \phi_n(\gamma_n) \) and \( \gamma \) corresponding under the parameter are distant in \( R \) by less than \( \varepsilon \). Then the pieces of the sides of \( \gamma_n \) corresponding under the parameter to the links of \( \gamma \) shortened by \( \varepsilon_i \) are distant from the other links of \( \gamma_n \) in \( R_n \) by more than \( 6\varepsilon \).

2. We may enclose the curve \( \gamma \) in \( R \) by an open region \( G \) homeomorphic to the plane annulus, with all the points of \( G \) distant from \( \gamma \) by less than \( \varepsilon \). We shall suppose that \( n \) is large and that \( \gamma_n' \) is so close to \( \gamma \) that \( \gamma_n' \subset G \). Then the regions \( \phi_n^{-1}(G) \) will encircle \( \gamma_n \) in \( R \) and all the points of these regions will be distant from \( \gamma_n \) by less than \( 3\varepsilon \).

3. From Lemma 2 of Chapter VI, the curve \( \gamma \) may be approximated by a simple broken curve \( l \) lying in \( G \) which, with respect to the parameter, will uniformly approximate \( \gamma \) with accuracy \( \varepsilon \) both in the position of corresponding points and in the lengths of corresponding pieces. The preimage \( \phi_n^{-1}(l) = l_n \) will be a simple curve (but no longer a broken curve) in \( R_n \).

4. Mark off on \( \gamma \) a finite number of points \( A_1' \), of which the end ones on each link are distant by \( \varepsilon_i \) from the vertices, and the remainder are arranged on each side so that the distance between them is not less than \( 9\varepsilon \) and not larger than \( 18\varepsilon \). The points \( A_i' \) on \( \gamma_n \) corresponding to them with respect to the parameter will be distant from one another at distances differing from theirs by not more than \( \pm 3\varepsilon \).

5. We join each point \( A_i' \) by a shortest path \( A_i'B_i' \) to the curve \( l_n \). Analogously we join \( A_i \) by a shortest path \( A'C_i \) to the curve \( l \). Since the points \( A_i \) are far from one another and close to \( l \), these shortest arcs will not intersect one another. In view of 1 above they cannot intersect the links of \( \gamma_n \) (or of \( \gamma \)) other than the link on which \( A_i' \) (or \( A_i \)) lies.

6. Replacing, if necessary, pieces of the shortest arc \( A_i'B_i' \) by a piece of a link of \( \gamma_n \), we may suppose that this shortest arc departs from the boundary \( \gamma_n \) at \( A_i' \) or close to this point. In the latter case we replace the piece of the shortest arc adjacent to \( \gamma_n \) by a piece of a polygonal curve of nearby length, which, with the exclusion of the endpoint \( A_i' \), lies inside \( P \). Analogously for \( A'C_i \).

7. The endpoints \( B_i, C_i \) need not correspond to one another under the homeomorphisms \( \phi_n \). Suppose that \( \bar{B}_i' = \phi_n(B_i) \). The points \( \bar{B}_i' \) lie on \( l \) near to \( C' \). We replace the shortest arcs \( A'C_i \) by the polygonal curves \( A_i\bar{B}_i' \).
closest in length and also lying in the strip between $\gamma$ and $l$.

8. Now we define a homeomorphism $\phi_n$ in the following way. Suppose
that in the region bounded by the curve $l$, the homeomorphism $\phi_n$
with $\phi_n$. The remaining piece of $P_n$ is decomposed into a region
$A_1^i A_1^{i+1} B_1^{i+1} B_1^i$ homeomorphic to the disc. We map each such region homeo-
morphically onto the region $A' A'^{i+1} B'_{i+1} B'_{i}$ corresponding to it, with
the preservation of the following relations on the boundary: the pieces $B_n^{i+1} B_n^{i+1} B'^{i+1} B'^{i}$ the same as under the mapping $\phi_n$; the pieces $A_n^{i+1} A_n^{i+1} A' A'^{i+1}$ the same as in the correspondence of the parameters $\gamma_n \leftrightarrow \gamma$, and the pieces
$B_n^{i+1} A_n^{i+1} \leftrightarrow B'_{n+1} A'$ according to the correspondence of the relative lengths.

The homeomorphism just constructed maps $P_n$ onto $P$ with the needed
relations of the boundaries. The metrics $\rho_n$ and $\rho$ can differ under this
relation by not more than $C_{\varepsilon_1}$. Beginning with some $n$, we pass to the
analogous construction of $\phi_n$ with a new $\varepsilon_2$ in place of $\varepsilon_1$ and so forth,
and thus construct the needed sequence of homeomorphisms. This proves
Theorem 3.

6. Convergence of the induced metrics. If the set $M$ in the metric space
$R$ is metrically connected (i.e., any two points $X, Y \in M$ may be joined
by a curve $\overline{XY}$ of finite length lying in $M$), then the selection of $M$ from
$R$ induces on $M$ the metric

$$\rho_M(X, Y) = \inf_{XY \in M} [s(\overline{XY})].$$

A polygon in a two-dimensional manifold of bounded curvature is always
metrically connected.

**Theorem 4.** Under the conditions of Theorem 3 the metrics $\rho_{R_n}$, induced
by the selection in $R_n$ of the polygons $P_n$, converge uniformly to the metric
$\rho_R$ induced by the selection of $P$ from $R$, this convergence being understood
in the following two senses:

a) for any $\varepsilon > 0$ and compact $K \subset P$, for $n \geq N(\varepsilon, K)$ any pair of points
$X, Y \in K$ falls within the region of values of the homeomorphism $\phi_n$ and

$$|\rho_R(X, Y) - \rho_R(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \varepsilon;$$

b) if $\phi_n$ are taken to be those homeomorphisms whose existence was as-
serted in Theorem 3, then for $n \geq N(\varepsilon)$ and for any $X, Y \in P$

$$|\rho_R(X, Y) - \rho_R(\phi_n^{-1}(X), \phi_n^{-1}(Y))| < \varepsilon.$$

**Proof.** 1. We first prove that for any fixed pair of points $A, B$ lying
within $P$, and their preimages $A_n = \phi_n^{-1}(A), B_n = \phi_n^{-1}(B)$, which for suf-
ficiently large \( n \) fall inside \( P_n \), we have the convergence
\[
\rho_p(A, B) = \lim_{n \to \infty} \rho_{P_n}(A_n, B_n).
\]

2. We join each pair \( A_n \) and \( B_n \) in \( P_n \) by a shortest arc \( \overline{A_nB_n} \).
Because of the uniform convergence \( \rho_n \to \rho \) and the convergence \( P_n \to P \) the diameters and perimeters of \( P_n \) are uniformly bounded, and therefore bounded in length \( s_n(\overline{A_nB_n}) \). From a sequence \( A_nB_n \) for which \( s_n(\overline{A_nB_n}) \to \liminf_{n \to \infty} s_n(\overline{A_nB_n}) \) we may by Theorem 2 select a subsequence converging to some curve \( \overline{AB} \). Then we obtain
\[
\rho_p(A, B) \leq s(\overline{AB}) \leq \liminf_{n \to \infty} s_n(\overline{A_nB_n}) = \liminf_{n \to \infty} \rho_{P_n}(A_n, B_n).
\]

3. Now join \( A \) and \( B \) by a shortest arc \( \overline{AB} \) in \( P \). It may be drawn so that this curve either does not intersect, or has one point in common with (or one common segment in common with) each of the links of the boundary \( \gamma \) of the polygon \( P \). Suppose that \( m \) of the vertices of \( P \) fall on \( \overline{AB} \). Choose an arbitrary \( \epsilon > 0 \) and choose \( \epsilon_i > 0 \) so small that \( \epsilon_i \leq \epsilon / 12m \), and, beginning with some \( n \), each of the vertices of \( P_n \) is distant in \( R_n \) by more than \( 6\epsilon_i \) from the boundary \( \gamma_n \) of the polygon \( P_n \), the sides issuing from this vertex being substracked. In what follows we shall consider only such \( n \).

4. Subdivide \( \overline{AB} \) by points \( A' \) into a finite number of pieces \( L_i = A_iA_{i+1} \) which are absolutely shortest arcs in \( R \) of the following four types:

I) \( L_i \) lies inside \( P \);
II) \( L_i \) lies inside \( P \) with the deletion of one endpoint, which lies inside a side of \( P \);
III) \( L_i \) lies entirely inside one side of \( P \);
IV) \( L_i \) arrives at a vertex of \( P \) at one of the endpoints.

There will not be more than \( 2m \) pieces of the fourth type. Adding if necessary division points, we may assume that each of the pieces of the fourth type is shorter than \( \epsilon_i \).

5. For each point \( A' \) falling on \( \gamma \) there is a point \( A'_i \in \gamma_n \) corresponding with respect to the parameter on the boundary. But if \( A' \) lies inside \( P_n \), we put it into correspondence with \( A'_i = \phi_n^{-1}(A') \). Beginning with some \( n \), all \( A'_i \in P_n \). We shall moreover suppose that \( n \) is so large that \( |\rho_n - \rho| < \epsilon_i \) and \( \rho(\gamma, \phi_n(\gamma_n)) < \epsilon_i \). Then for all points \( A_i \)
\[
\rho_n(A'_i, \phi_n^{-1}(A')) \leq \rho(\phi_n(A'_i), A') + \epsilon_i \leq 2\epsilon_i.
\]

6. For pieces of the fourth type we then have:
\[
\rho_n(A'_i, A'_{i+1}) \leq \rho_n(\phi_n^{-1}(A'), \phi_n^{-1}(A'_{i+1})) + 4\epsilon_i \leq \rho(A', A'_{i+1}) + 5\epsilon_i \leq 6\epsilon_i.
\]
One of the points $A'_n, A''_n$ is a vertex of $P_n$. It is distant from the sides not issuing from it by more than $6\varepsilon$, so that the absolute shortest arc $A'_n A''_n$ in $R_n$ may be regarded as not leaving $P_n$. Hence

\[ \rho_{P_n}(A'_n, A''_n) = \rho_n(A'_n, A''_n) \leq 6\varepsilon. \]

7. We shall show that on each of the pieces of the first three types

\[ \rho(A', A''^+) \geq \lim_{n \to \infty} \rho_n(A'_n, A''_n). \]

If $A'A''^+$ is a piece of the first type, i.e., $A_i A''_n$ lies inside $P$, then this piece may be subdivided into segments $A_i A''_n$ small in comparison with their distances to $\gamma$ and so small that for sufficiently large $n$ their preimages $a_i, a''_n$ lie inside $P_n$ substantially closer to one another than to $\gamma_n$. Then the absolute shortest arc $a_i a''_n$ passes inside $P_n$. Therefore for sufficiently large $n$

\[ \rho_{P_n}(A'_n, A''_n) \leq \sum_i \rho_{P_n}(a_i, a''_n) = \sum_i \rho_n(a_i, a''_n) \to \sum_i \rho(a_i, a''_n) = \rho(A', A''^+), \]

from which inequality (8) follows.

If $A'A''^+$ is a piece of the second or third type, i.e., $A_i A''_n$ has one endpoint in common with or entirely lies inside a side of $P$, then $A'A''^+$ may be subdivided into pieces $A_i A''_n$ small in comparison with their distance to the other sides of $P$. In this case one may mark off in $P_n$ points $a_i$ for which $\lim_{n \to \infty} \psi_n(a_i) = a_i$. For sufficiently large $n$ the absolute shortest arcs $a_i a''_n$ in $R_n$ cannot intersect any side of $P_n$ other than the one corresponding side, and they may be regarded as passing in $P_n$.

8. Thus we have

\[ \rho_{P}(A, B) = \sum \rho_{P}(A', A''^+) = \sum \rho(A', A''^+) \geq \sum' \rho(A', A''^+), \]

where in $\sum'$ we have admitted all the pieces of the fourth type. Taking into account (8), we may for large $n$ prolong this inequality

\[ \geq \sum' \rho(P(A_i, A''_n) - \varepsilon, \]

by adding terms corresponding to pieces of the fourth type, taking account of inequality (7) we obtain

\[ \geq \sum \rho(P(A_i, A''_n) - \varepsilon - 6\varepsilon, 2m \geq \rho_{P_n}(A_n B_n) - 2\varepsilon. \]

Therefore, in view of the arbitrary smallness of $\varepsilon$ it follows that
\[ \rho_f(A, B) \geq \limsup_{n \to \infty} \rho_{f_n}(A_n, B_n) \]

which along with inequality (6) proves (5).

9. We have proved that the convergence (5) holds for each fixed pair of points \( A, B \in P \). We choose an arbitrary compact \( K \subset P \). For \( n \) sufficiently large, \( \psi_n(P_n) \supset K \). We shall show that the convergence (5) is uniform for all \( A, B \in K \). Suppose the contrary. Then for some \( \varepsilon_0 > 0 \) there exist pairs of points \( X', Y' \in K \) (\( i = 1, 2, \ldots \)), for which

\[ |\rho_f(X', Y') - \rho_{f_n}(X'_n, Y'_n)| > 5\varepsilon_0, \]

as \( n \to \infty \). Because of the compactness of \( K \) we may suppose that we have already selected a subsequence such that \( X' \to X_0, Y' \to Y_0 \).

10. Suppose that \( Z \in P \) and \( \rho(Z, \gamma) \geq 6\varepsilon > 0 \). We choose an \( \varepsilon \)-neighborhood \( U \) of the point \( Z \), homeomorphic to the disc. Then for any \( x, y \in U \) the shortest arc \( xy \) does not issue from \( P \) and therefore \( \rho_f(x, y) = \rho(x, y) \).

Enclose \( \gamma \) in \( R \) by a strip \( G \) all of whose points are distant from \( \gamma \) by less than \( \varepsilon \). Suppose that \( n \) is so large that \( |\rho_n - \rho| < \varepsilon, \rho(\psi_n(\gamma_n), \gamma) < \varepsilon \) and \( \psi_n(\gamma_n) \) lies in \( G \). Then \( \rho(Z, \psi_n(\gamma_n)) > 5\varepsilon \).

The regions \( U_n = \psi^{-1}_n(U) \) lie in \( P_n \). For any \( x_n, y_n \in U_n \) we have

\[ \rho_n(x_n, y_n) \leq \rho(x, y) + \varepsilon \leq 3\varepsilon, \]

\[ \rho_n(x_n, \gamma_n) \geq \rho(x, \psi_n(\gamma_n)) - \varepsilon \geq \rho(Z, \psi_n(\gamma_n)) - 2\varepsilon > 3\varepsilon. \]

Therefore the shortest arc \( x_n\gamma_n \) cannot touch \( \gamma_n \) and lies in \( P_n \), so that \( \rho_{f_n}(x_n, y_n) = \rho_n(x_n, y_n) \).

11. Surround the points \( X^0 \) and \( Y^0 \) by neighborhoods \( U \) and \( V \) as was done under point 10 for the point \( Z \), putting \( \varepsilon \leq \varepsilon_0 \). Then in \( U, V \) we will have \( \rho_f = \rho \) and for sufficiently large \( n \) within the limits of \( U_n, V_n \) we will have \( \rho_{f_n} = \rho_n \).

Now suppose that \( X^i \) and \( Y^i \) are so close to \( X^0 \) and \( Y^0 \) respectively and \( n_i \) so large that the \( X^i \) and \( Y^i \) lie in \( U \) and \( V \) respectively. Their preimages \( X^0_n \) and \( Y^0_n \) in \( R_n \) lie in \( U_n \) and \( V_n \) respectively and are distant from \( X^0_n \) and \( Y^0_n \) respectively by less than \( \varepsilon_0 \). Moreover, because of the convergence (5), for \( A = X^0, B = Y^0 \)

\[ |\rho_f(X^0, Y^0) - \rho_{f_n}(X^0_n, Y^0_n)| < \varepsilon_0. \]

Then we have:

\[ |\rho_f(X^i, Y^i) - \rho_{f_n}(X^i_n, Y^i_n)| \leq |\rho_f(X^0, Y^0) - \rho_{f_n}(X^0_n, Y^0_n)| + |\rho_f(X^i, Y^i)| + |\rho_{f_n}(X^0_n, Y^0_n)| + |\rho_{f_n}(X^i_n, Y^i_n)| < 5\varepsilon_0, \]
which contradicts 10 above.

Thus assertion a) of Theorem 4 is completely proved.

12. Assertion b) may be proved in the following way. Consider the construction of the homeomorphisms \( \phi_n \) in Theorem 3. We may verify that for each \( \varepsilon > 0 \) there exists a number \( N \) and a compact \( K \subset P \) such that for \( n \geq N \) and for any \( A, B \subset P \) there exist \( A', B' \subset K \) distant from them in \( \rho_P \) by less than \( \varepsilon \), while the preimages \( \phi_n^{-1}(A) \), \( \phi_n^{-1}(B) \) will be distant from \( \phi_n^{-1}(A') \), \( \phi_n^{-1}(B') \) in \( \rho_P \), by less than \( \varepsilon \).

Moreover we may suppose that for \( n \geq N \) the homeomorphisms \( \phi_n \) and \( \phi_n \) coincide on \( K \) and that the metrics \( \rho_P \) and \( \rho_P \) on \( K \) and \( \phi_n^{-1}(K) \), from the already proved a) of Theorem 4, differ by less than \( \varepsilon \). Then for any \( A, B \subset P \) we will have

\[
| \rho_P(A, B) - \rho_P(\phi_n^{-1}(A), \phi_n^{-1}(B)) | \\
< | \rho_P(A', B') - \rho_P(\phi_n^{-1}(A'), \phi_n^{-1}(B')) | + 2\varepsilon < 3\varepsilon,
\]

which proves assertion b) of Theorem 4.

3. **Charges and weak convergence.** In this section we shall formulate the definitions and some known properties of charges and weak convergence of set functions, which will be needed in this chapter. The proofs of these results, along with other properties of charges and of weak convergence, may be found in a paper of A. D. Aleksandrov [1].

We should like to emphasize that the formulation presented below is related to fully normal\(^2\) topological spaces \( R \), which are certainly metrizable manifolds.

**7. Charges.**

**Definition.** By a charge in a space \( R \) we mean a function \( \mu(M) \) defined on Borel sets \( M \subset R \) which satisfies the following three conditions:

1) \( \mu(M_1 \cup M_2) = \mu(M_1) + \mu(M_2) \) if \( M_1 \cap M_2 = 0 \);
2) \( |\mu(M)| \leq N < +\infty \);
3) for each \( \varepsilon > 0 \) there exists a closed \( F \subset M \) for which \( |\mu(M) - \mu(F)| < \varepsilon \).

**Lemma 1.** A bounded completely additive function defined on a field of Borel sets in \( R \) is always a charge.

For example, if from a two-dimensional manifold of bounded curvature

\(^1\) See also V. I. Glivenko [42] and Ju. V. Prohorov [60].

\(^2\) The space \( R \) is called normal if in it any two nonintersecting closed sets \( F \) are separable by nonintersecting open sets, and fully normal, if moreover each set \( F \) in it is a set of type \( G_\delta \).
we distinguish a region with a compact closure, then within the limits of this region the functions \( \omega, \omega^+, \omega^-, \Omega \) are charges.

**Lemma 2.** For each charge \( \mu \) there exist nonnegative charges \( \mu^+ \) and \( \mu^- \), called the positive and negative parts of the charge \( \mu \), for which the equations

\[
\begin{align*}
\mu^+(M) &\equiv \sup_{E \subseteq M} \{ \mu(E) \} = \sup_{F \subseteq M} \{ \mu(F) \}, \\
\mu^-(M) &\equiv \sup_{E \subseteq M} \{ -\mu(E) \} = \sup_{F \subseteq M} \{ -\mu(F) \}
\end{align*}
\]

(11)

hold, where \( E \) are Borel sets and \( F \) closed sets.

The identities (11) constitute definitions of \( \mu^+ \) and \( \mu^- \).

**Lemma 3.** The representation \( \mu = \mu^+ - \mu^- \) is valid. It is minimal in the following sense: for any representation \( \mu = \mu_1 - \mu_2 \), where \( \mu_1 \) and \( \mu_2 \) are nonnegative charges, the inequality

\[
\mu^+(M) \leq \mu_1(M), \quad \mu^-(M) \leq \mu_2(M)
\]

(12)

holds.

**Definition.** The variation of the charge \( \mu(M) \) is the charge

\[
|\mu|(M) = \mu^+(M) + \mu^-(M).
\]

**Lemma 4.** For each nonnegative charge \( \mu \)

\[
\mu(M) = \inf_{G \supseteq M} \{ \mu(G) \},
\]

(13)

where \( G \) are open sets.

8. Weak convergence.

**Definition.** We say that the sequence of charges \( \mu_n \) converges weakly if for any bounded continuous function \( f(x) \), \( X \subseteq L \), the limit

\[
\lim_{n \to \infty} \int f(x) \mu_n(dE)
\]

exists. The integral is understood in the sense of Lebesgue-Stieltjes.

**Lemma 5.** If the charges \( \mu_n \) converge weakly, then they converge to some uniquely defined charge \( \mu \), i.e., for each continuous bounded function \( f(x) \)

\[
\lim_{n \to \infty} \int f(x) \mu_n(dE) = \int f(x) \mu(dE).
\]

(14)

In this case we write \( \mu_n \to \mu \).
Lemma 6. The charges of a weakly converging sequence are uniformly bounded, i.e., $|\mu_n(M)| \leq N < +\infty$ for all $M$ and $n$.

Lemma 7 (Kolmogorov’s test). For a sequence of charges $\mu_n$ to converge weakly to the charge $\mu$ it is necessary and sufficient that they should be bounded uniformly, and that for each open $G$ and closed $F_0 \subset G$, the condition

$$\lim_{n \to \infty} \inf_{F \subset G} |\mu(G) - \mu_n(G)| = 0$$

should be satisfied.

Lemma 8 (Weak compactness). From a sequence of charges which are uniformly bounded and given on a compact space one may select a weakly converging subsequence.

Lemma 9. If $\mu_n \rightharpoonup \mu$, then for any open set $G$

$$\mu^*(G) \leq \liminf_{n \to \infty} \mu_n^*(G), \quad \mu^*(G) \leq \liminf_{n \to \infty} \mu_n^*(G).$$

Lemma 10. If the charges $\mu_n, \mu$ are nonnegative, then for weak convergence $\mu_n \rightharpoonup \mu$ it is necessary and sufficient that the two following conditions be observed:

1) in the space in the large

$$\mu(R) = \lim_{n \to \infty} \mu_n(R);$$

2) for each closed $F \subset R$

$$\mu(F) \geq \limsup_{n \to \infty} \mu_n(F)$$

or, which in this case is the same, for any open $G \subset R$

$$\mu(G) \leq \liminf_{n \to \infty} \mu_n(G).$$

Lemma 11. If the $\mu_n$ and $\mu$ are nonnegative, then for weak convergence $\mu_n \rightharpoonup \mu$ it is necessary and sufficient that the equation

$$\mu(G) = \lim_{n \to \infty} \mu_n(G)$$

hold for all $G$ for which $\mu(\overline{G}) = \mu(G)$, i.e., $\mu(\text{Fr. } G) = 0$. Here $\overline{G}$ is the closure in $R$.

Lemma 12. If the $\mu_n$ are completely additive and $\mu_n \rightharpoonup \mu$, then $\mu$ is also a completely additive charge.

9. Escaping loads.

Definition. A sequence of nonempty closed sets $F_n$ is said to be di-
vergent if these sets do not intersect one another and the sum of any collection (including infinite collections) of sets $F_n$ is closed in $R$. We shall say that in a system of charges there is an escaping load, if there exists a diverging sequence of closed sets $F_n$ and a number $a \neq 0$ such that for all $n$

$$\frac{\mu_n(F_n)}{a} \geq 1.$$  

**Lemma 13.** In a weakly converging sequence of charges there is no escaping load.

**Lemma 14.** If in a sequence of charges $\mu_n$ there is no escaping load, then for any fixed diverging sequence $F_n$ and $\epsilon > 0$, for $n > N(\epsilon)$

$$|\mu_n|(F_n) < \epsilon.$$ 

**Lemma 15.** In a locally compact $R$, for the weak compactness of a system of charges $\mu_n$ it is necessary and sufficient that the $\mu_n$ should be uniformly bounded and that there should be no escaping load.

10. Local weak convergence.

**Definition.** A sequence of charges $\mu_n$ converges locally weakly to the charge $\mu$ if for any continuous function $f(x)$ distinct from zero only on a set with compact closure

$$\lim_{n \to \infty} \int f(x) \mu_n(dE) = \int f(x) \mu(dE).$$

**Lemma 16 (Kolmogorov’s test).** For the local weak convergence of $\mu_n$ to $\mu$ it is necessary and sufficient that in each region $G_0$ with compact closure $\bar{G}_0$ the charges $\mu_n$ and $\mu$ should be bounded uniformly and that equation (15) should be satisfied for each $F_0 \subset G_0$.

**Lemma 17.** From local weak convergence and the absence of an escaping charge follows weak convergence on all of $R$.

4. Curvatures of converging metrics

11. Curvature as a charge. In a two-dimensional manifold of bounded curvature with a compact closure the completely additive functions $\omega, \omega^+, \omega^-$ and $\Omega$ are bounded and are therefore charges. The last three of these are nonnegative charges.

**Theorem 5.** The positive and negative parts $\omega^+, \omega^-$ of the curvature $\omega$ coincide with the positive and negative parts of $\omega$ in the sense of the theory of charges, in other words
\begin{align*}
\left\{ \begin{array}{c}
\omega^+(M) = \omega^p(M) \equiv \sup_{E \in M} \{ \omega(E) \}, \\
\omega^-(M) = \omega^p(M) \equiv \sup_{E \in M} \{-\omega(E)\}.
\end{array} \right.
\end{align*}

**Proof.** Since \( \omega = \omega^p - \omega^s \) and \( \omega = \omega^+ - \omega^- \), it follows from Lemma 3 that \( \omega^s \leq \omega^+ \), \( \omega^s \leq \omega^- \), so that it suffices to prove the converse inequalities. Because of the fact that relation (13) is valid for \( \mu = \omega^p, \omega^s, \omega^+, \omega^- \), it is sufficient to prove equation (20) for open \( M \).

Suppose that \( G \) is an open set with compact closure. For any \( \varepsilon > 0 \), from the definition of \( \omega^+(G) \), there exists in the set \( G \) a system of triangles homeomorphic to the disc \( \{ T_i \} \) with excesses \( \delta(T_i) \geq 0 \), for which
\[ \omega^+(G) < \sum_i \delta(T_i) + \varepsilon. \]

From Theorem 9 of Chapter VI, for a triangle \( T_i \) homeomorphic to the disc
\[ \delta(T_i) = \omega(T_{(-\gamma)}) + \sum_{i=1}^3 \tau_i, \]
where \( \tau_1, \tau_2, \tau_3 \) are the certainly nonpositive rotations of the open sides of \( T_i \) from the side of the interior region of \( T_i \).

From the additivity of \( \omega \) and the nonpositiveness of \( \tau_i \) we conclude that
\[ \omega^+(G) < \omega(\sum T_{(-\gamma)}) + \varepsilon, \]
so that in view of the arbitrariness of \( \varepsilon > 0 \) it follows that
\[ \omega^+(G) \leq \omega^p(G). \]

Analogously for each \( \varepsilon > 0 \), from the definition of \( \omega^-(G) \), there exists a system of reduced triangles \( T_i \) in the set \( G \) with excesses \( \delta(T_i) \leq 0 \) for which
\[ \omega^-(G) < -\sum \delta(T_i) + \varepsilon. \]
But this time there may be triangles of five types among the \( T_i \), as depicted in Figure 89.

![Figure 89](image)

If in triangles of types two through four we reject “exterior tails”, thus replacing the \( T_i \) by triangles homeomorphic to the disc, and drop
triangles of the fifth type altogether, but here also add $-\sum \omega(A_j)$, to $-\sum \delta(T_i)$, where the $A_j$ are the points serving as the bases of the rejected "tails", then we will have

$$-\sum \delta(T_i) \leq -\sum \delta(t_i) - \sum \omega(A_j).$$

For each $t$, homeomorphic to the disc we have:

$$\delta(t_i) = \omega(t_{c-\gamma}) + \sum_{i=1}^{3} \tau_i,$$

where the $\tau_i$ are the rotations of the sides of $t_i$. Here, on each piece of a shortest arc its rotations are nonpositive, and the sum of the right and left rotations is the curvature of that portion of the shortest arc. Therefore we may write

$$\omega^{-}(G) < -\omega \left( \sum t_{c-\gamma} + \sum \text{sides } t_i + \sum A_j \right) + \varepsilon.$$

Because of the arbitrariness of $\varepsilon > 0$ it therefore follows that

$$\omega^{-}(G) \leq \omega^{*}(G).$$

Theorem 5 is proved.

12. **Angles of sectors with a common vertex.** Suppose that in a neighborhood $U$ of the point $O$, homeomorphic to the disc, there are given metrics $\rho_n$, uniformly converging to the metric $\rho$, with the absolute curvatures of all these metrics bounded uniformly by a small number $\varepsilon$:

$$\Omega(U) < \varepsilon, \quad \Omega_n(U) < \varepsilon, \quad (0 < \varepsilon < \pi).$$

(21)

Suppose that we have drawn from the point $O$ in the neighborhood $U$ two shortest arcs $L$ and $M$ in the metric $\rho$ and shortest arcs $L_n, M_n$ in the metrics $\rho_n$, with each pair $L, M; L_n, M_n$ of these shortest arcs having no common points other than $O$ or a common initial segment issuing from $O$. Suppose finally that in the metric $\rho$ the curves $L_n$ converge to $L$ and the $M_n$ to $M$.

Supposing that the neighborhood $U$ is oriented, we may for each pair of shortest arcs $L_n, M_n$; $L, M$ distinguish two sectors: the first in the order of the circuit of the point $O$ from $L_n$ or $L$ to $M_n$ or $M$, and the second the complementary sector. The angles of the first sectors will be denoted by the superscript 1, and those of the second sectors by the superscript 2. If one of the sectors $L, M$ is degenerate, we take it to be the first one.

**Lemma 18.** With the conditions stated, i.e., given the uniform convergence of metrics and shortest arcs and with condition (21) observed, for the angles
\( \alpha, \alpha_n \) between the shortest arcs and the angles \( \tilde{\alpha}^1, \tilde{\alpha}^2, \tilde{\alpha}^3, \tilde{\alpha}^n \) of the sectors between them, the following inequalities are valid for sufficiently large \( n \):

\[
\begin{align*}
|\alpha - \alpha_n| &< 5\varepsilon, \\
|\tilde{\alpha}^1 - \tilde{\alpha}^n| &< 9\varepsilon, \\
|\tilde{\alpha}^2 - \tilde{\alpha}^n| &< 9\varepsilon.
\end{align*}
\]

**Proof.** 1. Choose points \( X \) and \( Y \) distinct from \( O \) on \( L \) and \( M \), and on \( L_n \) and \( M_n \) points \( X_n \) and \( Y_n \) that correspond to them with respect to the parameter on the converging curves. We may choose the points \( X \) and \( Y \) so close to \( O \) that any shortest arcs joining points of the segments \( OX, OY \) in \( \rho \) or \( OX_n, OY_n \) in \( \rho_n \) will not leave \( U \).

Join \( X \) and \( Y \) in \( \rho \) and \( X_n \) and \( Y_n \) in \( \rho_n \) by shortest arcs, forming with \( L, M \) and \( L_n, M_n \) reduced triangles \( T, T_n \). These triangles are then developed on the plane, with \( \alpha^0 \) and \( \alpha^0_n \) the angles of these plane triangles corresponding to the vertex \( O \). From Lemma 23 in subsection 12 of Chapter V we have

\[
|\alpha - \alpha^0| \leq 2\Omega(U), \quad |\alpha_n - \alpha^0_n| \leq 2\Omega_n(U).
\]

Because of the uniform convergence \( \rho_n \rightarrow \rho \) and \( X_n \rightarrow X, Y_n \rightarrow Y \), the lengths of the sides of \( T_n \) converge to the lengths of the sides of \( T \). Therefore for sufficiently large \( n \) we will have the inequality

\[
|\alpha^0 - \alpha^0_n| < \varepsilon.
\]

Then

\[
|\alpha - \alpha_n| \leq |\alpha - \alpha^0| + |\alpha^0 - \alpha^0_n| + |\alpha^0_n - \alpha_n| \leq 2\Omega(U) + \varepsilon + 2\Omega_n(U) < 5\varepsilon,
\]

which proves inequality (22).

2. In order to prove inequality (23), we consider first the case when for the construction just realized

\[
0 \leq \alpha^0 \leq \pi - 2\varepsilon.
\]

In this case the shortest arc \( XY \) does not pass through \( O \). Suppose for definiteness that it passes in the sector with angle \( \tilde{\alpha}^1 \) and forms with \( L \) and \( M \) a triangle \( T \). From Theorem 10 of Chapter VI we have

\[
|\tilde{\alpha}^1 - \alpha^0| \leq \Omega(T) \leq \Omega(U) < \varepsilon.
\]

In the case at hand there cannot exist another shortest arc \( \tilde{XY} \) lying in the sector with the angle \( \tilde{\alpha}^2 \). For otherwise we would have also

\[
|\tilde{\alpha}^2 - \alpha^0| < \varepsilon,
\]

which would lead to the inequality

\[
\Omega(O) = |2\pi - (\tilde{\alpha}^1 + \tilde{\alpha}^2)| = |2(\pi - \alpha^0) - (\tilde{\alpha}^1 + \tilde{\alpha}^2 - 2\alpha^0)| \geq 4\varepsilon - 2\varepsilon = 2\varepsilon,
\]

\[
(\Omega(O) = |2\pi - (\tilde{\alpha}^1 + \tilde{\alpha}^2)| = |2(\pi - \alpha^0) - (\tilde{\alpha}^1 + \tilde{\alpha}^2 - 2\alpha^0)| \geq 4\varepsilon - 2\varepsilon = 2\varepsilon)
\]

\[
(\Omega(O) = |2\pi - (\tilde{\alpha}^1 + \tilde{\alpha}^2)| = |2(\pi - \alpha^0) - (\tilde{\alpha}^1 + \tilde{\alpha}^2 - 2\alpha^0)| \geq 4\varepsilon - 2\varepsilon = 2\varepsilon)
\]
in contradiction to the fact that $\Omega(O) \leq \Omega(U) < \varepsilon$.

For sufficiently large $n$, when condition (25) is satisfied, the angles $\alpha_s^0 < \pi$ and the shortest arcs $X_s, Y_s$ in $\rho_s$ also do not pass through $O$. Moreover, beginning with some $n$, all of these shortest arcs pass through the sectors with angles $\bar{\alpha}_s^1$. For otherwise we would have a subsequence of shortest arcs which would converge to a shortest arc $\bar{XY}$ in the metric $\rho$, lying in the sector with angle $\bar{\alpha}^2$. But as we have shown above such a shortest arc does not exist.

For the triangle $T_s$ excised in $\rho_s$ by the shortest arcs $X_s, Y_s$, we have

$$|\bar{\alpha}_s^1 - \alpha_s^0| \leq \Omega_s(T_s) \leq \Omega_n(U) < \varepsilon.$$

Finally

$$|\bar{\alpha} - \bar{\alpha}^1_s| \leq |\bar{\alpha}^1 - \alpha^0| + |\alpha^0 - \alpha_s^0| + |\alpha_s^0 - \bar{\alpha}_s^1| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

so that in the case at hand certainly the first of the inequalities (23) holds.

The second inequality follows from the relations

$$|\bar{\alpha}^2_s - \bar{\alpha}^1_s| = |(\bar{\alpha}^1 + \bar{\alpha}^2 - 2\pi) - (\bar{\alpha}_s^1 + \bar{\alpha}_s^2 - 2\pi) - (\bar{\alpha}^1 - \bar{\alpha}_s^1)|$$

$$< \varepsilon + \varepsilon + 3\varepsilon = 5\varepsilon.$$

3. Now we shall prove inequality (23) in the case when

$$\pi - 2\varepsilon < \alpha^0 \leq \pi.$$  \hspace{1cm} (27)

In this case, from inequality (24), $\pi - 4\varepsilon < \alpha$, so that $\pi - 4\varepsilon < \bar{\alpha}^1$, $\pi - 4\varepsilon < \bar{\alpha}^2$. Along with relation $\bar{\alpha}^1 + \bar{\alpha}^2 < 2\pi + \varepsilon$, resulting from the bound $\Omega(O) = |\bar{\alpha}^1 + \bar{\alpha}^2 - 2\pi|$ on the curvature, this shows that

$$\pi - 4\varepsilon < \bar{\alpha}^1 < \pi + 5\varepsilon, \hspace{0.5cm} \pi - 4\varepsilon < \bar{\alpha}^2 < \pi + 5\varepsilon. \hspace{1cm} (28)$$

From the fact that $\alpha_s^0 \to \alpha^0$ as $n \to \infty$ it follows that for sufficiently large $n$ the condition

$$\pi - 2\varepsilon < \alpha_s^0 \leq \pi,$$

analogous to (27), holds.

From this, as above, it follows that

$$\pi - 4\varepsilon < \alpha_s^1 < \pi + 5\varepsilon, \hspace{0.5cm} \pi - 4\varepsilon < \alpha_s^2 < \pi + 5\varepsilon. \hspace{1cm} (29)$$

From relations (28) and (29) follow inequalities (23). The lemma is completely proved. Of course, the constants in inequalities (22) and (23) may be decreased.

13. **Weak convergence of curvature.**

**Theorem 6.** Suppose that on one and the same two-dimensional manifold $R$ there are given metrics $\rho_s, \rho$ of bounded curvatures $\omega_s, \omega$, with the metrics
\( \rho_n \) converging uniformly to the metric \( \rho \) and the absolute curvatures \( \Omega_n \) of the metrics \( \rho_n \) bounded uniformly. Then the curvatures \( \omega_n \), given as set functions on Borel sets, converge locally weakly to \( \omega \), i.e., for every continuous function \( f(x) \), distinct from zero only on a set with compact closure, the equation

\[
\lim_{n \to \infty} \int f(x) \omega_n(dE) = \int f(x) \omega(dE)
\]

is valid.

Proof. 1. From Lemma 16, for the proof of Theorem 6 it is sufficient to verify that the Kolmogorov test is fulfilled, i.e., to prove that for each open \( \mathcal{G}_0 \) with compact closure and each closed \( F_0 \subset \mathcal{G}_0 \) the relation

\[
\lim_{n \to \infty} \inf_{F_0 \subset G \subset \mathcal{G}_0} |\omega(G) - \omega_n(G)| = 0
\]

is fulfilled.

Suppose that for some pair \( F_0 \subset \mathcal{G}_0 \), where \( \mathcal{G}_0 \) is compact, this last relation is not true, i.e., for some \( \varepsilon_0 > 0 \) and some subsequence \( n \to \infty \), to which we shall hold in what follows, keeping to the notation 1,2,\ldots,\( n \), the condition

\[
(30) \quad \inf_{F_0 \subset G \subset \mathcal{G}_0} |\omega(G) - \omega_n(G)| > \varepsilon_0
\]

is fulfilled.

2. From Lemma 7, on a compact set \( \mathcal{G}_0 \), from a uniformly bounded system \( \Omega_m \) it is possible to select a uniformly convergent subsequence. Keeping only to this subsequence and retaining for it the numeration 1,2,\ldots,\( n \), we will suppose that \( \Omega_n \to \Omega' \). Here, from Lemma 12, the function \( \Omega' \) will in its turn be a completely additive nonnegative charge, defined on the Borel subsets of the set \( \mathcal{G}_0 \).

3. Select an open set \( \mathcal{G} \) such that \( F_0 \subset G \subset \mathcal{G} \subset \mathcal{G}_0 \) and \( \Omega'(Fr. G) = 0 \). Such sets \( \mathcal{G} \) certainly exist. It suffices to consider various \( \xi \)-neighborhoods of the set \( F_0 \). For sufficiently small \( \xi \) they are contained in \( \mathcal{G}_0 \), and their boundaries are distinct for distinct \( \xi \). Because of the complete additivity of \( \Omega' \) only for a finite number of values of \( \xi \) can the value of \( \Omega' \) on these boundaries differ from zero.

Take now an arbitrarily small \( \varepsilon > 0 \). We choose a closed set \( F \) such that \( F_0 \subset F \subset G \), \( \Omega'(G - F) < \varepsilon \), \( \Omega(G - F) < 2\varepsilon \). This is possible since \( \Omega' \) and \( \Omega \) are charges.

Under these conditions, for sufficiently large \( n \), using property (18), we will have
\[ \Omega_n(G - F) < \Omega'(G - F) + \varepsilon < 2\varepsilon. \]

4. From Theorem 9 of Chapter V,
\[ \omega(G) = \lim_{P \subseteq G, F \subseteq G} \delta(P), \]
so that there exists a polygon \( P, \subseteq G \) such that for all polygons \( P \) with \( P, \subseteq P \subseteq G \)
\[ |\omega(G) - \delta(P)| < \varepsilon. \]

We shall moreover suppose \( P, \) chosen so that \( F \) is contained inside \( P, \)
\[ F \subseteq P, \subseteq G. \]

5. Select a polygon \( P \) so that \( P, \subseteq P \subseteq P \subseteq G. \) The polygon \( P \) may
be made up of arbitrarily many (but finitely many) connected components
and will be bounded by a finite number of simple closed polygonal curves
\( \gamma. \) For simplicity we shall deal in what follows with one of these, but
with the understanding that the construction is carried out simultaneously
for all of these polygonal curves.

We enclose each polygonal curve \( \gamma \) in a region \( K, \) homeomorphic to
a plane ring and entirely lying in \( G - P. \) We take the regions \( K, \)
for different \( \gamma \) to be nonoverlapping.

Suppose that \( 2\varepsilon, \) is the distance in the metric \( \rho \) from \( \gamma \) to the boundary
of \( K, \) and that \( 2\varepsilon, \) is the lower bound for the lengths of curves in \( K, \)
homologous to \( \gamma. \)

We arrange on \( \gamma \) a finite system of points \( A, \) in the following way.
First we mark the vertices of \( \gamma. \) Then on the sides of \( \gamma \) we mark off,
two at a time, points distant from the endpoints by distances less than
\( \min(\varepsilon, \varepsilon, \varepsilon). \) Suppose further that \( 2\varepsilon, \) is the smallest of the distances from
a link of the polygonal curve \( \gamma, \) shortened from the endpoints, to the
remaining links of \( \gamma. \) We decompose the already shortened links into
segment smaller than \( \min(\varepsilon, \varepsilon, \varepsilon, \varepsilon). \) Finally, we enumerate all the points
\( A \) thus marked off in cyclic order around the contour \( \gamma. \)

We join the points \( A, \) in each metric \( \rho, \) successively by shortest arcs
\( (A,A_{i+1}), \) drawing them so that two successive shortest arcs will have
only one common endpoint or only a common segment adjacent to that
point. Then we choose a subsequence \( n \) for which the thus-drawn shortest
arcs \( (A,A_{i+1}), \) on each piece converge to some shortest arc \( A,A_{i+1}, \) in the
metric \( \rho. \) In what follows we shall suppose that the numeration of \( n \)
refers only to this subsequence.

On a piece corresponding to one link of the polygonal curve \( \gamma, \) the
shortest arcs $A_iA_{i+1}$, from Theorem 1 of this chapter, constitute a single shortest arc in the metric $\rho$, so that they \textit{a fortiori} form a simple curve. Now close to a vertex of the polygonal curve $\gamma$ two shortest arcs $A_iA_{i+1}$ merging at that vertex may have a common piece adjacent to the vertex, as depicted on Figure 90 near the points $B_1, B_2$. In Figure 90 the broken curves denote a polygonal curve $\gamma$ and the continuous lines the shortest arcs $A_iA_{i+1}$. Thus, the limiting shortest arcs $A_iA_{i+1}$ in the metric $\rho$ bound a polygon $P_0$ with the indicated possible singularities close to those points of $A_i$ which were vertices of $\gamma$.

The broken curves $(A_iA_{i+1})_n$ converge to the contour $P_0$ and for sufficiently large $n$ bound a polygon $P_n$ which may have singularities of the indicated kind at each of the vertices $A_n$, as in Figure 91.

6. We encircle the vertices of the polygon $P_0$ by nonintersecting neighborhoods $U_i$. We select and hold to some fixed subsequence of $n$ for which the quantity $\Omega_n(U_i)$ approaches a limit for each $i$. Since for each $n$

$$\sum_i \Omega_n(U_i) \leq \Omega(G - F) < 2\varepsilon, \quad \sum_i \Omega(U_i) \leq \Omega(G - F) < 2\varepsilon,$$

then we may suppose that $n$ is so large that

$$\Omega_n(U_i) < \varepsilon_i, \quad \Omega(U_i) < \varepsilon_i,$$

where $\sum \varepsilon_i \leq 4\varepsilon$.

Then, by Lemma 18 on close sectors with common vertices, for sufficiently large $n$ we will have

$$|\delta(P_0) - \delta(P_n)| < \sum_i 9\varepsilon_i \leq 36\varepsilon.$$

7. If the sides of the polygon issuing from the vertex $B$ coincide on an initial piece $BB'$, and thus form a "protuberant tail" or a "reentrant tail", then, dropping the piece $BB'$, we replace the angle $\alpha$ of the sector of the polygon at the vertex $B$ by a new sector angle $\alpha'$ at the vertex.
$B'$, where in the case of a protuberant tail
\[ |\alpha - \alpha'| \leq \Omega(B') \]
and in the case of a reentrant tail
\[ |\alpha - \alpha'| \leq |\alpha - 2\pi| + |2\pi - \alpha'| \leq \Omega(B) + \Omega(B'). \]
Therefore, rejecting the tails from the polygons $P_0, P_\infty$, we obtain polygons $P'_0, P'_\infty$ for which
\begin{align*}
|\delta(P'_0) - \delta(P_0)| &\leq \Omega(G - F) < 2\varepsilon, \\
|\delta(P'_\infty) - \delta(P_\infty)| &\leq \Omega_\infty(G - F) < 2\varepsilon.
\end{align*}
Moreover, from Theorem 9 of Chapter VI, the excesses computed with respect to the sector angles of a polygon differ from its curvature by no more than the curvature at its vertices and the rotations of its sides. Hence
\[ |\delta(P'_0) - \omega_\infty(P'_0)| \leq \Omega_\infty(Fr. P'_0) \leq \Omega_\infty(G - F) < 2\varepsilon. \]
We note further that
\[ |\omega_\infty(P'_0) - \omega_\infty(G)| = |\omega_\infty(G - P'_0)| \leq \Omega_\infty(G - F) < 2\varepsilon. \]
8. Finally, by the last inequalities and inequalities (31)–(34) we have
\begin{align*}
|\omega(G) - \omega_\infty(G)| &\leq |\omega(G) - \delta(P'_0)| + |\delta(P'_0) - \delta(P_0)| + |\delta(P_0) - \delta(P_\infty)| \\
&+ |\delta(P_\infty) - \delta(P'_\infty)| + |\delta(P'_\infty) - \omega_\infty(P'_\infty)| + |\omega_\infty(P'_0) - \omega_\infty(G)| \\
&< \varepsilon + 2\varepsilon + 36\varepsilon + 2\varepsilon + 2\varepsilon + 2\varepsilon.
\end{align*}
Thus
\[ |\omega(G) - \omega_\infty(G)| < 45\varepsilon, \]
which with $45\varepsilon < \varepsilon$ contradicts inequality (30). Thus Theorem 6 is proved.

Remarks. 1) If under the conditions of Theorem 6 the sequence $\omega_\infty$ does not contain escaping loads, which a fortiori is the case for compact $R$, then from Lemma 17 and Theorem 6 it follows that $\omega_\infty \rightarrow \omega$ on all of $R$.

2) Under the conditions of Theorem 6 the requirement of boundedness of the curvature of the metric $\rho$ may be deduced from the uniform boundedness of the curvatures of the metrics $\rho_\infty$ (see Chapter IV).

5. Regular convergence.

14. Regularly converging metrics. Earlier we have proved that under uniform convergence of metrics with absolute curvatures bounded uniformly, the curvatures $\omega_\infty$ of these metrics converge locally weakly. But the positive and negative parts $\omega^+_\infty, \omega^-_\infty$ of these curvatures may fail to
converge locally weakly, and therefore a fortiori may fail to converge locally weakly to the functions $\omega^+, \omega^-$ for the limiting metric. We shall show this with a simple example.

Suppose that the surface $\Phi_n$ is an open plane square, at the middle $O$ of which there is a protuberance in the form of a lateral surface of a right circular cone with height $1/n$ and complete angle $\theta = \pi$ at the vertex.

The metric $\rho_n$ is defined as the intrinsic of the surface $\Phi_n$. As $n \to \infty$ the surfaces $\Phi_n$ and their metrics $\rho_n$ converge uniformly to the plane square $\Phi$ with its metric $\rho$. Evidently, on the limit surface $\omega^+ = \omega^- = 0$. On the surfaces $\Phi_n$ the positive curvature is concentrated at the point $O$, where $\omega_n^+(0) = \omega_n(0) = 2\pi - \theta = \pi$, and the negative part of the curvature, $\omega_n^-$, is concentrated along the curve of the base of the conical protuberance, where it also adds up to $\pi$. In the limit $\omega_n^+$ and $\omega_n^-$ absorb each other so to speak, and, in spite of the convergence $\omega_n \to \omega$, there is no convergence $\omega_n^+ \to \omega^+$ or $\omega_n^- \to \omega^-$.

**Definition.** We shall say that the metrics $\rho_n$ with curvatures $\omega_n, \omega_n^+$, $\omega_n^-$, $\Omega_n$ converge regularly to the metric $\rho$ with curvatures $\omega, \omega^+, \omega^-, \Omega$, if $\rho_n$ converges locally uniformly to $\rho$ and if we have also

\[
\omega_n^+ \to \omega^+, \quad \omega_n^- \to \omega^-.
\]

Evidently in this case also

\[
\omega_n \to \omega, \quad \Omega_n \to \Omega.
\]

In Theorem 7 we shall establish regular convergence under certain special approximations of metrics of bounded curvature. In this connection we shall find the following result of use.

**Lemma 19.** Suppose that the polygon $P$ in a metric $\rho$ of bounded curvature is subjected to a triangulation $Z$, i.e., is decomposed into triangles $t_i$. We replace each of the triangles $t_i$ by a plane triangle with sides of the same length. In accordance with the triangulation $Z$ there is constituted from the plane triangles a polyhedral development $P_Z$. Then

\[
(35) \quad \omega^+(P_Z) \leq \omega^+(P_-), \quad \tilde{\omega}^-(P_Z) \leq \tilde{\omega}^-(P),
\]

where $\omega^+(P_-)$ and $\omega^-(P_{\subset Z})$ are the positive parts of the curvature of the interior regions of $P$ and $P_Z$, and

\[
\tilde{\omega}^-(P) = \omega^-(P_-) + \sum_i \tau_i(a_i),
\]

\[
\tilde{\omega}^-(P_Z) = \omega^-(P_{\subset Z}) + \sum_i \tau_i(a_{id}),
\]
where $a_i$ are the sides of $P$, $a_{i2}$ polygonal curves in $P_2$ corresponding to the sides of $P$, $\tau_i$ the rotations from the side of $P$ and $P_2$.

Proof. At each interior vertex $A$ of the development $P_2$

\begin{equation}
\omega_2(A) = \omega(A) + \sum_j (\bar{\alpha}_j - \alpha_j^0),
\end{equation}

where $\bar{\alpha}_j$ are the angles of the sectors of the triangles $t_j$ adjacent to $A$, and $\alpha_j^0$ the corresponding angles in the plane triangles. We add the equations (36) over all those vertices $A^+$ at which $\omega_2(A) > 0$. Recalling from Theorem 11 of Chapter VI that

$$\sum_A \sum_j (\bar{\alpha}_j - \alpha_j^0) \leq \sum_j \omega^+(t_{i-1}),$$

we thus obtain the first of inequalities (35).

For a vertex $B$ of the development $P_2$ which lies on the boundary of $P_2$ inside the polygonal curve corresponding to the side of $P$ we have

\begin{equation}
\tau_2(B) = \bar{\phi}_2 - \pi = \bar{\phi}_i - \pi + \sum_k (\bar{\alpha}_k - \alpha_k^0),
\end{equation}

where $\bar{\phi}_2$ and $\bar{\phi}$ are the sector angles of $P_2$ and $P$ at the vertex $B$, $\bar{\alpha}_k$ the angles of the sectors of the triangles $t_k$ adjacent to $B$, and $\alpha_k^0$ the corresponding angles of the plane triangles.

Add equations (36) over the interior vertices $A^-$ at which $\omega_2(A) < 0$, and equations (37) over the vertices $B^-$ at which $\tau_2(B) < 0$. Recalling from Theorem 11 of Chapter VI

$$\sum_A \sum_j (\bar{\alpha}_j - \alpha_j^0) + \sum_B \sum_k (\bar{\alpha}_k - \alpha_k^0) \geq - \sum_i \bar{\omega}^-(t_i),$$

we obtain after a change of sign

$$- \sum_A \omega_2(A) - \sum_B \tau_2(B) \leq - \sum_A \omega(A) - \sum_B \tau(B) + \sum_i \bar{\omega}^-(t_i),$$

from which follows the second of the inequalities (35).

Lemma 19 is proved.

Theorem 7. If the polyhedral metrics $\rho_n$, locally weakly converging to a metric $\rho$ of bounded curvature, are obtained from $\rho$ by rectifying triangles for ever finer triangulations $Z_n$, as was described in subsections 16, 17 of Chapter III and subsections 1–3 of Chapter IV, then $\rho_n$ converges regularly to $\rho$, i.e.,

$$\omega_n^+ \rightarrow \omega^+, \quad \omega_n^- \rightarrow \omega^-.$$

We shall carry out all the discussion within the limits of an arbitrary
region $Q$ with compact closure. In making up the developments the region $Q$ is topologically mapped on a region in a space with a polyhedral metric. We shall suppose that the polyhedral metrics $\rho_n$ are simply given in the same region $Q$.

From Theorem 6, within the limits of the region $Q$ we have weak convergence $\omega_n \to \omega$. We assert that in the case at hand also $\omega^+_n \to \omega^+$. From Lemma 11, for the proof of this fact it suffices to prove that for each open $G \subseteq \overline{G} \subseteq Q$ for which $\omega^+(\text{Fr. } G) = 0$ we have

$$\lim_{n \to \infty} \omega^+_n (G) = \omega^+(G).$$

But because of the general Lemma 9, from the weak convergence $\omega_n \to \omega$ it follows that

$$\liminf_{n \to \infty} \omega^+_n (G) \geq \omega^+(G),$$

so that it is sufficient for us to prove that if $\omega^+(\text{Fr. } G) = 0$ we have the reverse inequality

$$\limsup_{n \to \infty} \omega^+_n (G) \leq \omega^+(G).$$

Because of the complete additivity of $\omega^+$, for any $\varepsilon > 0$ in $Q$ there is a neighborhood $G$, such that $\overline{G} \subseteq G \subseteq Q$ and $\omega^+(G, \overline{G}) < \varepsilon$. For sufficiently large $n$ the triangulations $Z_n$ are made so fine that from them we may make up polygons $P_n$ for which $G \subseteq P_{(\rightarrow n)} \subseteq P_n \subseteq G$. Then we will have

$$\omega^+(G) = \omega^+(\overline{G}) > \omega^+(P_{(\rightarrow n)}) - \varepsilon.$$

Moreover, as we know from Lemma 19, $\omega^+(P_n) \geq \omega^+_n (P_{(\rightarrow n)})$, so that the preceding inequality may be put in the form

$$\omega^+(G) > \omega^+_n (P_{(\rightarrow n)}) - \varepsilon \geq \omega^+_n (G) - \varepsilon.$$

From this last inequality, because of the arbitrary smallness of $\varepsilon > 0$, relation (40) follows, which along with inequality (39) gives equation (38) and proves Theorem 7.

**Theorem 8.** If under the conditions of Theorem 7 one and the same polygon $P$ undergoes the triangulations then

$$\omega^+(P_{-}) = \lim_{n \to \infty} \omega^+_n (P_{-}),$$

$$\omega^{-}(P) = \lim_{n \to \infty} \omega^{-}_n (P),$$

where

$$\omega^{-}(P) = \omega^{-}(P_{-}) + \sum_{i} \tau^{-}(a_i) \quad (a_i \text{ a side of } P),$$
\[ \omega^-(P) = \omega_n^-(P_\cdot) + \sum \tau^-(a_n) \]

(_a_n - polygons corresponding to the sides of P_)

**Proof.** From the general Lemma 9,

\[ \omega^+(P_\cdot) \leq \liminf_{n \to \infty} \omega_n^+(P_\cdot), \]

and from Lemma 19 in the case at hand

\[ \omega^+(P_\cdot) \geq \omega_n^+(P_\cdot). \]

(41) follows from these two inequalities.

From the generalized Gauss-Bonnet theorem,

\[ \delta(P) = \omega(P) + \sum \tau(a) = \omega^+(P) - \omega^-(P) - \sum \tau^-(a) = \omega^+(P) - \omega^-(P). \]

Analogously, if in considering P in the metric \( \rho_n \) the \( a_n \) are regarded as complete "sides", then we will have

\[ \delta_n(P) = \omega_n(P) + \sum \tau(a_n) = \omega^+(P) - \omega^-(P) + \sum \tau^+(a_n) - \sum \tau^-(a_n) \]

\[ = \omega^+(P) - \omega^-(P) + \sum \tau^+(a_n). \]

But by the construction at each vertex of P we have convergence of the corresponding sector angles, so that

\[ \lim_{n \to \infty} \delta_n = \delta(P). \]

Moreover, the positive rotations \( \tau^+(a_n) \) could appear only at the expense of a decrease in the sector angles for the swung triangles, adjacent to the boundary. These contractions do not exceed the positive curvature of the developed triangles, more precisely of their interior regions. But for large \( n \) we are dealing with triangles whose interior regions lie in vanishing neighborhoods of the boundary P, where in view of the complete additivity of \( \omega^+ \) the values of \( \omega^+ \) become small. Therefore

\[ \lim_{n \to \infty} \sum \tau^+(a_n) = 0. \]

Finally, we have already proved equation (41).

From equations (43), (44), and the limiting relations (45), (46), and (41) follows the validity of relation (42). The theorem is proved.

**Remark.** Because of equation (46), we may consider \( \sum \tau(a_n) \) in place of \( \sum \tau^-(a_n) \).

15. **Variation of the angle \( \gamma \).** The regular approximations described in subsection 14 are an important means of investigation of general metrics.
As an example we consider for any metric the variation of the angle \( \gamma \), which for the polyhedral metric was studied in §2 of Chapter IV.

Suppose that \( T \) is a reduced triangle without interior tails in a two-dimensional manifold of bounded curvature. We shall take \( T \) to be convex, so that all distances are measured in \( T \) itself. This may always be done by pasting the triangle \( T \) into a plane in the place of the plane triangle \( T_0 \) having sides of the same length.

To each pair of points \( X \pm A, \ Y \pm A \) on the sides \( AB, \ AC \) of the triangle \( T \) there corresponds an angle \( \gamma(X, Y) \) at the vertex \( A_0 \) in the plane triangle with sides \( AX, \ AY, \ XY \).

**Theorem 9.** If the pairs of points \( X_i, Y_i \) \( (i = 1, \ldots, r + l) \) form on the sides \( AB, \ AC \) of the triangle \( T \) an increasing sequence \( i.e., \ X_{i+1} \in [X, B], \ Y_{i+1} \in [Y, C] \), then the sum of the positive increments of the angle \( \gamma_T \) does not exceed \( \bar{\omega}^- (T) \):

\[
\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^+ \leq \bar{\omega}^- (T),
\]

and the sum of the absolute values of the negative increments does not exceed \( \omega^+ (T) \):

\[
\sum_{i=1}^{r} (\gamma_T(X_{i+1}, Y_{i+1}) - \gamma_T(X_i, Y_i))^- \leq \omega^+ (T).
\]

**Proof.**

1. We carry out some triangulation \( Z_n \) of the triangle \( T \) into triangles of diameters less than \( d_n = 1/n \). Over these triangles we construct a polyhedral development \( Q_n \). From Lemma 19,

\[
\omega^+ (Q_{(\to \infty)}) \leq \omega^+ (T), \quad \bar{\omega}^- (Q_n) \leq \bar{\omega}^- (T).
\]

2. The polygonal curve \( \widehat{AB} \), which corresponds in \( Q \) to the side \( AB \) of the triangle \( T \), might turn out not to be a shortest arc in \( Q_n \). However, from Theorem 9 of Chapter III, it cannot exceed in length that of a shortest arc in \( Q_n \) by more than \( \varepsilon < C \). In this case we construct a plane isosceles triangle with the sum of its lateral sides equal to \( \widehat{AB} \) and a base equal to a shortest arc in \( Q_n \). Suppose that \( \alpha_{AB} \) is the exterior angle at a vertex of this triangle. We paste this triangle along the lateral sides to the broken curve \( \widehat{AB} \) in \( Q_n \). We carry out if necessary an analogous pasting for the polygonal curves \( \widehat{AC} \) and \( \widehat{BC} \). We obtain a polyhedral triangle \( Q'_n \), in which the sides are already shortest arcs. The original broken curves \( \widehat{AB}, \widehat{AC}, \widehat{BC} \) compare uniformly with them in length. Here evidently we will have
\[ \omega^+(Q_n^0) \leq \omega^+(T_\sim) + 3 \beta_n, \quad \omega^-(Q_n^0) \leq \omega^-(T_\sim) \]

where

\[ \beta_n = \max(\alpha_{AB}, \alpha_{AC}, \alpha_{BC}). \]

3. If we carry out the indicated constructions for ever finer triangulations \( Z_n \), then the metrics in the developments \( Z_n \) will converge to the metric in \( T \). Therefore \( \gamma_{Q^0}(X_m, Y_1) \to \gamma(T)(X_n, Y_1) \).

4. Using for \( Q'_n \) Theorem 1 of Chapter IV, and also taking account of inequalities (49) and the fact that \( \beta_n \to 0 \), we obtain assertions (47) and (48) of Theorem 9.

Remark. It follows from Theorem 9 that for any monotone variation \( X(t), Y(t) \) from \( A \) towards \( B \) and from \( A \) towards \( C \) along \( AB, AC \) the function \( \gamma(t) = \gamma(T)(X(t), Y(t)) \) has bounded variation, with

\[ \Var^+ \gamma(t) \leq \omega^-(T), \]
\[ \Var^- \gamma(t) \leq \omega^+(T_\sim). \]

These results strengthen the assertions of Theorem 10 of Chapter VI.

Analogously Theorem 9 may be used to extend Theorems 2 and 3 of Chapter IV to the case of nonpolyhedral metrics.


16. Angles of converging sectors. Suppose that within the limits of a two-dimensional region \( G \) there are defined metrics \( \rho_n, \rho \) of bounded curvature, with \( \rho_n \) uniformly converging to \( \rho \). Suppose moreover that in the metric \( \rho \) the points \( O_n \to O \) and the shortest arcs \( L_n, M_n \) issuing from the points \( O_n \) in the metrics \( \rho \) converge as curves in the metric \( \rho \) to shortest arcs \( L, M \) issuing from \( O \). We suppose moreover that each pair of shortest arcs \( L_n, M_n \) and \( L, M \) either has no common points besides a common endpoint \( O_n \) or an initial segment adjacent to it, and thus subdivide a small neighborhood of \( O_n, O \) into two sectors. Let \( V_n \) and \( V \) be distinguished sectors in the order of circuiting the vertex \( O_n \) or \( O \) from \( L_n \) or \( L \) to \( M_n \) or \( M \). Then \( V_n \to V \) in the sense of Definition 3' in subsection 3 of this chapter.

Let \( G'_n \) be any neighborhood of the point \( O_n \) homeomorphic to the disc, with all the points of \( G'_n \) distant from \( O_n \) by not more than \( \varepsilon \). Suppose that \( V'_n \) is the portion of \( G'_n \) related to the sector \( V_n \). We introduce the following characteristics of the subsequence \( \{V_n\} \):
\[
\begin{align*}
\omega^*(V_n) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \omega^+_n(V_n^\varepsilon), \\
\omega^-(V_n) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \omega^-_n(V_n^\varepsilon), \\
\bar{\omega}^-(V_n) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \bar{\omega}^-_n(V_n^\varepsilon),
\end{align*}
\]

where
\[
\bar{\omega}^-_n(V_n^\varepsilon) = \omega^-_n(V_n^\varepsilon) + \tau^-_n(L_n \cap V_n^\varepsilon) + \tau^-_n(M_n \cap V_n^\varepsilon),
\]
\(\tau^-_n\) being the rotation on the side of \(V_n\) in the metric \(\rho_n\).

**Theorem 10.** If, as described above, \(\rho_n \to \rho\), \(O_n \to O\), \(L_n \to L\), \(M_n \to M\), \(V_n \to V\), then for the angles \(\bar{\alpha}_n\) and the sectors \(V_n, V\) the following relations hold:
\[
\limsup_{n \to \infty} \bar{\alpha}_n - \omega^*(V_n) \leq \bar{\alpha} \leq \liminf_{n \to \infty} \bar{\alpha}_n + \bar{\omega}^-(V_n).
\]

**Proof.** First suppose that \(\bar{\alpha} < \pi\). Then the excision of the sector \(V\) from the space with metric \(\rho\) generates a metric \(\rho V\), in which the angle between the sides is equal to \(\bar{\alpha} = \min[\bar{\alpha}, \pi] = \bar{\alpha} < \pi\). Therefore for any points \(X,Y\) on the sides of \(V\) any shortest arc \(XY\) joining them in \(V\) cannot pass through \(O\). Let \(OX = x\), \(OY = y\), \(XY = z\), and \(\alpha^\pi(x,y)\) be the corresponding angle in the plane triangle with sides \(x, y, z\).

We excise analogously from spaces with metrics \(\rho_n\) sectors \(V_n\), and obtain in them induced metrics \(\rho_V\). From Theorem 4, the induced metrics converge uniformly: \(\rho_{V_n} \to \rho_V\). Suppose that \(X_n, Y_n\) are points on \(L_n, M_n\) corresponding in parameter on the shortest arcs to the points \(X, Y\) on \(L, M\). The relative shortest arcs \(X_nY_n\) in the sectors \(V_n\), beginning with some \(n\), cannot pass through \(O_n\), for otherwise the limit of a subsequence of these shortest arcs would lead to a shortest arc \(XY\) passing through \(O\).

We mark off \(X_nY_n\), excising from \(V_n\) reduced triangles \(T_n\), and we develop \(T_n\) onto the plane. Then from Theorem 10 of Chapter VI,
\[
\bar{\alpha}_n - \omega^+_n(T_n) \leq \alpha^\pi_n(x, y) \leq \bar{\alpha}_n + \bar{\omega}^-_n(T_n).
\]

For \(x\) and \(y\) small in comparison with \(\varepsilon\) we have moreover
\[
\bar{\alpha}_n - \omega^+_n(V_n) \leq \alpha^\pi_n(x, y) \leq \bar{\alpha}_n + \bar{\omega}^-_n(V_n).
\]

We pass in the left inequality to the limit through the same subsequence of \(n\) for which \(\limsup \bar{\alpha}_n\) is realized, and in the right inequality through that subsequence for which \(\liminf \bar{\alpha}_n\) is realized. Then we strengthen these inequalities by passing to the least upper bounds with respect to
$V_n$ and the lower limit as $n \to \infty$ in the remaining terms. Finally, we let $x, y,$ and $z$ tend to zero. Then, recalling that $\lim_{n \to \infty} \alpha^b(x, y) = \bar{\alpha} = \bar{\alpha}$, we obtain inequality (52).

Now we suppose that $\bar{\alpha} > \pi$. We subdivide the sector $V$ by a finite number of shortest arcs $L_k$ into sectors with angles $\beta$'s, with $0 < \beta < \pi$. Then we choose a $\beta$ for which the condition $0 < 2 \beta < \bar{\alpha} < \pi - 2 \beta$ is satisfied for all $k$. Finally, on each of the shortest arcs $L_k$ we mark a point $A^k$ so close to $O$ that no pair of shortest arcs $A^kO, A^kO$ can form at the vertex $O$ an angle larger than $\beta$. For sufficiently large $n$ all the points $A^k$ fall inside $V_n$. In what follows we will suppose that this construction is carried out separately for the subsequences of $n$ realizing $\lim \sup_{n \to \infty} \bar{\alpha}_n$ and $\lim \inf \bar{\alpha}_n$. We choose a subsequence for which all the $L_k$ converge to some shortest arc $A^kO = L'$. These shortest arcs may fail to coincide with segments of the shortest arcs $L_k$, but because of the special choice of the points $A^k$ they also will subdivide the sector $V$ into sectors with the angles $0 < \bar{\alpha}' < \pi$, while these sectors will be limiting for the sectors $V_n$.

Because the characteristics $\omega^+ (V_n^+), \omega^- (V_n^-)$, in distinction from $\omega^+ (V_n^-)$, $\omega^- (V_n^+)$, are not additive under combination of sectors, we may not use directly the result (52) for separate sectors $\bar{\alpha}'$. But we turn to inequalities (53), true for each $k$:

$$\bar{\alpha}_n^k - \omega^+ (T_{k-1}^-) \leq \alpha_n^k (x, y) \leq \bar{\alpha}_n^k + \omega^- (T_k^+).$$

For simplicity we shall suppose that for each $k$ we have chosen one and the same $x = y$. Adding these inequalities with respect to $k$ and using the additivity of $\omega^+$ and $\omega^-$ under adjoinment of the $T_k^+$ along entire sides, we obtain

$$\bar{\alpha}_n^k - \omega^+ (V_{k-1}^-) \leq \sum \alpha_n^k (x, y) \leq \bar{\alpha}_n^k + \omega^- (V_n^+).$$

Passing as in inequality (54) to the limit as $n \to \infty$ on the left through the subsequence realizing $\lim \sup \bar{\alpha}_n$ and on the right $\lim \inf \bar{\alpha}_n$, then letting $x, y$ and $\varepsilon$ tend to zero, and taking account of the fact that $\sum \bar{\alpha}' = \sum \bar{\alpha} = \bar{\alpha}$, we obtain also in this case inequality (52).

Theorem 10 is proved.

**Corollary 1.** If we denote by $W_n$ and $W$ the sectors which serve as complete neighborhoods of the points $O_nO$ with the corresponding points $O_nO$ deleted, then evidently

$$\lim \sup \bar{\alpha}_n^k - \omega^+ (W_n) \leq \bar{\alpha} \leq \lim \inf \bar{\alpha}_n + \omega^- (W_n).$$
COROLLARY 2. For the complete angles \( \theta_n, \theta \) around the points \( O_n, O \) the following inequality is also valid:

\[
\limsup_{n \to \infty} \theta_n - \omega^+ (W_n) \leq \theta \leq \liminf_{n \to \infty} \theta_n + \omega^- (W_n).
\]

THEOREM 11. Under the hypotheses of Theorem 10, i.e., as \( n \to \infty \), \( O_n \to O \), \( L_n \to L \), \( M_n \to M \), where \( L_n, M_n \) and \( L, M \) bound sectors with vertices \( O_n, O \), for the angles \( \alpha_n, \alpha \) between \( L_n, M_n \) and \( L, M \) the following inequality holds:

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ (W_n) \leq \alpha \leq \liminf_{n \to \infty} \alpha_n + \omega^- (W_n).
\]

PROOF. Suppose that \( \tilde{\alpha}_n^1, \tilde{\alpha}_n^2 \) and \( \alpha_n^1, \alpha_n^2 \) are the angles of the corresponding sectors with the sides \( L_n, M_n \), and \( L, M \). We know that

\[
\alpha_n = \min \{ \tilde{\alpha}_n^1, \tilde{\alpha}_n^2, \pi \}, \quad \alpha = \min \{ \alpha_n^1, \alpha_n^2, \pi \},
\]

so that it is not difficult to obtain Theorem 11 from Theorem 10.

First we prove the right inequality (57). If \( \lim \inf_{n \to \infty} \alpha_n = \pi \), then that inequality is trivial. Suppose \( \lim \inf_{n \to \infty} \alpha_n < \pi \). Then in the sequence realizing \( \lim \inf \alpha_n \), beginning with some \( n \), all \( \alpha_n < \pi - \varepsilon \). In this sequence we may select a subsequence \( n_k \) for which some of the sectors, which we will denote by \( \tilde{\alpha}_n^1 \), satisfy \( \tilde{\alpha}_n^1 = \alpha_n \). Then

\[
\alpha \leq \tilde{\alpha}^i \leq \liminf_{k \to \infty} \tilde{\alpha}_n^1 + \omega^- (W_n) = \liminf_{n \to \infty} \alpha_n + \omega^- (W_n),
\]

which proves the right inequality in (57).

The proof of the left inequality (57) is somewhat more complicated. We hold to that sequence of \( n \) for which \( \limsup_{n \to \infty} \alpha_n \) is realized.

If that limit is equal to \( \pi \), then for any \( \varepsilon > 0 \) beginning with certain \( n \), all \( \alpha_n \geq \pi - \varepsilon \), \( \tilde{\alpha}_n^1 \geq \pi - \varepsilon \), \( \tilde{\alpha}_n^2 \geq \pi - \varepsilon \). Then from Theorem 10

\[
\pi - \varepsilon - \omega^+ (W_n) \leq \tilde{\alpha}^1,
\]

\[
\pi - \varepsilon - \omega^+ (W_n) \leq \tilde{\alpha}^2,
\]

\[
\pi - \varepsilon - \omega^+ (W_n) \leq \pi,
\]

from which, from (58), we have

\[
\pi - \varepsilon - \omega^+ (W_n) \leq \alpha,
\]

which, because of the arbitrariness of \( \varepsilon > 0 \), yields the left inequality (57).

If now \( \lim \sup_{n \to \infty} \alpha_n < \pi \), then, beginning with some \( n \), all the \( \alpha_n < \pi \). In the sequence realizing \( \lim \sup \alpha_n \) we select a subsequence of the \( n_k \) for which \( \tilde{\alpha}_n^1 = \alpha_n \). Then from Theorem 10

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ (W_n) = \limsup_{n \to \infty} \tilde{\alpha}_n^1 + \omega^+ (W_n) \leq \tilde{\alpha}^1,
\]

\[
\limsup_{n \to \infty} \alpha_n - \omega^+ (W_n) = \limsup_{k \to \infty} \tilde{\alpha}_n^2 - \omega^+ (W_n) \leq \limsup_{k \to \infty} \tilde{\alpha}_n^2 - \omega^+ (W_n) \leq \tilde{\alpha}^2,
\]
so that the left inequality in (57) follows by (58).

Theorem 11 is proved.

17. Special cases of convergence of angles. The assertions enumerated below follow directly from Theorems 10 and 11 and the complete additivity of $\omega^+$ and $\omega^-$.

**Corollary 1.** If $\omega^+ [W_n] = \omega^- [W_n] = 0$, then from inequalities (55)—(57) it follows that in this case

$$\bar{\alpha}^1 = \lim_{n \to \infty} \bar{\alpha}^1_n, \quad \bar{\alpha}^2 = \lim_{n \to \infty} \bar{\alpha}^2_n, \quad \theta = \lim_{n \to \infty} \theta_n, \quad \alpha = \lim_{n \to \infty} \alpha_n.$$  

**Corollary 2.** If all the shortest arcs $L_n, M_n$ are drawn in one and the same metric, i.e., $\rho_n \equiv \rho$, and moreover from one and the same point, then from the complete additivity of $\omega^+$ and $\omega^-$ it follows that the hypotheses of Corollary 1 are satisfied, and therefore (59).

**Corollary 3.** If $\rho_n \equiv \rho$ and $\omega(O) = 0$, then the hypotheses of Corollary 1 are also satisfied and (59) is true.

**Corollary 4.** If $\rho_n \equiv \rho$ and all the sectors $V_n$ do not contain the point $O$ in their interior or on their sides, then because of the complete additivity of the curvature $\omega^+ [V_{\infty}] = \omega^- [V_n] = 0$ and therefore (52) implies

$$\bar{\alpha} = \lim_{n \to \infty} \bar{\alpha}_n.$$  

**Corollary 5.** If $\rho_n \equiv \rho$ and all the sectors $V_n$ contain the point $O$ inside themselves, then the hypotheses of Corollary 4 are satisfied for the complementary sectors. For these inequality (52) is satisfied. Hence for the angles of the sectors $V_n, V$ themselves, we have

$$\bar{\alpha} = \lim_{n \to \infty} \bar{\alpha}_n - \omega(O).$$  

**Corollary 6.** If $\rho_n \equiv \rho$ and $O$ might lie on the boundary of $V_n$, then always $\omega^+ [V_n] = 0$, $\omega^- [V_n] \leq -\omega(O)$, so that if $\rho_n \equiv \rho$ always

$$\lim \sup_{n \to \infty} \bar{\alpha}_n \leq \bar{\alpha} \leq \lim \inf_{n \to \infty} \bar{\alpha}_n - \omega(O).$$  

**Corollary 7.** If all the metrics $\rho_n$ have only positive curvature, then $\omega^- [V_n] = 0$ and it follows from (52) that

$$\bar{\alpha} \leq \lim \inf_{n \to \infty} \bar{\alpha}_n.$$  

If all the $\rho_n$ are of negative curvature, then $\omega^+ [V_n] = 0$ and

$$\bar{\alpha} \geq \lim \sup_{n \to \infty} \bar{\alpha}_n.$$