CHAPTER VIII

Area

1. Triangles in polyhedral metrics.

1. Preparatory remarks. Suppose that $T$ is a triangle in a polyhedral metric. Excising $T$ and pasting it into the plane in the place of the plane triangle having sides of the same length, we may consider $T$ as a convex triangle in the resulting new polyhedral metric.

**Lemma 1.** If the convex triangle $T$ in a polyhedral metric contains within itself no vertex of positive curvature, then the diameter $d$ of the triangle $T$ is equal to the length of the largest of its sides.

**Proof.** In the absence of vertices of positive curvature any two points may be joined in $T$ by a unique shortest arc, and each shortest arc may be prolonged to the boundary in $T$. Otherwise there would be a two-gon in $T$, in which there must be at least one vertex of positive curvature. Suppose $X, Y \in T$ are those points for which $\rho(X, Y) = d$. From what has been said above it follows that $X$ and $Y$ lie on the boundary of $T$. Suppose that one of these points, say $T$, does not coincide with a vertex of $T$. Then at the point $Y$ the shortest arc $XY$ forms with the side $AC$ angles $\xi + \eta \geq \pi$, as depicted in Figure 92. Suppose for definiteness that $\eta \geq \pi/2$. Shifting $Y$ along the boundary of $T$ a short distance to a close position $Y'$, we increase the length of $XY$, thus contradicting the equation $\rho(X, Y) = d$. Consequently $X$ and $Y$ coincide with vertices of $T$, and Lemma 1 is proved.

**Lemma 2.** Suppose that a two-gon $D$ is excised from a compact region $\mathcal{G}$ in a two-dimensional manifold of bounded curvature. On identifying the edges of $D$ we obtain a new region $\mathcal{G}'$. When this is done the diameter of the region does not increase:
\[ d(\overline{G}') \leq d(\overline{G}). \]

**Proof.** In studying the diameters it suffices to consider the shortest possible curves joining pairs of points. The length of each curve in \( \overline{G} \) which does not pass within \( D \) is preserved in \( \overline{G}' \). Suppose that a curve joining the points \( X \) and \( Y \) has a piece \( MN \) in \( D \), as in Figure 93. We then may suppose that \( MN \) is a shortest arc in \( D \). Then from the triangle \( MON \) follows \( MN + NO \geq MO \), so that \( MN' = MO - NO \leq MN \), so that after excision of \( D \) the points \( X, Y \) are joined by a curve \( XM(N'N)Y \) in \( \overline{G}' \) of no greater length than \( XMNY \). Hence \( d(\overline{G}') \leq d(\overline{G}) \).

**Lemma 3.** Suppose that a “triangle” \( T \) is situated on the plane, its sides being broken curves convex toward the interior of \( T \). (Not excluding the possibility that the pieces of the sides adjacent to a vertex may coincide, as depicted on Figure 92 at the vertex \( B \).) Suppose that \( \sigma(T) \) is the area of \( T \) and \( \sigma(T_0) \) the area of the ordinary plane triangle with sides of the same lengths as the broken curves serving as the “sides” of \( T \). Then

\[ 0 \leq \sigma(T_0) - \sigma(T) \leq \frac{1}{2} \bar{\omega}^{-}d^2, \tag{1} \]

where \( \bar{\omega}^{-} \) is the total negative rotation of the sides of \( T \), in absolute value, and \( d \) is the intrinsic diameter of \( T \).

**Proof.** 1. Excision of \( T \) induces in \( T \) an intrinsic metric in which the distance between vertices coincides with the lengths of the “sides” of \( T \). Therefore these three lengths satisfy the triangle inequality and \( T_0 \) may be constructed.

2. As shown in sub-
section 9 of Chapter II, in a plane quadrilateral with reentrant angle $\delta$ and three salient angles $\alpha, \beta, \gamma$ as in Figure 94, if we rectify the reentrant angle $\delta$ each of the angles $\alpha, \beta, \gamma$ increases. Moreover, for a small partial increase $\Delta \delta$ we will have all of $\Delta \alpha, \Delta \beta, \Delta \gamma > 0$. This is easily verified by first straightening the polygon CDM (Figure 94) and then forming the broken curve BMA.

Even a slightly rectified quadrilateral contains the original quadrilateral, cut along $AD$, as shown in Figure 95. Therefore the increase of the area $\Delta \sigma > 0$. With the notations indicated in Figure 95, we have for a differential change of area

$$d\sigma = \frac{1}{2} a^2 d\alpha + \frac{1}{2} b^2 d\beta + \frac{1}{2} c^2 d\gamma.$$ 

Hence

$$d\sigma \leq \frac{1}{2} (d\alpha + d\beta + d\gamma) [\max(a, b, c)]^2 = \frac{1}{2} d\delta [\max(a, b, c)]^2.$$

(2)

3. Now we shall prove Lemma 3. Rectification of $T$ into $T_0$ may be carried out by rectifying in turn the reentrant angles at the vertices of the polygons serving as the "sides" of $T$. From Lemma 1, the diameter of the transformed triangle remains equal to the length of the largest of its sides. Therefore all the time

$$d\sigma \leq \frac{1}{2} d^2 d\delta.$$ 

But $d\delta > 0$ and $d\delta = -d\omega^\gamma$. Therefore inequality (1) follows from (2).

Lemma 3 is proved.

3. **Comparison with plane triangles.** Suppose that $T$ is a triangle in the polyhedral metric, $\omega^+ \omega^-$ the total positive curvature of the vertices lying within $T$, $\omega^+$ the absolute value of the total negative curvature of the vertices lying within $T$, augmented by the rotation of the sides of $T$ on the side of the triangle. The area $\sigma(T)$ is defined as the total area of all the plane pieces of which $T$ consists, and $\sigma(T_0)$ the area of the plane triangle $T_0$ with sides of the same length as $T$.

**Theorem 1. For every triangle $T$ in the polyhedral metric**

$$\frac{1}{2} \omega^- d^2 \leq \sigma(T) - \sigma(T_0) \leq \frac{1}{2} \omega^+ d^2,$$

(3)
where $d$ is the intrinsic diameter of $T$.

**Proof.** Without loss of generality we may suppose $T$ convex. We consider separately four cases.

**Case 1.** If there is no vertex of the metric inside $T$, Theorem 1 follows from Lemma 3. In this case $T$ is isometric to the figure indicated in Lemma 3.

**Case 2.** Suppose that within $T$ there are only vertices with negative curvature. From the connection between the curvature and the rotation of the sides with the excess of the triangle,

$$
\delta(T) + \omega^+(T_+) - \omega^-(T_-) - \sum_{i=1}^{3} \tau_i = \bar{\alpha} + \bar{\beta} + \bar{\gamma} - \pi \geq -\pi
$$

we conclude that if $\omega^+(T_+) = 0$ always $\omega^-(T_-) \leq \pi$. Therefore for any of the $n$ interior vertices $O$ we have $-\pi \leq \omega(O) < 0$.

If $n = 0$ the theorem is already proved. Suppose that it is true for $(n-1)$ vertices. We shall prove it for $n$ vertices. Suppose that $O$ is the closest of these vertices to $A$. Draw the shortest arc $AO$ and extend it through the point $O$ so that it bisects the angle $2\pi + 2\alpha$ of the complete sector around the point $O$ and, further, in passing any vertices with negative curvature leaves a sector $\pi$ to the right of itself. Thus we prolong $AO$ as a shortest arc $AP$, where $P$ lies on the boundary of $T$. (We observe that $AP$ remains a shortest arc and $P \neq B, P \neq C$, for otherwise there would be a two-gon containing inside itself vertices with curvature $\omega^+ > 0$.) We make a cut along $AP$ and paste along it the plane quadrilateral depicted in Figure 96 with sides of lengths $O_1A = O_2A = OA$, $O_1P = O_2P = OP$ and obtuse angles $\pi - \alpha$ at the vertices $O_1, O_2$.

As is clear from Figure 96, the pasted area

$$
\Delta \sigma < \frac{1}{2} AP \xi < \frac{1}{2} (AO + OP) \xi \leq \frac{1}{2} \xi d^2 = \frac{1}{2} \Delta \omega^{-} d^2.
$$

where $d$ is the diameter of $T$, which from Lemma 1 does not change under the pasting. After the pasting the number of vertices decreases by exactly one, which along with the induction hypothesis and the inequality $\Delta \sigma < (1/2) \Delta \omega^{-} d^2$ proves Theorem 1 for Case 2.

**Case 3.** Suppose that inside $T$ there are also vertices of the metric with $\omega^+ > 0$. Suppose that some pair of vertices of the triangle $T$ may
be joined in \( T \) by a shortest arc distinct from a side of \( T \). Then there exists a two-gon \( D_1 \) with vertices \( M, N \), (Figure 97) which does not contain other shortest arcs \( MN \). Inside \( D_1 \) there is at least one vertex of the metric with \( \omega^+ > 0 \).

The proof of the theorem in Case 3 will be carried out by induction on the number \( n \) of vertices with \( \omega^+ > 0 \) inside \( T \). Suppose that \( O_i \) is one such vertex, lying in \( D_i \). If \( O_i \) can be joined in \( D_i \) to \( M \) by more than one shortest arc, then in \( D_i \) there will appear a new two-gon \( D_2 \) with vertices \( M \) and \( O_i \). Again inside this there is a vertex \( O_i \) with \( \omega^+ > 0 \). Thus in a finite number of steps we arrive at a vertex \( O_k \) with \( \omega^+ > 0 \) which is joined to \( M \) in the corresponding two-gon \( D_k \subset D \) by a unique shortest arc \( MO_k \).

Continue the segment \( O_kS \) so that it bisects along with \( MO_k \) the complete angle of the sector around \( O \), as in Figure 98. For small \( O_kS \) the point \( S \) will be joined in \( D_k \) to \( M \) by exactly two shortest arcs, which pass to the right and left of \( MO_k \). Suppose that \( R' \) and \( R'' \) are the points at which these shortest arcs depart from \( MO_k \). If we excise the two-gon \( MR'SR''M \) thus formed and identify its edges, then the vertex \( O_k \) is moved into the point \( S \), the positive curvature decreases by no less than \( \alpha + \beta \) (Figure 98), and the area of the polygon decreases by the sum of the areas of the two plane triangles \( SR'O_k, SR''O_k \). Since the sides of these triangles do not exceed the diameter of the whole figure, and the latter from Lemma 2 remains for such excisions less than or equal to \( d \), where \( d \) is the intrinsic diameter of the original triangle, therefore for each such excision the inequality

\[
-\Delta \sigma \leq \frac{1}{2} (-\Delta \omega^+)d^2
\]

will be preserved.

Repeating such excisions a finite number of times, we will sooner or later liquidate the entire two-gon \( D \). This process will certainly decrease the number of vertices with \( \omega^+ > 0 \). Taking account of the induction hypothesis and inequality (4) we establish Theorem 1 for this case.
Case 4. We suppose finally that inside \( T \) there are vertices of the metric with \( \omega^+ > 0 \) and vertices of \( T \) not joinalbe in \( T \) by shortest arcs distinct from the sides of \( T \). Then we consider in \( T = ABC \) the closest vertex \( O_1 \) to \( A \) with \( \omega^+ > 0 \). As in Case 3, we carry out excisions of the corresponding two-gons. This time it may happen that further excisions of the two-gon are not possible, because there appears in \( T \) a shortest arc distinct from the sides of \( T \). But then we fall into the conditions of Case 3.

Inequality (4) is observed under each excision. This completes the proof of Theorem 1.

2. Definition of area.

3. Area of a polygon.

**Definition.** The area \( \sigma(E) \) of a polygon \( P \) is the limit of the sums \( \sum_{t \in Z} s(t) \) of areas of plane triangles with sides of the same length as the triangles \( t \) of triangulations \( Z \) of the polygon \( P \), the limit being taken under the condition that the triangulations \( Z \) become ever finer:

\[
\sigma(E) = \lim_{\max d(t) \to 0} \sum_{t \in Z} s(t).
\]

**Lemma 4.** Suppose that \( T \) is a triangle, \( \omega^+ \) and \( \omega^- \) the positive and negative parts of its curvature, the latter being augmented by the rotations of its sides turned towards \( K \), and \( d \) its intrinsic diameter. Then for any \( \epsilon > 0 \) there exists an arbitrarily fine triangulation of the triangle \( T \) under which the total area \( s(Q) \) of the plane triangles in the polyhedral development \( Q \) corresponding to \( Z \) satisfies the inequality

\[
-\frac{1}{2} \omega^- d^2 - \epsilon < s(Q) - s(T) < \frac{1}{2} \omega^+ d^2 + \epsilon,
\]

where \( T \) is a plane triangle with sides of the same length as the triangle \( T \).

**Proof.** Suppose that \( x, y, \) and \( z \) are the lengths of the sides of \( T \). Taking an \( \epsilon > 0 \), we choose a \( \delta > 0 \) such that the area of the plane triangle \( T^0 \) with sides \( x, y, z \) differs from the area of the plane triangle \( T^0 \) with sides \( x \cos \delta, y \cos \delta, z \cos \delta \) by less than \( \epsilon/3 \), and moreover the total area \( s(t_1 + t_2 + t_3) \) of the three isosceles triangles \( t_1, t_2, t_3 \) with angles \( \delta \) at the bases and lateral sides equal respectively to \( x/2, y/2, z/2 \) satisfies the condition

\[
s(t_1 + t_2 + t_3) < \frac{\epsilon}{3}.
\]

For a fixed \( \delta > 0 \) we may in accordance with Lemma 10 of Chapter VI
construct an arbitrarily fine triangulation of the triangle \( T \) such that the corresponding polyhedral development \( Q \) after pasting the triangles \( t_1, t_2, t_3 \) to it will form a polyhedral development \( T \) in which the bases of \( t_1, t_2, \) and \( t_3 \) will be shortest arcs.

We may take it for granted that
\[
\omega^+(R) \leq \omega^+(Q) + 6\delta \leq \omega^+(T_-) + 6\delta, \\
\tilde{\omega}^-(R) \leq \tilde{\omega}^-(Q) \leq \tilde{\omega}^-(T).
\]

Here in each row the first inequality follows trivially from the structure of the pasting and the second from Theorem 11 of Chapter VI. Moreover, on refinement of the development \( Q \) we see that the diameter \( d(Q) \to d(T) \) and for sufficiently small \( \delta \) this diameter differs little from \( d(R) \). Therefore we may suppose that \( \delta \) is so small and the triangulation with respect to which \( Q \) was constructed so fine that
\[
\frac{1}{2} \omega^+(R)d^2(R) \leq \frac{1}{2} \omega^+(T_-)d^2(T) + \frac{\varepsilon}{3}, \\
\frac{1}{2} \tilde{\omega}^-(R)d^2(R) \leq \frac{1}{2} \tilde{\omega}^-(T)d^2(T) + \frac{\varepsilon}{3}.
\]

Now for the proof of Lemma 4 it is sufficient to apply Theorem 1 to the development \( R \):

\[
-\frac{1}{2} \tilde{\omega}^-(R)d^2(R) \leq s(R) - s(T^0) \leq \frac{1}{2} \omega^+(R)d^2(R).
\]

Replacing in (7) \( s(R) \) by \( s(Q) \), \( s(T^0) \) by \( s(T^0) \), \( 1/2\omega(R)d^2(R) \) by \( 1/2\omega(T)d^2(T) \)

Therefore (6) follows from inequality (7). Lemma 4 is proved.

We note that the construction of the triangulation may without loss of property (5) be constructed so that it is a finer subdivision of any given triangulation.

**Theorem 2.** Every polygon \( P \) has a definite area \( \sigma_0(P) \), i.e., the limit (5) exists.

**Proof.** Suppose that \( P \) is a polygon and \( Q_1, Q_2 \) polyhedral developments constructed with respect to triangulations \( Z_1, Z_2 \) of \( P \). If in these triangulations the sides have multiple intersections, then this situation may be avoided without changing the structure of the triangulation nor the lengths of the sides of the triangles (i.e., without changing \( Q_1, Q_2 \)) and if the greatest of the intrinsic diameters of the triangles of the triangulations \( Z_1, Z_2 \) is increased then this happens only a finite number of times. After this we
may carry out a common subdivision of the resulting \( Z'_1, Z'_1 \) and a finer triangulation \( Z_1 \), satisfying for each triangle \( T \) of \( Z'_1, Z'_1 \) the requirements of Lemma 4. Then we will have

\[
|s(Q_1) - s(Q_3)| \leq \frac{1}{2} \Omega(P) d^2(Z'_1) + \varepsilon,
\]

\[
|s(Q_1) - s(Q_3)| \leq \frac{1}{2} \Omega(P) d^2(Z'_1) + \varepsilon,
\]

where \( d(Z'_1) = \max_{T \in Z'_1} d(T) \) is the "diameter" of the triangulation \( Z'_1 \), so that, in view of the arbitrary smallness of \( \varepsilon > 0 \), the boundedness of \( \Omega(P) \), and the fact that \( d(Z'_1) \) tends to zero along with \( d(Z) \), it follows that the limit (5) exists.

Theorem 2 is proved.

**Theorem 3.** If \( P \) is a polygon and \( Q_z \) the development obtained by rectification of the triangles of the triangulation \( Z \) of the polygon \( P \) with intrinsic diameters no greater than \( d(Z) \), then

\[
-\frac{1}{2} \tilde{\omega}^- (P) d^2(Z) \leq \sigma_0 (P) - s(Q_z) \leq \frac{1}{2} \omega^+ (P) d^2(Z).
\]

*Here it may happen that the structure of \( \omega^+(P) \) does not include the curvatures of points which serve as vertices of the triangulation, but that the curvatures of such points and the rotations of the sides of \( P \) at such points are included in \( \tilde{\omega}^-(P) \).*

The assertion of Theorem 3 follows immediately from Lemma 4 and Theorem 2.

4. **Area of arbitrary sets.**

**Definition.** By the area \( \sigma(M) \) of an arbitrary set \( M \) we shall mean the quantity

\[
\sigma(M) = \inf_{G \supseteq M} \sup_{P \subset G} \sigma_0 (P),
\]

where \( G \) are open sets and \( P \) polygons. Disconnected polygons are admitted. From definition (9) it follows that \( \sigma(M) \) is a negative monotone (with respect to set inclusion) function. The values \( \sigma = \infty \) are not excluded. For open sets we evidently have

\[
\sigma(G) = \sup_{P \subset G} \sigma_0 (P).
\]

It follows from Theorem 2 that all the conditions of Theorem 2 of Chapter V are satisfied for \( \sigma(M) \), so that the following important theorem is valid.
**Theorem 4.** The area \( \sigma(M) \) defined by equation (9) is a regular Carathéodory measure. In particular, it is completely additive on the ring of Borel sets.

**Lemma 5.** For every set \( M \) which lies strictly inside a polygon \( P \), i.e., \( M \subset P_- \), the inequality

\[
\sigma(M) \leq \sigma_0(P)
\]

(11)

holds.

Indeed, suppose that the polygon \( P_1 \subset P_- \). Then any triangulation of \( P_1 \) may be complemented to a triangulation of \( P \), so that \( \sigma_0(P_1) \leq \sigma_0(P) \). It follows from this that \( \sigma(M) \leq \sigma(P_-) = \sup_{\sigma \leq \sigma_0} \sigma_0(P_1) \leq \sigma_0(P) \), i.e., inequality (11).

**Theorem 5.** The areas \( \sigma \) of one-point sets, of shortest arcs, of polygonal curves, and of geodesics are all equal to zero.

**Proof.** 1. Suppose that \( A \) is a one-point set, and \( \theta \) the complete angle around \( A \). From Theorem 1 of Chapter III, \( A \) may be enclosed in a convex polygon \( P \) homeomorphic to the disc and lying in a neighborhood \( U \) of radius \( r \) of the point \( A \). The shortest arcs going from \( A \) to the vertices of \( P \) decompose \( P \) into triangles. On swinging these onto the plane the total area \( s(Q) \) of the resulting development \( Q \) will satisfy the inequality

\[
s(Q) \leq \frac{1}{2} [\theta + \omega^+(U)] r^2,
\]

and moreover, from Theorem 3,

\[
|s(Q) - \sigma_0(P)| \leq \frac{1}{2} \Omega(U)(2r)^2.
\]

Since \( r \) may be taken arbitrarily small without decreasing \( \theta \), \( \omega^+(U) \), \( \Omega(U) \), we therefore conclude that \( A \) can be imbedded in a polygon with arbitrarily small area \( \sigma_0(A) \). Therefore it follows from Lemma 5 that \( \sigma(A) = 0 \).

2. Suppose given an arbitrary shortest arc. It may be decomposed into a finite number of pieces on each of which its curvature and right and left rotations (all three quantities are not larger than zero) do not exceed a certain \( \epsilon_1 > 0 \) in absolute value. In view of the additivity of \( \sigma \) it is sufficient to prove that on each such piece \( L \) the area \( \sigma(L) = 0 \).

Fix a small \( \epsilon_2 > 0 \) and encircle \( L \) by a region \( U \) homeomorphic to the disc within which \( \Omega(U - L) < \epsilon_2 \). Then divide \( L \) by points \( A' \) into segments
of small length $r$.

If $\varepsilon_1$ is sufficiently small, then around each point $A'$ the complete angle is close to $2\pi$ and from it on each side of $L$ we can pass two shortest arcs $l_i^1, l_i^2$ (Figure 99) such that they form with $l$ and with one another sectors with angles close to $\pi/6$. Moreover, if $r$ is small, we may suppose $U$ so small that all these shortest arcs leave the limits of $U$. Moreover, because of the sufficient smallness of the original $\varepsilon_1$ and the rational compatibility of the choice of $\varepsilon_2, r$ and the width of $U$ we can arrange things so that the shortest arcs $\cdots, l_i^1, l_i^{i+1}, \cdots$ of one type do not intersect the shortest arcs $\cdots, l_i^1, l_i^{i+1}, \cdots$ within the limits of $U$ and each shortest arc $l_i$ with $i \geq 2$ intersects at least two of the preceding shortest arcs $l_i^{i-1}, l_i^{i-2}$.

In the resulting quadrilaterals $A^i A^{i-1} B^i C B^{i-1} A^{i-1}$ (Figure 99) we draw diagonals $B^i A^{i-1}$. For sufficiently small $\varepsilon_2$, we may for arbitrarily small $\varepsilon_2, r$ and width of $U$ attain in fact the situation depicted in Figure 99, with the triangles $A^i A^{i-1} B^i$, $B^{i-1} A^{i-1} B^i$ close to equilateral both in the lengths of sides and in sector angles. We leave it to the reader to verify the validity of these assertions. We note only that it suffices to make repeated use of Theorem 11 of Chapter VI on the variations of the angles when the triangles are swung onto the plane.

If now, fixing $\varepsilon_2$, we decrease $\varepsilon_2$ and $r$, then the curve $\cdots B_{i-1} B_i B_{i+1} \cdots$ with the adjunction of polygonal pieces enveloping the endpoints of $L$ bounds a polygon $P$, with its area $\sigma(P)$ tending to zero as $\varepsilon_2, r \to 0$. Therefore, from Lemma 5, it follows that $\sigma(L) = 0$.

3. The equation $\sigma(\gamma) = 0$ for any polygon or geodesic $\gamma$ follows from the equation $\sigma(L) = 0$ for shortest arcs $L$ and the additivity of $\sigma$.

This completes the proof of Theorem 5.

Theorem 6. For polygons $P$ the area $\sigma(P)$ in the sense of definition (9) coincides with the earlier definition of the area $\sigma(P)$ by equation (5):
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\[ \sigma(P) = \sigma_{0}(P). \]

**Proof.** By Theorem 5, for any \( \varepsilon > 0 \) the boundary of \( P \) may be enclosed in an open set \( g \) for which \( \sigma(g) < \varepsilon \). Suppose that \( P_- \) is the interior region of \( P \), and put \( G = P_- + g \). From (11) and (10) we have

\[ \sigma(P_-) \leq \sigma_{0}(P) \leq \sigma(G), \]

but

\[ \sigma(G) - \sigma(P_-) = \sigma(G - P_-) \leq \sigma(g) < \varepsilon; \]

so that

\[ \sigma(P) = \inf_{g \in P} \sigma(G) \leq \sigma_{0}(P). \]

But it follows from equation (10) that always \( \sigma(P) \geq \sigma_{0}(P) \), which along with the preceding proves equation (12).

It follows from Theorem 6 that defining the area of a set using (9) does not in the case of polygons contradict the earlier definition (5). Therefore we shall not henceforth distinguish the areas \( \sigma_{0} \) and \( \sigma \).

5. **Supplementary remarks.**

**Theorem 7.** If the set \( M \) has interior points, then \( \sigma(M) > 0 \).

**Proof.** Choose in \( M \) an interior point \( O \) with a nonzero complete angle. Draw from \( O \) two shortest arcs, forming some angle \( \alpha > 0 \). We mark points \( X, Y \) at distances \( r \) from \( O \) on these shortest arcs. The triangle \( T = OXY \) on being swung onto the plane gives a triangle \( T_0 \) with area \( s(T_0) = (r^2/2) \sin \alpha \). By Theorem 3,

\[ \sigma(T) \geq s(T_0) - \frac{1}{2} \bar{\omega}^{-}(T)d^{2}(T). \]

As \( r \to 0 \) we have \( s(T_0) \approx (r^2/2) \sin \alpha \), \( d(T) \leq Cr \), \( \bar{\omega}^{-} \to 0 \). Therefore for sufficiently small \( r \) we obtain \( \sigma(T) > 0 \). This proves Theorem 7.

**Remarks.** 1) In the case of a repeatedly differentiable two-dimensional Riemannian surface the area in the sense of Definition (9) coincides on measurable sets with the area in the sense of the usual definitions of differential geometry. We shall not give the proof of this quite natural assertion.

2) Definition (9) was constructed in complete correspondence with the general scheme of §1, Chapter V. The role of the elementary sets \( t \) was played by the triangle, the role of the function \( \phi(t) \) by the area \( s(t_0) \) of the triangle developed onto the plane, and the role of the resulting measure \( \mu \) by the area \( \sigma \).
In this case the intermediate figures $P$ (polygons), which figured in the
general scheme of §1 of Chapter V, play an essential role. For the
curvature the situation was different. There the curvature of open sets
could be defined directly without the intermediate sets $P$. The following
example shows that this is not what happens in the case of area, i.e.,
we never define the area of open sets by the equation

$$
\sigma^*(G) = \limsup_{\max \delta(t_i) \to 0} \sum_{t_i \subset G} s(t_i),
$$

where $\{t_i\}$ are finite systems of nonoverlapping triangles $t_i \subset G$.

Example. Consider a sequence of plane equilateral triangles decreasing
unboundedly in measure, whose total area is infinite. We enclose each
such triangle $t^n_i$ in three isosceles triangles, cross-hatched in Figure 100. Suppose that the
altitude of these triangles decreases so fast that the total area of the cross-hatched triangles is
finite, taken together.

We suppose that in an open plane square we have made cuts $X_sY_s$ converging to the edge.
Each such triple of triangles $AC'B$, $BA'C$, $CB'A$ is excised from the plane and the segments
$AB''$, $AC''$, $BC''$, $BA''$, $CA''$, $CB''$ are pasted
pairwise together. The resulting figure is pasted along the exterior contour
$AC'BA'CB'A$ to the corresponding cut $X_sY_s$.

There results a locally polyhedral surface with an infinite number of
boundaries condensing towards the edge. The area introduced above for
it, if it is the ordinary area on the plane portions and is completely
additive, must turn out to be finite in the large. At the same time the
Definition (13) would lead us in this case to an infinite area, since for
any $\varepsilon > 0$ it is possible to find a collection of triangles $t_i$ with diameters
less than $\varepsilon$ for which the sum $\sum s(t_i)$ will be larger than any number
given in advance.

3) The definition of area adopted by us is based on a specific construction. But noting that our space is metric, we could have defined the area as the Hausdorff two-dimensional measure $H^2$. In a two-dimensional manifold of bounded curvature these measures will always coincide on Borel sets: $H^2 = \sigma$. This was proved in [70].
3. **Convergence of areas.**

6. **Areas of converging polygons.** Suppose that on a set $M$ there are given a metric $\rho$ and a converging sequence of metrics $\rho_n \to \rho$, with all the metrics $\rho_n$ defining the same topology on $M$ and converting $M$ into two-dimensional manifolds $R_n$ of bounded curvature. Suppose that the absolute curvatures $\Omega(R_n)$ are bounded throughout. In this case the limit metric $\rho$ in its turn defines a two-dimensional manifold of bounded curvature. Suppose further that polygons $P_n$ are given in the metrics $\rho_n$, which in the metric $\rho$ are figures which may be considered as curvilinear "polygons" $P'_n$, distinguishing in them the original vertices and sides of the polygons $P_n$. Suppose finally that in the metric $\rho$ the figures $P'_n$ together with vertices and sides converge to some polygon $P$. Then the areas $\sigma_n(P_n)$ of the polygons $P_n$ in the spaces $R_n$ converge to the area $\sigma(P)$ in the space $R$.

**Theorem 8.** If in the converging metrics $\rho_n \to \rho$ the polygons $P_n$ converge to $P$ in the sense of Definition 2 of Chapter VII, then their areas converge:

$$
\sigma_n(P_n) \to \sigma(P).
$$

Here, as in the case of reduced triangles, we do not in general include polygons "with tails," i.e., with sides which coincide on some initial segment on issuing from the common vertex. But in the proof, for simplicity, we suppose that all the $P_n$ and $P$ are bounded only by simple curves. The more general case requires some evident stipulations, the general course of the proof being preserved.

**Proof.** Suppose that relation (14) is not so. Then there exists a sequence $P_n \to P$ for which

$$
|\sigma_n(P_n) - \sigma(P)| \geq \epsilon_0 > 0.
$$

We decompose $P$ into triangles $T'$ with intrinsic diameters not larger then $d'$. For any finite system of points on the sides of $T'$ there will be two types of points, those lying on the boundary of $P$ and those lying inside $P$. To points of the first type there correspond (with respect to the parameter on the converging boundary curves) points on the boundaries of $P_n$, whereas points of the second type for sufficiently large $n$ all fall inside $P_n$ and may therefore be put into correspondence with the same points but already in $P_n$. Thus to each point of a finite collection of points in $P$ there corresponds a definite point in each $P_n$. We include in the chosen collection the following three types of points: 1. All vertices $A$ of the
triangles $T'$. 2. Points $A'$ close to $A$, one on each of the sides of $T'$ issuing from $A$. If we join the points $A_n, A'_n$ corresponding to $A, A'$ in $P_n$ by shortest arcs in $R_n$, passing them along the side of $P$ if the corresponding points $A, A'$ lie on one side of $P$, and otherwise drawing them in such a way that there will be no superfluous intersections of shortest arcs, then because of the closeness of the selected points $A'$ to the corresponding point $A$ we can arrange that the shortest arcs $AA'$ in $R$ which are limiting for $A_n A'_n$ (convergence may be assured by an appropriate choice of a subsequence of numbers $n$) will be situated around each point $A$ in the same order as the shortest arcs $AA'$. Here it is of course not excluded that these shortest arcs acquire common initial pieces close to the points $A$. 3. Finally, on the pieces $A'A'_n$ of each side of each of the triangles $T'$ we choose a finite net of points $B$ sufficiently close to one another that, successively joining the points in $P_n$ corresponding to them, we obtain polygons in $P_n$ which converge (convergence being guaranteed by the choice of a subsequence of the numbers $n$) to shortest arcs $A'A'$ which along with the shortest arcs $AA'$ obtained earlier form, possibly a new net $T$, not coinciding with the sides of $T'$, which decomposes $P$ into triangles $T$. The $T$ have the same vertices $A$ as the $T'$, and form a net of the same structure and each side of $T$ coincides with the previous side of $T'$ at least at the points $AA'BB\cdots BA'A$. To each triangle $T$ in the polygon $P_n$ (more precisely in a subsequence selected from them) there corresponds a polygon $Q_n$, bounded by three polygons $A_n A'_n, B_n B'_n, \cdots, B_n A_n A'_n$, each of which converges in the metric $p$ to the corresponding side of $T$. Thus the net of triangles $T'$ is replaced by a net of triangles $T$, limiting for the polygons $Q_n$ of subdivisions of $P_n$. The maximum $d_t$ of the intrinsic diameters of the triangles $T$ will be arbitrarily small along with $d'$. We draw in $Q_n$ without superfluous mutual intersections, three relative shortest arcs, joining vertices which correspond to the vertices of $T$. These shortest arcs decompose $Q_n$ into triangles $T_n$ and three "crusts" $\Gamma_n$. By Theorem 4 of Chapter VII, the metrics $p_{Q_n}$ induced by the excision of the polygons $Q_n$ in $R_n$ converge to a metric $p_T$ induced by the excision of $T$ in $R$. Therefore we will have the following convergences:

1) the intrinsic diameters $d_{Q_n} \to d_T$;

2) the lengths of the sides of $T_n$ converge to the lengths of the sides of $T$, and therefore also the areas of the plane triangles with sides of the same length also converge: $s(T_n) \to s(T)$;

3) the lengths of the "sides" of the polygons $Q_n$, by the construction,
also converge to the lengths of the sides of $T$, so that if any of the three crusts $\Gamma_n$ is decomposed into triangles by relative shortest arcs in $Q_n$, drawn from the vertices of $\Gamma_n$ and these triangles developed onto the plane, then from them one may make up a plane polygon $\Gamma_0$ whose area $s_n(\Gamma_0) \to 0$ with increasing $n$.

Now we denote by $\Omega$ the upper bound of all $\Omega(P), \Omega(P_n), \Omega(P^*_n)$. Then on the basis of Theorem 3, in which we may because of Theorem 6 regard $o_n$ as simple area, we will have

$$|\sigma_n(P_n) - \sigma(P)| = \left| \sum_{\gamma_n} \sigma_n(Q_n) - \sum_T (T) \right| = \left| \sum_{\gamma_n} \sigma_n(\Gamma_n) + \sum_T \sigma_n(T_n) - \sum_T (T) \right|$$

$$\leq \left| \sum_{\gamma_n} \sigma_n(\Gamma_n) - \sum_T s(\Gamma_n^e) \right| + \left| \sum_{\gamma_n} \sigma_n(T_n) - \sum_T s(T_n^e) \right|$$

$$+ \left| \sum_T s(T_n^e) - \sum_T s(T^e) \right|$$

$$\leq \frac{1}{2} \Omega \max d_{\gamma_n}^2 + \varepsilon + \frac{1}{2} \Omega \max d_{\gamma_n}^2 + \varepsilon + \frac{1}{2} \Omega \max d_T^2.$$

(16)

For sufficiently small $d'$ and sufficiently large $n$ the right side of inequality (16) becomes less than $\varepsilon_n$, leading to a contradiction with the supposition (15). This proves Theorem 8.

7. Weak convergence of areas.

**Theorem 9.** If on the set $M$ there are given metrics $\rho_n$ with absolute curvatures bounded uniformly, the $\rho_n$ defining on $M$ the same topology and uniformly converging to a metric $\rho$, then the areas $\sigma_n$ defined in the metrics $\rho_n$ converge locally weakly as set functions to the area $\sigma$ in the metric $\rho$:

$$\sigma_n \rightharpoonup \sigma.$$

(17)

It is also sufficient to have local boundedness of the $\Omega_n$, and local uniform convergence $\rho_n \to \rho$.

**Corollary.** Under the conditions of Theorem 9, on each open set $G$ whose boundary has zero area in the metric $\rho$, we have the usual convergence

$$\sigma_n(G) \to \sigma(G).$$

(18)

**Proof.** Using Kolmogorov’s test, Lemma 16 of Chapter VII, for the proof of weak convergence it suffices to show that for any $F_0 \subset G_0$, where $F_0$ is a closed set and $G_0$ an open set with a compact closure, the relation

$$\lim_{n \to \infty} \inf_{F \subset G \subset G_0} |\sigma_n(G) - \sigma(G)| = 0$$

holds. But since for such $F_0$ and $G_0$ there is always an open set $G$ satisfying the conditions
\[ \sigma(F_{\text{r., } G}) = 0, \quad F_{0} \subset G \subset \bar{G} \subset G_{\circ}, \]
it suffices to prove the convergence (18) formulated above in order to show the validity of Theorem 9.

Thus suppose that \( \sigma(F_{\text{r., } G}) = 0 \). Taking an arbitrarily small \( \varepsilon > 0 \), we choose two polygons \( p, P \) such that the conditions
\[ p \subset G \subset \bar{G} \subset P, \quad \sigma(G) - \varepsilon < \sigma(p) < \sigma(P) < \sigma(G) + \varepsilon \]
are satisfied. Such a choice for a set \( G \) with a compact closure is possible since
\[ \sigma(G) = \sup_{p \subset \bar{G}} \sigma(p), \quad \sigma(G) = \sigma(\bar{G}) = \inf_{P_{\text{r., } G}} \sigma(P_{\text{r., } G}). \]

We choose further polygons \( q, Q \) such that
\[ p \subset q_{-} \subset q \subset G \subset \bar{G} \subset Q_{-} \subset Q \subset P_{-} \subset P. \]

On the boundaries of \( q, Q \) we choose a sufficiently dense net of points and, successively joining them by shortest arcs in the metric \( \rho_{*} \) without superfluous intersections, we obtain polygons \( q_{n}, Q_{n} \). This is done analogously to the proof of Theorem 8, in the construction of the polygons \( Q_{n} \).

Then we choose a subsequence of numbers \( n_{i} \) such that on this subsequence the polygons \( q_{n_{i}}, Q_{n_{i}} \) converge to some limiting polygons \( q', Q' \), and such that the quantities \( \sigma_{n}(q_{n}) \) and \( \sigma_{n}(Q_{n}) \) should have limits. Then, using Theorem 8 for converging polygons, we have
\[ \sigma(G) - \varepsilon < \sigma(p) \leq \sigma(q') = \lim \sigma_{n}(q_{n}) \leq \lim \sup \sigma_{n}(q_{n}) \leq \lim \inf \sigma_{n}(G) \]
\[ \leq \lim \sup \sigma_{n}(G) \leq \lim \sup \sigma_{n}(\bar{G}) \leq \lim \inf \sigma_{n}(Q_{n}) \]
\[ \leq \lim \sigma_{n}(Q_{n}) = \sigma(Q') \leq \sigma(P) < \sigma(G) + \varepsilon,^{1} \]
i.e.,
\[ \sigma(G) - \varepsilon < \lim \inf \sigma_{n}(G) \leq \lim \sup \sigma_{n}(G) < \sigma(G) + \varepsilon, \]
which, in view of the arbitrary smallness of \( \varepsilon > 0 \), proves relation (18) and with it Theorem 9.

The following example shows that under the conditions of Theorem 9 there is generally speaking only weak convergence \( \sigma_{n} \rightarrow \sigma \), and not convergence of these functions on each Borel set.

**Example.** Consider a square \( K \) with side 1. On one side of the square we remove a center segment of length 1/4. In the remaining two pieces of the side we again remove from each piece an open segment of length

---

1. Here we are using a strengthened formulation of Theorem 8, since the polygons \( q_{n} \rightarrow q' \) and \( Q_{n} \rightarrow Q' \) may have "tails" close to the vertices.
1/4^n \cdot 1/2. At the third stage we remove from each of the 2^n \cdot 1/2^n \cdot 1/2 remaining pieces an open middle segment of length 1/4^n \cdot 1/2^n, and so forth, on the \( n \)th step removing from each of the 2^{n-1} remaining pieces an open middle segment of length 1/4^n \cdot 1/2^{n-1}. Over each removed segment we construct a strip through the entire square. All of these open strips, taken together, form a set \( G \) with area

\[
\sigma(G) = \frac{1}{4} + \frac{1}{4^2} + \cdots = \frac{1}{3}.
\]

Moreover, consider a sequence \( K_1, K_2, \cdots \) of such squares. For each square we repeat a similar construction with the differences that in the square \( K_n \) the removal of strips up to the \( n \)th step is done in the same way as for the square \( K \), and, beginning with the \((n+1)\)th step we successively remove middle pieces equal to one half of the pieces from which they are removed. The sum of the strips removed from \( K_n \) forms a set \( G_n \), but this time all the areas \( \sigma(G_n) \neq 1 \).

Now suppose that \( \phi_n \) is a mapping of the square \( K \) onto \( K_n \) which consists of mapping each strip of the set \( G \) (in case of necessity by uniform expansion) onto the corresponding strip of the set \( G_n \), and then this mapping is extended onto all of \( K \) by continuity.

For each \( n \) we can introduce a metric \( \rho_n \) on \( K \), defining \( \rho_n(X,Y) \) as the distance between the points \( \phi_n(X) \) and \( \phi_n(Y) \) in \( K_n \). All the metrics \( \rho_n \) are metrics of bounded curvature, since they are metrics of the plane squares \( K_n \). Moreover, on all of \( K \) there is uniform convergence \( \rho_n \to \rho \), where \( \rho \) is the usual metric in \( K \). At the same time

\[
1 = \sigma_n(G) \leftrightarrow \sigma_n(G) = \frac{1}{3}.
\]