7. Computing the Symmetric Ring of Quotients

Let $R$ be a prime ring. We recall that $Q_l(R)$, the left Martindale ring of quotients of $R$, is defined as the set of equivalence classes of left module homomorphisms $f: R A \to R R$ where $A$ runs over the nonzero two-sided ideals of $R$. Therefore, the following abstract characterization comes as no surprise.

**Proposition 1.** Let $R$ be prime. Then $Q_l = Q_l(R)$ has the following four properties.

(i) $Q_l \ni R$ with the same 1.
(ii) If $q \in Q_l$, then there exists $0 \neq A \subset R$ with $A q \subset R$.
(iii) If $q \in Q_l$ and $0 \neq A \subset R$, then $A q = 0$ implies $q = 0$.
(iv) Let $0 \neq A \subset R$ and let $f: R A \to R R$ be an $R$-module homomorphism. Then there exists $q \in Q_l$ with $a q = a f$ for all $a \in A$.

Furthermore, $Q_l$ is uniquely characterized by these properties.

Of course, the right ring of quotients $Q_r(R)$ also exists and has an analogous characterization. Here are some examples of $Q_l$.

1. Let $R$ be a simple ring. Then $Q_l(R) = R$. Indeed (i), (ii) and (iii) are obvious and (iv) follows since the only left module maps $f: R R \to R R$ are right multiplication by elements of $R$.

2. Let $R$ be a commutative domain with field of fractions $F$. Then $Q_l(R) = F$. Again (i), (ii) and (iii) are trivially satisfied by $F$. For (iv), suppose $f: R A \to R R$ is given. If $a, b \in A \setminus 0$, then

$$a(f) = (ab)f = (ba)f = b(a f)$$

so $a^{-1}(a f) = b^{-1}(b f)$ and we see that $a^{-1}(a f)$ is the same element $q \in F$ for all $a \in A \setminus 0$.

3. Let $R = K \langle x, y \rangle$ be the free algebra on two variables and let $I$ be the ideal generated by $x$ and $y$. Then $I = Rx + Ry$ is the free $R$-module on $x$ and $y$ and we can define $f: R I \to R R$ by $x f = 0$ and $y f = 1$. Thus, by (iv), there exists $q \in Q_l(R)$ with $x q = 0$ and $y q = 1$. In particular, $Q_l(R)$ has zero divisors even though $R$ itself is a domain.

It is because of example (2) that $Q_l$ is a ring of quotients and it is because of (3) that $Q_l$ is really too large.

Martindale [3] introduced $Q_l$ to study generalized polynomial identities. For the most part, he restricted his attention to the smaller ring $R C$ where $C = Z(Q_l)$.

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is the extended centroid of $R$. $RC$ is called the central closure of $R$. It is prime and centrally closed. Martindale used this extension ring to handle certain types of linear identities. In particular he proved (i) below and (ii) is an easy consequence.

**Proposition 2** [3]. Let $R$ be a prime ring and $C$ its extended centroid.

(i) Let $a_i, b_i \in R \setminus 0$ and suppose $\sum_1^n a_i x b_i = 0$ for all $x \in R$. Then $a_1, a_2, \ldots, a_n$ are linearly dependent over $C$.

(ii) Assume that $R$ is centrally closed and let $E$ be an algebra over $C$. If $P$ is a prime ideal of $R \otimes_C E$ with $P \cap R = 0$, then $P = R \otimes Q$ with $Q$ a prime ideal of $E$.

The following computations concern group rings. Here $Q_{cl}$ denotes the classical ring of quotients.

**Theorem 3** [1, 8]. Let $K[G]$ be a prime group ring with center $Z$.

(i) If $K[G]$ is an Ore ring, then $\mathcal{Z}(Q_{cl}(K[G])) = Q_{cl}(Z)$.

(ii) $Q_{cl}(Z) = C$, the extended centroid of $K[G]$.

Kharchenko observed that $Q_l$ can be used to develop a Galois theory of prime rings. He actually worked with semiprime rings (there is an appropriate definition of $Q_l$ in this case) but the subject becomes much more unpleasant. Recall that $\sigma \in \text{Aut } R$ is $X$-inner if its unique extension to $Q_l$ is inner. A group $G$ of automorphisms is $X$-outer if only the identity is $X$-inner. He showed

**Lemma 4.** Let $R$ be a prime ring with $a_i, b_i \in R$ and $\sigma_i \in \text{Aut } R$ for $i = 0, 1, \ldots, n$. Assume that $a_0, b_0 \neq 0$ and $\sigma_0 = 1$. If $\sum_0^n a_i x^{a_i} b_i = 0$ for all $x \in R$, then, for some $i \neq 0$, $\sigma_i$ is $X$-inner.

**Corollary 5** [2]. If $G$ is a group of $X$-outer automorphisms of $R$, then the elements of $G$ are linearly independent automorphisms.

The independence of automorphisms is of course a key ingredient in the ordinary Galois theory of fields. This led Montgomery to define the normal closure $RN$ of $R$ to be the subring of $Q_l$ generated by $R$ and

$$N = \{\text{units } q \in Q_l | q^{-1} R q = R\}.$$ 

That is, $N$ is the set of units which define $X$-inner automorphisms. It follows that $RN$ is an extension ring of $R$ which is still prime, but it is not normally closed in general (example later). Since $RN$ is large enough for the study of crossed products and Galois theory, numerous special cases have been computed. We start with

**Theorem 6** [5]. Let $A$ be a filtered algebra, so that $A = \bigcup_{m=0}^{\infty} A_n$ with $A_n A_m \subseteq A_{n+m}$, and assume that the associated graded algebra $\overline{A}$ is a commutative domain. If $\sigma$ is an $X$-inner automorphism of $\overline{A}$, then $\sigma$ preserves the filtration and acts trivially on $\overline{A}$.

**Proof (Sketch).** If $a \in A_n \setminus A_{n-1}$, let $\overline{a} \in A_n / A_{n-1} \subseteq \overline{A}$ be its leading term. Since $\overline{A}$ is a domain it follows that $\overline{a} \overline{b} = \overline{ab}$. Now let $\sigma$ be an $X$-inner
automorphism. Then there exist $a, b, c, d \in A \setminus 0$ with $arb = cr^\sigma d$ for all $r \in A$. We therefore have $a \bar{r}b = c\bar{r}d$ for all $r \neq 0$. In particular, setting $r = 1$ yields $\bar{a}b = \bar{c}d$ and thus since $\bar{A}$ is a commutative domain we see that $\bar{r} = \bar{r}^\sigma$. Thus $\sigma$ preserves the filtration and acts trivially on $\bar{A}$.

This of course applies to $U = U(L)$, the universal enveloping algebra of a Lie algebra $L$ over $K$. Indeed we know that $U$ is filtered and that $\bar{U}$ is a commutative polynomial ring. Also, if $L$ is finite-dimensional, then $U$ is an Ore domain.

Let $q$ be a unit of $Q_1(U)$ which gives rise to an $X$-inner automorphism $\sigma$. Then the above implies that $q^{-1}lq = l + \lambda(l)$ for all $l \in L$ where $\lambda : L \to K$. Thus $\lambda \in \text{Hom}(L,K)$ and $[l,q] = lq - ql = \lambda(l)q$ so $q$ is a semi-invariant for $L$ with $q \in Q_1(U)_\lambda$. Conversely, suppose $0 \neq q \in Q_1(U)$ with $[l,q] = \lambda(l)q$ for all $l \in L$. It follows that $U(L)q = qU(L)$ and hence that $q$ is a unit of $Q_1(U)$. Multiplying by $q^{-1}$ then yields $q^{-1}lq = l + \lambda(l)$ and again we have an $X$-inner automorphism. We conclude that

**Corollary 7 [5].** Let $L$ be a Lie algebra over $K$ and let $U = U(L)$ be its enveloping algebra.

(i) The group of $X$-inner automorphisms of $U(L)$ is isomorphic to the additive subgroup of $\text{Hom}(L,K)$ of those $\lambda$ with $Q_1(U)_\lambda \neq 0$.

(ii) The semi-invariants for $L$ in $U$ are precisely the normal elements of $U$. Hence, the semicenter is a characteristic subring of $U$.

Filtered rings occur naturally in the study of coproducts. Let $R_1$ and $R_2$ be rings containing a common division ring $D$. Then $R = R_1 \boxplus R_2$, the coproduct over $D$, is filtered by $F^0 = D$ and $F^n = (R_1 + R_2)^n = \sum R_{i_1}R_{i_2}\cdots R_{i_n}$. The $X$-inner automorphisms of such rings have been studied in a series of papers by Lichtman, Martindale and Montgomery (in various combinations). The best result now is

**Theorem 8 [4].** Assume that each $R_i > D$, at least one of the dimensions over $D$ is larger than $2$, and one-sided inverses in $R_i$ are two-sided. Then every $X$-inner automorphism of $R = R_1 \boxplus R_2$ is inner unless one of the following occurs.

(i) Each $R_i$ is primary, that is, $R_i = D + T_i$ with $T_i^2 = 0$.

(ii) One $R_i$ is primary and the other is $2$-dimensional.

(iii) $\text{char} D = 2$, one $R_i$ is not a domain, and one is quadratic.

In the course of the proof one shows that $X$-inner automorphisms $\sigma$ are strongly bounded; that is, there exists an integer $k \geq 0$ with $\deg r^\sigma \leq \deg r + k$ for all $r \in R$.

As we observed above, if $q$ induces an $X$-inner automorphism of $U(L)$, then $q$ normalizes $L + K$. As an analog in crossed products we consider those $X$-inner automorphisms of $R \star G$ which normalize the group of trivial units. The following result gives the idea without all the tedious details. Recall that the condition $G_{\text{inn}} \cap \Delta^+ = 1$ implies that $R \star G$ is prime.
THEOREM 9 [6]. Let $R \ast G$ be a crossed product with $R$ prime and $G_{\text{inn}} \cap \Delta^+ = 1$. Suppose $q$ is a unit of $Q_l(R \ast G)$ and $\sigma \in \text{Aut } G$ with
\[ q^{-1}R\sigma q = R\sigma^x \quad \forall x \in G. \]
Then $\sigma = \sigma_1\sigma_2$ where $\sigma_1$ centralizes a subgroup of $G$ of finite index and $\sigma_2$ is an inner automorphism of $G$. Furthermore, there exists a fairly tight description of the element $q$.

COROLLARY 10 [6]. Let $R \ast G$ be given with $R$ prime and $G_{\text{inn}} \cap \Delta^+ = 1$ and let $S \ast G$ be the unique extension of $R \ast G$ with $S = Q_l(R)$. Suppose $X$ is the group of $X$-inner automorphisms of $R \ast G$ which normalize both $R$ and the group of trivial units of $R \ast G$. If $X_0$ is the subgroup of $X$ consisting of those automorphisms induced by trivial units of $S \ast G$, then $X/X_0$ is a periodic abelian group.

We remark that we frequently see theorems which assert that $X$-Inn $R/\text{Inn } R$ is a periodic abelian group. This is certainly a special situation since, for domains $R$, $X$-Inn $R/\text{Inn } R$ can be any group.

Let us return again to enveloping algebras and use them as the coefficient ring of a skew group ring $H = U(L)G$. Then $H$ is actually a co-commutative Hopf algebra. Conversely, a result of Kostant asserts that every co-commutative Hopf algebra over an algebraically closed field of characteristic 0 is of this form. Moreover, the Hopf algebra structure essentially picks out $L$ and $G$.

Note that since $U(L)$ is a domain and we know its group of $X$-inner automorphisms, it is easy to determine when $H$ is prime. For example, in characteristic 0 this occurs if and only if $C_G(U(L)) \cap \Delta^+ = 1$.

Let $0 \neq q \in Q_l(H) = Q_l(U(L)G)$. We say that $q$ is a semi-invariant for $L$ and $G$ if and only if there exists $\mu: L \rightarrow K$ with
\[ [l, q] = \mu(l)q \quad \forall l \in L \]
and there exists $\lambda: G \rightarrow K \backslash 0$ with
\[ x^{-1}qx = \lambda(x)q \quad \forall x \in G. \]
That is, $q$ is a common eigenvector for the natural actions of $L$ and of $G$ on $Q_l$. This actually makes sense in Hopf algebra terms. Indeed $q$ is a common eigenvector for the Hopf inner action of $H$ on $Q_l(H)$.

Note that if $q$ is a semi-invariant, then $q$ is a unit of $Q_l(H)$ normalizing $U(L)$ and the group of trivial units of $U(L)G$. Thus the above applies. That along with known results on the semi-invariants of $U(L)$ then yields

COROLLARY 11 [6]. Let $U(L)G$ be a skew group ring of $G$ over the universal enveloping algebra $U(L)$ of the finite $K$-dimensional Lie algebra $L$. Assume that $H = U(L)G$ is prime and that char $K = 0$ and let $SZ$ denote the linear span of all semi-invariants for $L$ and $G$ contained in $H$.

(i) $SZ$ is a commutative integral domain.
(ii) Every semi-invariant for $L$ and $G$ in $Q_l(H)$ is a quotient of semi-invariants in $SZ$.

As we said earlier, the normal closure $RN$ is large enough to handle problems in crossed products and Galois theory. But it is not large enough for derivations. Let $\delta : R \to R$ be a derivation. Then $\delta$ extends uniquely to a derivation $\delta : Q_l(R) \to Q_l(R)$. As usual, we say that $\delta$ is $X$-inner if it becomes inner on $Q_l$, that is, if there exists $q \in Q_l$ with $\delta(r) = [r, q] = qr - q r$ for all $r \in R$ (or $r \in Q_l(R)$). This concept is needed, for example, in

LEMMA 12. Let $R$ be a prime algebra over the rationals and let $R[x; \delta]$ be a differential operator ring in the variable $x$. If $\delta$ is not $X$-inner, then for any ideal $0 \neq I \subset R[x; \delta]$ we have $I \cap R \neq 0$.

If $\delta$ is an $X$-inner derivation induced by $q \in Q_l(R)$, then $q$ need not belong to the normal closure $RN$. However it does belong to a larger and still well behaved subring of $Q_l$. Rather than merely adding all such $q$ to $RN$, we take a slightly different point of view.

Let $R$ be a prime ring. We have already defined $Q_l(R)$ and its right analog $Q_r(R)$. In view of Proposition 1, we might expect a symmetric Martindale ring of quotients to be characterized as follows.

PROPOSITION 13. Let $R$ be prime. Then the ring $Q_s = Q_s(R)$ is uniquely determined by the following four properties.

(i) $Q_s(R) \supseteq R$ with the same 1.
(ii) If $q \in Q_s$, then there exist $0 \neq A, B \subset R$ with $A q, q B \subseteq R$.
(iii) Let $q \in Q_s$ and let $0 \neq I \subset R$. Then $q I = 0$ or $I q = 0$ implies that $q = 0$.
(iv) Let $0 \neq A, B \subset R$ and suppose $f : R A \to_R R$ and $g : B R \to_R R$ are module homomorphisms satisfying

$$(a f)b = a (g b) \quad \forall a \in A, b \in B.$$ Then there exists $q \in Q_s$ with $a f = a q$ and $g b = q b$ for all $a \in A$, $b \in B$.

Notice that the balanced or associative condition $(a f)b = a (g b)$ is necessary in (iv) since it is equivalent to the associativity statement $(a q)b = a (q b)$ in $Q_s$. Notice also that while $Q_s$ is unique, at this point we do not know that it exists. One approach, of course, is to consider the set of all equivalence classes of pairs $(f, g)$ of balanced maps. However, we avoid this by identifying $Q_s$ as a subring of $Q_l$.

PROPOSITION 14. If $R$ is a prime ring, then $Q_s(R)$ exists. Indeed

$$Q_s(R) = \{ q \in Q_l | q B \subseteq R \text{ for some } 0 \neq B \subset R \}$$

and

$$Q_s(R) = \{ q \in Q_r | A q \subseteq R \text{ for some } 0 \neq A \subset R \}.$$

PROOF. Set $S = \{ q \in Q_l | q B \subseteq R \text{ for some } 0 \neq B \subset R \}$. Since $R$ is prime, it follows easily that $S$ is a subring of $Q_l(R)$ containing $R$. The goal is to show that
S satisfies (i)-(iv) of Proposition 13. Of course, (i) and (ii) follow by definition and for (iii) we know at least, since \( q \in Q_I \), that \( Iq = 0 \) implies \( q = 0 \). In the other direction, if \( qI = 0 \), then choose \( 0 \neq Aq \triangleleft R \) with \( Aq \subseteq R \). Then \((Aq)I = 0\) so \(Aq = 0\) since \( R \) is prime and hence \( q = 0 \). Finally, let \( f, g \) be given as in (iv). Then we know at least that there exists \( q \in Q_I \) with \( af = aq \) for all \( a \in A \). The balanced condition and associativity now imply that \( a(gb) = (af)b = (aq)b = a(qb) \) so \( A(gb - qb) = 0 \) and hence \( gb = qb \) for all \( b \in B \). Finally, \( qB = gB \subseteq R \) so \( q \in S \).

This subring of \( Q_I \) was used by Kharchenko in his work on Galois theory, although he did not really stress the symmetric aspects of the ring. The symmetric formulation can presumably simplify some arguments—my later work with Montgomery, in particular.

Two remarks are now in order. First, if \( R \) is a domain, then so is \( Q_s(R) \). Thus \( Q_s \) is not too big. Second, \( Q_s(R) \supseteq RN \) and it contains all \( q \) inducing \( X \)-inner derivations. Thus it is big enough. Let us consider some specific computations.

1. Let \( R = K\langle x, y, \ldots \rangle \) be a free algebra on at least two generators. Then \( Q_s(R) = R \). More generally Kharchenko showed that if \( R \) satisfies a 2-term weak algorithm and if \( R \neq D[x; \sigma] \), a skew polynomial ring over a division ring, then \( Q_s(R) = R \).

2. This is the example used by Bergman to show that \( RN \) need not be normally closed. Let \( R = K[t][x, y|xy = tyx] \). Then we have

\[
Q_I(R) = K(t)[x^{-1}, x, y|xy = tyx],
Q_r(R) = K(t)[x, y, y^{-1}|xy = tyx],
Q_s(R) = K(t)[x, y|xy = tyx].
\]

Thus \( Q_s(R) = RC = RN \). Also if \( S \) is any of the three rings above, then

\[
Q_I(S) = Q_r(S) = Q_s(S) = K(t)[x^{-1}, x, y, y^{-1}|xy = tyx],
\]

a simple ring. This shows that none of \( Q_I, Q_r, Q_s, RN \) is a closure operator. On the other hand, these examples terminate in two steps. One wonders whether there is an example which keeps growing for infinitely many steps.

3. Let \( M_\infty(K) \) be the set of all \( \infty \times \infty \) matrices over \( K \). Then this is not a ring but it does contain the rings of row finite matrices and of column finite matrices. Let \( I \) be the set of all finite matrices (that is having only finitely many nonzero entries) and let \( R = K + I \). Then

\[
Q_I(R) = \{ \text{row finite matrices} \},
Q_r(R) = \{ \text{column finite matrices} \},
Q_s(R) = \{ \text{row and column finite matrices} \}.
\]

One would like to say in general that \( Q_s = Q_I \cap Q_r \). However this example shows that \( Q_I \) and \( Q_r \) are not subrings, in a natural manner, of a common ring.
Now let us look at group rings. First, $K[G]$ must be prime so $\Delta^+(G) = 1$. Next we have

**Lemma 15.** If $Q_s(K[G]) = K[G]$, then $\Delta(G) = 1$.

This is proved by first showing that $\mathcal{I}(K[G])$ must be trivial. Thus if $K[G]$ is symmetrically closed, then $G$ has no nonidentity finite conjugacy classes. It turns out that the countable classes are the interesting ones since

**Theorem 16 [7].** If all nonidentity conjugacy classes of $G$ are uncountable, then $K[G]$ is symmetrically closed.

What about the countable classes? They each generate, of course, a countable normal subgroup of $G$ and we have at least

**Proposition 17 [7].** Let $\Delta^+(G) = 1$ and let $1 \neq N$ be a countable locally finite normal subgroup of $G$. Then $K[G]$ is not symmetrically closed.

**Corollary 18 [7].** Let $G$ be a locally finite group with $\Delta^+(G) = 1$. Then $K[G]$ is symmetrically closed if and only if all nonidentity conjugacy classes of $G$ are uncountable.

Let $0 \neq I \triangleleft K[G]$ and let $H \triangleleft G$. Then an intersection theorem asserts that under suitable hypotheses $I \cap K[H] \neq 0$. Such theorems can be used to compute $Q_s(K[G])$. We will skip the precise details here and just mention some groups to which they apply.

**Theorem 19 [7].** (i) If $G$ is a nonabelian free group or an algebraically closed group, then $K[G]$ is symmetrically closed.

(ii) If $G$ is a polycyclic-by-finite group with $\Delta^+(G) = 1$, then $Q_s(K[G]) = Z^{-1}K[G]$, the central closure of $K[G]$, where $Z = \mathcal{I}(K[G])$.

We remark that if $R$ is a prime Noetherian ring, then $Q_s(R)$ need not equal the central closure $RC$. Thus part (ii) above does have content. Finally, it appears that the problem of finding necessary and sufficient conditions for $K[G]$ to be symmetrically closed will prove to be elusive. Aspects of it seem to be related to the zero divisor problem.

**References**

4. _____, *The normal closure of the coproduct of rings over a division ring*.