2. TAME QUOTIENTS

In this chapter we define the "tame quotients" of a finite algebra and their minimal sets (Definitions 2.5 and 2.6) and prove two principal theorems about them (Theorems 2.8 and 2.11).

We recall from Chapter 0 that the congruence lattice of an algebra \( A \), or \( \text{Con} \ A \), is a complete, 0,1-sublattice of the full partition lattice \( \Pi_A \) of all equivalence relations on the base set of \( A \). That is to say, the join and meet of any subset of \( \text{Con} \ A \), computed in \( \Pi_A \), belong to \( \text{Con} \ A \), and \( \text{Con} \ A \) contains the least and largest elements, \( 0_A \) and \( 1_A \), of \( \Pi_A \). The members of \( \text{Con} \ A \) are precisely the equivalence relations on \( A \) that are also subalgebras of \( A \times A \). The congruences on \( A \), or members of \( \text{Con} \ A \), can also be defined as the equivalence relations \( \alpha \) on \( A \) such that \( f(\alpha) \subseteq \alpha \) for every unary polynomial \( f \) of \( A \). (By \( f(\alpha) \subseteq \alpha \) we mean that whenever \( (x, y) \in \alpha \), then \( (f(x), f(y)) \in \alpha \).) Thus \( \text{Con} \ A = \text{Con} (A, \text{Pol}_1 A) \); and in congruence theory it is often convenient to work directly with unary algebras (whose basic operations are 1-ary).

The basic (or given) operations of an algebra \( A \) determine the set \( \text{Pol}_1 A \), and this monoid determines the congruence lattice of \( A \). But often when examining an algebra, its set of unary polynomials and its congruence lattice are unknown. In this chapter, as we begin to present the basics of tame congruence theory, we will consider the congruence lattice and the unary polynomials of any algebra to be known entities. The principal thrust of our theory will be to reveal subtle ways in which the congruence lattice of a finite algebra \( A \), either considered as an abstract lattice or as a specific set of equivalence relations, influences all of the operations (not just the unary operations) which preserve these equivalence relations. In the beginning, however, we shall be looking only at the interaction between \( \text{Con} \ A \) and \( \text{Pol}_1 A \).

**DEFINITION 2.1.** The set of all \( e \in \text{Pol}_1 A \) such that \( e = e^2 \) (\( = e \circ e \)) will be denoted by \( \text{E}(A) \). (This is the set of idempotents, or projections, in \( \text{Pol}_1 A \).)

Our symbol for restriction is \( | \). The several ways in which this symbol will be used are defined below.

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DEFINITION 2.2. Suppose that $A$ is a nonvoid set, $\emptyset \neq U \subseteq A$, $\vartheta \in \Pi_A$, $f$ is a function with domain $A$, $h$ is an $n$-ary operation on $A$, and $U_0 \cup \cdots \cup U_{n-1} \subseteq A$. We define

1. $\vartheta|_U \overset{\text{def}}{=} \vartheta \cap (U \times U)$;
2. $f|_U \overset{\text{def}}{=} \{(x, f(x)) : x \in U\}$;
3. $h|_{U_0 \times \cdots \times U_{n-1}} \overset{\text{def}}{=} \{(x_0, \ldots, x_{n-1}, h(x_0, \ldots, x_{n-1})) : x_i \in U_i \text{ for } 0 \leq i < n\}$;
4. $h|_U \overset{\text{def}}{=} h|_{U^n}$.

If $A = \langle A, \cdots \rangle$ is any algebra with base set $A$, then we define:

5. $(\text{Pol } A)|_U$ is the set of all $h|_U$ such that $h \in \text{Pol}_n A$ for some $n$, and $h(U^n) \subseteq U$;
6. $A|_U \overset{\text{def}}{=} \langle U, (\text{Pol } A)|_U \rangle$, called the algebra induced on $U$ by $A$ (or an induced algebra of $A$).

We can now state and prove an easy but very useful lemma discovered by P.P. Pálfy and P. Pudlák [27].

LEMMA 2.3. Suppose that $A$ is an algebra, $e \in \text{E}(A)$, and $U = e(A)$. The mapping $|_U$ is a lattice homomorphism of $\text{Con } A$ onto $\text{Con } A|_U$.

$$\text{Con } A \xrightarrow{\theta|_U} \text{Con } A|_U$$

PROOF. The restriction map of $\Pi_A$ to $\Pi_U$ trivially preserves meets. It is also obvious that $\vartheta|_U \in \text{Con } A|_U$ whenever $\vartheta \in \text{Con } A$. To see that $|_U$ preserves joins of pairs of elements of $\text{Con } A$, and that it maps $\text{Con } A$ onto $\text{Con } A|_U$, we define for each $\alpha \in \text{Con } A|_U$:

$$\hat{\alpha} = \{(x, y) \in A^2 : \langle ef(x), ef(y) \rangle \in \alpha \text{ for all } f \in \text{Pol}_1 A\}.$$

Now if $\alpha \in \text{Con } A|_U$ and $(x, y) \in \hat{\alpha}$ and $g \in \text{Pol}_1 A$, then for every $f \in \text{Pol}_1 A$,

$$\langle ef(g(x)), ef(g(y)) \rangle = \langle e(fg)(x), e(fg)(y) \rangle \in \alpha;$$

thus $(g(x), g(y)) \in \hat{\alpha}$. Since $\hat{\alpha}$ is obviously an equivalence relation on $A$, it follows that $\hat{\alpha} \in \text{Con } A$.

The equation $\hat{\alpha}|_U = \alpha$ is easily demonstrated to be true. First, if $x, y \in U$ and $(x, y) \in \hat{\alpha}$, then $(x, y) = \langle ee(x), ee(y) \rangle \in \alpha$ by definition of $\hat{\alpha}$. Thus $\hat{\alpha}|_U \subseteq \alpha$. Second, if $(x, y) \in \alpha$ and $f \in \text{Pol}_1 A$, then $\langle ef \rangle|_U$ is a polynomial of $A|_U$, and this gives $\langle ef(x), ef(y) \rangle \in \alpha$. Consequently, $\alpha \subseteq \hat{\alpha}$. We have now shown that $|_U$ maps $\text{Con } A$ onto $\text{Con } A|_U$. 


Just as easily, we see that for $\theta \in \text{Con} A$ and $\alpha \in \text{Con} A|_U$ we have $\theta|_U \leq \alpha$ iff $\theta \leq \check{\alpha}$. Finally, we show that $|_U$ preserves joins. Let $\theta_1, \theta_2 \in \text{Con} A$ and put $\alpha = \theta_1 \lor \theta_2$ and $\beta = \theta_1 \lor \theta_2$. We have to show that $\beta|_U = \alpha$. Clearly $\beta|_U \geq \alpha$. Conversely, since $\theta_i|_U \leq \alpha$ we have $\theta_i \leq \check{\alpha}$ ($i = 1, 2$). From this, it follows that $\beta \leq \check{\alpha}$, and thus $\beta|_U \leq \alpha$. The proof is finished.

A useful extension of the last lemma is the following one. In this lemma, $I[0_A, \theta]$ denotes the interval sublattice of $\text{Con} A$ consisting of all congruences $\alpha$ such that $\alpha \leq \theta$.

**Lemma 2.4.** Let $A$ be an algebra, $e \in E(A)$, $U = e(A)$, $\theta \in \text{Con} A$, and $N \subseteq U$ be such that $N$ is a union of $\theta|_U$-equivalence classes. Then $A|_N = (A|_U)|_N$ and restriction is a lattice homomorphism of $I[0_A, \theta]$ into the interval $I[0_N, \theta|_N]$ in $\text{Con} A|_N$. If $N^2 \subseteq \theta$, this homomorphism is onto $\text{Con} A|_N$.

**Proof.** Every operation of $(A|_U)|_N$ is clearly an operation of $A|_N$. On the other hand, let $g$ be an operation of $A|_N$. Then $g = f|_N$ for some polynomial operation $f$ of $A$ under which $N$ is closed. $U$ is closed under $ef$, and $(ef|_U)|_N = f|_N$. Thus $g$ is an operation of $(A|_U)|_N$. So the two non-indexed algebras are equal.

Now the mapping $\alpha \mapsto \alpha|_N$ is the composition of the lattice homomorphism $\alpha \mapsto \alpha|_U$ (Lemma 2.3) mapping $I[0_A, \theta] \subseteq \text{Con} A$ onto $I[0_U, \theta|_U] \subseteq \text{Con} A|_U$, with the restriction map from $I[0_U, \theta|_U]$ into $\Pi N$. Thus we may as well assume that $U = A$ and $e = \text{id}$. With these assumptions, it is clear that $|_N$ maps $\text{Con} A$ into $\text{Con} A|_N$, preserves meets, and preserves joins of congruences in the interval $I[0_A, \theta]$.

Now assume that $N^2 \subseteq \theta$, i.e., that $N$ is a $\theta$-equivalence class. Let $\alpha \in \text{Con} A|_N$, and put

$$\check{\alpha} = \{ (x, y) \in \theta : \{ f(x), f(y) \} \cap N \neq \emptyset \quad \text{implies} \quad (f(x), f(y)) \in \alpha \},$$

for all $f \in \text{Pol}_1 A$.

It is easily seen that $\check{\alpha} \in \text{Con} A$, $\check{\alpha} \leq \theta$, and $\check{\alpha}|_N \leq \alpha$. To see that $\alpha \leq \check{\alpha}|_N$, let $(x, y) \in \alpha$, $f \in \text{Pol}_1 A$, $f(x) \in N$ (or $f(y) \in N$). Then $f(N) \subseteq f(x) = N$ (since $f$ preserves $\theta$), so $f|_N \in \text{Pol}_1 A|_N$, and it follows that $(f(x), f(y)) = (f|_N(x), f|_N(y)) \in \alpha$. Thus we see that $(x, y) \in \check{\alpha}$.

**Exercises 2.5**

1. Prove that the lattice homomorphisms of Lemmas 2.3 and 2.4 preserve all infinite joins and meets.
2. Prove that if $e \in E(A)$ and $\emptyset \neq N \subseteq U = e(A)$, then $A|_N = (A|_U)|_N$.
3. Prove that for any algebra $A$ and $\emptyset \neq B \subseteq A$, $\text{Pol}(A|_N) = (\text{Pol} A)|_N$. 


In the remainder of this chapter and throughout Chapters 3 and 4, all algebras considered will be assumed to be finite. The concept of a minimal set relative to a pair of congruences is fundamental for our work.

**Definition 2.5.** Let $\mathbf{A}$ be a finite algebra and let $\alpha < \beta$ be two congruences of $\mathbf{A}$. We define $U_\mathbf{A}(\alpha, \beta)$ to be the set of all sets of the form $f(A)$ where $f \in \text{Pol}_1 \mathbf{A}$ and $f(\beta) \nsubseteq \alpha$. We define $M_\mathbf{A}(\alpha, \beta)$ to be the set of all minimal members of $U_\mathbf{A}(\alpha, \beta)$; i.e., $U \in M_\mathbf{A}(\alpha, \beta)$ iff $U \in U_\mathbf{A}(\alpha, \beta)$ and there does not exist $V \in U_\mathbf{A}(\alpha, \beta)$ with $V \subseteq U$, $V \neq U$. The members of $M_\mathbf{A}(\alpha, \beta)$ are called $(\alpha, \beta)$-minimal sets of $\mathbf{A}$.

Observe that in the framework of the definition, $M_\mathbf{A}(\alpha, \beta)$ is non-empty and for each $(\alpha, \beta)$-minimal set $U$, we have $\alpha|_U \neq \beta|_U$.

By a **quotation** in a lattice $\mathbf{L}$, we shall mean simply a pair $(x, y)$ of elements of $\mathbf{L}$ with $x < y$. A **prime quotient** is a quotient $(x, y)$ where $x < y$. The interval lattice $I[x, y]$ with a quotient $(x, y)$ is the sublattice of $\mathbf{L}$ consisting of all elements $z$ such that $x \leq z \leq y$. By a **quotient (of congruences) in an algebra** $\mathbf{A}$ we mean any quotient $(\alpha, \beta)$ in $\text{Con} \mathbf{A}$.

The concept of a tame quotient is technical, but quite easy to define. Recall the notion of $0,1$-separating homomorphism, from Chapter 1 (Definition 1.5).

**Definition 2.6.** Let $\mathbf{A}$ be a finite algebra and let $(\alpha, \beta)$ be a quotient of congruences in $\mathbf{A}$. We call $(\alpha, \beta)$ **tame** if there exists $V \in M_\mathbf{A}(\alpha, \beta)$ and $e \in E(\mathbf{A})$ such that $e(A) = V$ and $I[\alpha, \beta] \xrightarrow{\text{[V]}} I[\alpha|_V, \beta|_V]$ is a $0,1$-separating lattice homomorphism.

In a short time we shall see that a quotient $(\alpha, \beta)$ in a finite algebra is tame if the interval lattice $I[\alpha, \beta]$ is tight. (Thus, for instance, $(\alpha, \beta)$ is tame whenever $\alpha < \beta$.) But first, we shall prove some facts which explain why the concept of tameness is natural and important. To do this, we need the concept of (internal) polynomial isomorphism.

**Definition 2.7.** Let $\mathbf{A}$ be any algebra and let $B$ and $C$ be nonvoid subsets of $\mathbf{A}$. We say that $B$ and $C$ are **polynomially isomorphic in $\mathbf{A}$**, and we write $B \cong C$ or simply $B \simeq C$ for this, iff there exists $f, g \in \text{Pol}_1 \mathbf{A}$ with

\[
\begin{align*}
  f(B) &= C, \\
  g(C) &= B, \\
  g|_B &= \text{id}_B, \\
  f|_C &= \text{id}_C.
\end{align*}
\]

We write $f : B \simeq C$ iff $f \in \text{Pol}_1 \mathbf{A}$ and there exists $g \in \text{Pol}_1 \mathbf{A}$ such that the above equations hold.

It is important to observe that when $B \cong C$, the induced algebras $\mathbf{A}|_B$ and $\mathbf{A}|_C$ are isomorphic non-indexed algebras. If $f : B \simeq C$ then, where $\pi = f|_B$, we have $\pi(B) = C$ and $\pi((\text{Pol} \mathbf{A})|_B) = (\text{Pol} \mathbf{A})|_C$. [This last equation is understood to mean...
that for any operation \( h \) on \( B \) (say \( h \) is \( n \)-ary), \( h \) is an operation of \( A|_B \) if there exists a 
(unique) \( n \)-ary operation \( h' \) of \( A|_C \) such that 
\[\pi h(x_0, \ldots, x_{n-1}) = h'(\pi x_0, \ldots, \pi x_{n-1}) \]
for all \( x_0, \ldots, x_{n-1} \in B \). Moreover, \( \pi(\theta|_B) = \theta|_C \) for every congruence \( \theta \) of \( A \).

The relation \( \simeq \) is, of course, an equivalence relation on the set of non-void subsets of 
\( A \). We shall now see that when \( \langle \alpha, \beta \rangle \) is tame, the set \( M_A(\alpha, \beta) \) defined in Definition 
2.5 is an equivalence class under \( \simeq \).

**Theorem 2.8.** Let \( \langle \alpha, \beta \rangle \) be a tame quotient of a finite algebra \( A \). The following hold.

1. For all \( U, V \in M_A(\alpha, \beta) \), \( U \simeq V \).

2. For all \( U \in M_A(\alpha, \beta) \) there is \( e \in E(A) \) with \( e(A) = U \); moreover, the map 
\[|_U : I[\alpha, \beta] \to I[\alpha|_U, \beta|_U] \]
is 0,1-separating.

3. For all \( U \in M_A(\alpha, \beta) \) and \( f \in \text{Pol}_1A \), if \( f(\beta|_U) \not\subseteq \alpha \) then \( f(U) \in M_A(\alpha, \beta) \) 
and \( f : U \simeq f(U) \).

4. If \( (x, y) \in \beta - \alpha \) and \( U \in M_A(\alpha, \beta) \) then for some \( f \in \text{Pol}_1A \), 
\( f(A) = U \) and \( \langle f(x), f(y) \rangle \in \beta|_U - \alpha|_U \).

5. For each \( U \in M_A(\alpha, \beta) \), \( \beta \) is the transitive closure of 
\[\alpha \cup \{(g(x), g(y)) : (x, y) \in \beta|_U \text{ and } g \in \text{Pol}_1A\} \]

6. For all \( f \in \text{Pol}_1A \), if \( f(\beta) \not\subseteq \alpha \) then for some \( U \in M_A(\alpha, \beta) \), 
\( f : U \simeq f(U) \).

**Proof.** From the definition (i.e., 2.6) we can choose \( V_0 \in M_A(\alpha, \beta) \) and \( e_0 \in E(A) \) 
such that \( V_0 = e_0(A) \) and \( |V_0| \) is 0,1-separating on the interval sublattice \( I[\alpha, \beta] \). We 
first establish the truth of (4) and (5) just for this \( \langle \alpha, \beta \rangle \)-minimal set.

To prove (4) for \( V_0 \), we consider the congruence 
\[\theta = \alpha|_{V_0} \cap \beta = \{(x, y) \in \beta : \langle e_0 f(x), e_0 f(y) \rangle \in \alpha \text{ for all } f \in \text{Pol}_1A\} \] .

Now \( \theta \in I[\alpha, \beta] \) and \( \theta|_{V_0} = \alpha|_{V_0} \). This implies that \( \theta = \alpha \), 
since \( |V_0| \) is 0-separating on \( I[\alpha, \beta] \). Thus for each pair \( (x, y) \in \beta - \alpha \) 
we have that \( (x, y) \in \beta - \theta \); and so, by the definition of \( \theta \), there must exist \( g \in \text{Pol}_1A \) 
with \( \langle e_0 g(x), e_0 g(y) \rangle \notin \alpha \). The function 
\( f = e_0 g \) satisfies \( f(A) \subseteq V_0 \) and \( \langle f(x), f(y) \rangle \in \beta|_{V_0} - \alpha|_{V_0} \). It follows that \( f(A) = V_0 \) 
by the \( \langle \alpha, \beta \rangle \)-minimality of \( V_0 \). We have proved that (4) holds for \( V_0 \).

To prove (5) for \( V_0 \), we notice that the transitive closure of the relation 
\[\alpha \cup \{(g(x), g(y)) : (x, y) \in \beta|_{V_0} \text{ and } g \in \text{Pol}_1A\} \]
is the congruence \( \alpha \cup \Theta(\beta|_{V_0}) \) of \( A \). This congruence belongs to the interval 
lattice \( I[\alpha, \beta] \), and obviously \( (\alpha \cup \Theta(\beta|_{V_0}))|_{V_0} = \beta|_{V_0} \). Since \( |V_0| \) is 1-separating on \( I[\alpha, \beta] \), it 
follows that \( \beta = \alpha \cup \Theta(\beta|_{V_0}) \), as stated by (5).
Now let $U$ be any $(\alpha, \beta)$-minimal set. We shall prove that $U \simeq V_0$. This will prove (1), and in the process we shall prove (2), (4), and (5). We choose a $\mu \in \text{Pol}_1 A$ with $\mu(A) = U$ and $\mu(\beta) \not\subseteq \alpha$. By (5) for $V_0$, there must exist $a, b \in V_0$ and $g \in \text{Pol}_1 A$ such that $(a, b) \in \beta$ and $(\mu g(a), \mu g(b)) \not\subseteq \alpha$. (We use that $\mu^{-1}(\alpha)$ is an equivalence relation and that $\alpha \vee \Theta(\beta|V_0) = \beta \not\subseteq \mu^{-1}(\alpha)$.) The function $\mu_1 = \mu g e_0$ satisfies $\mu_1(A) \subseteq U$ and $\mu_1(\beta) \not\subseteq \alpha$ (since $(\mu_1(a), \mu_1(b)) \not\subseteq \alpha$). Thus $\mu_1(A) = U$ by the $(\alpha, \beta)$-minimality of $U$. Since $\mu_1 = \mu_1 e_0$, we actually have that $\mu_1(V_0) = \mu_1(A) = U$. Next, to get a polynomial function mapping $U$ onto $V_0$, we apply (4) for $V_0$ with $(x, y) = (\mu_1(a), \mu_1(b))$. This produces a $v \in \text{Pol}_2 A$ with $v(A) = V_0$ and $(\nu \mu_1(a), \nu \mu_1(b)) \not\subseteq \alpha$. Thus $\nu \mu_1(A) = V_0 = \nu(A)$ by the $(\alpha, \beta)$-minimality of $V_0$. Since $\mu_1(A) = U$, we actually have $\nu(U) = \nu(A) = V_0$ as well as $\mu_1(V_0) = \mu_1(A) = U$. The argument can now be completed easily. The function $\mu_1|U$ is a member of the finite group of all permutations on $U$, and so there exists an integer $k > 1$ such that $(\mu_1|U)^k|U = \text{id}_U$.

We write $e = (\mu_1|U), f = \nu, g = (\mu_1|U)^{k-1}\mu_1$. It is now trivial to check that $e = e^2, e(A) = U, f(U) = V_0, g(V_0) = U, gf|U = \text{id}_U, fg|V_0 = \text{id}_V$. Thus we have established that $U \simeq V_0$. In this situation, for all $\theta \in \text{Con} A$ we must have $\theta|U = g(\theta|V_0)$ and $\theta|V_0 = f(\theta|U)$. Statement (2) is an obvious corollary of these equalities. Statements (4) and (5) must be true for $U$, since in proving their validity for $V_0$, we used only that (2) holds for $V_0$.

The statements (3) and (6) still remain to be proved. Assume that $U \in M_\alpha(\alpha, \beta), f \in \text{Pol}_1 A$, and $f(\beta|U) \not\subseteq \alpha$. Choose $(a, b) \in \beta|U$ with $(f(a), f(b)) \not\subseteq \alpha$ and apply (4) for $(x, y) = (f(a), f(b))$. This gives a $g \in \text{Pol}_1 A$ satisfying $g(A) = U$, $(gf(a), gf(b)) \not\subseteq \alpha$. We choose $e \in \text{E}(A)$ such that $e(A) = U$ (by (2)). Thus $gf(U) = gfe(A) = U$, by the $(\alpha, \beta)$-minimality of $U$ (since $(gf e(a), gf e(b)) \not\subseteq \alpha$). From $gf(U) = U$ it follows that $|f(U)| = |U|$ and that $f$ maps $U$ one-to-one onto $f(U)$. The calculation that finished the proof of (1) (taking $\mu_1 = g, \nu = f$) will show that the inverse function of $f|U$ is the restriction to $f(U)$ of some polynomial. Hence $f : U \simeq f(U)$, and this fact certainly implies that $f(U) \in M_\alpha(\alpha, \beta)$. This finishes the proof of (3).

To prove (6), let $f$ be any unary polynomial of $A$ such that $f(\beta) \not\subseteq \alpha$. By (5) there is a $g \in \text{Pol}_1 A$ and $(x, y) \in \beta|V_0$ such that $(g(x), g(y)) \not\subseteq \alpha$. Now this implies that $(g(x), g(y)) \not\subseteq \alpha$, and so $g(\beta|V_0) \not\subseteq \alpha$. By (3), we have that $g(V_0)$ is an $\langle \alpha, \beta \rangle$-minimal set $U$. Furthermore, $(g(x), g(y)) \in \beta|U$ implies that $(\beta|U) \not\subseteq \alpha$. Now (6) follows by an application of (3).

Here is a brief history of the origins of tame congruence theory. In [27], Pálfy and Pudlák observed that if $\text{Con} A$ is a simple lattice and if $e$ is an idempotent polynomial function of $A$, then $\text{Con} A$ must be isomorphic to $\text{Con} A|e(A)$ under the restriction map of Lemma 2.3, provided that $|e(A)| > 1$. They took $e$ to be non-constant with a minimal range (here it was necessary to assume that $A$ is finite) and proved, under
some further hypotheses on $\text{Con } A$, that $A|_{c(A)}$ must be permutational: every one of its non-constant unary polynomials is a permutation of the set $c(A)$. From here, their reasoning led to a proof of the equivalence of these statements: (i) every finite lattice is isomorphic to the congruence lattice of some finite algebra; (ii) every finite lattice is isomorphic to an interval sublattice of the lattice of subgroups of some finite group. Whether these equivalent statements are actually true is still unknown.

Four years later, McKenzie found another way to exploit these ideas, and developed the first rudimentary version of tame congruence theory (which was reported in [22]). Involved was the discovery that under mild assumptions, all the “minimal sets” $c(A)$ are “polynomially isomorphic”, and the collection of these sets behaves somewhat like a geometric structure on the base set of the algebra. Shortly later, during Spring 1982, Pálfy succeeded in writing down a complete list of all the finite permutational algebras (reported in [26]). All of these developments paved the way for an evolution of ideas that accelerated rapidly. Most of the theory presented in this book was discovered during 1983.

Exercises 2.9

1. Let $\langle \alpha, \beta \rangle$ be a congruence quotient of a finite algebra. Show that if 2.8(4) and 2.8(5) are both valid for one $\langle \alpha', \beta' \rangle$-minimal set $U$ then $\langle \alpha, \beta \rangle$ is tame.

2. Let $\langle \alpha, \beta \rangle$ be a tame quotient of a finite algebra $A$. Show that this modified form of 2.8(4) is valid.

$$(4') \text{ For each } (x, y) \in \beta - \alpha \text{ there exist } U \in M_A(\alpha, \beta) \text{ and } e \in E(A) \text{ such that } e(A) = U \text{ and } (e(x), e(y)) \in \beta|_U - \alpha|_U.$$

Show that $(4')$ cannot be strengthened to read “For each $(x, y) \in \beta - \alpha$ and $U \in M_A(\alpha, \beta)$ there exists $e \in E(A)$ such that ...” by constructing a three-element unary algebra with tame quotient $\langle 0_A, 1_A \rangle$ for which the strengthened form of $(4')$ is false.

3. Assertion (3) of Theorem 2.8 can be strengthened. Show that if $\langle \alpha, \beta \rangle$ is tame, $U \in M_A(\alpha, \beta)$, $f \in Pol_1 A$ then $f(\beta|_U) \not\subset \alpha$ iff $f(U) \in M_A(\alpha, \beta)$. Show that if $f|_U$ is one-to-one it need not follow that $f(U) \in M_A(\alpha, \beta)$. (There is a three-element unary algebra with exactly three congruences, $0_A < \alpha < 1_A$, in which $(\alpha, 1_A)$ is tame and the implication fails for this tame quotient.)

4. Construct a three-element algebra with precisely three congruences such that $M_A(0_A, 1_A) = \binom{2}{3}$ (the set of two-element subsets of $A$) and $\langle 0_A, 1_A \rangle$ is not tame. Which parts of Theorem 2.8 fail to be true in your example?

5. Give a detailed proof of the following fact. If $B$ and $C$ are nonvoid subsets of an algebra $A$ and if $f : B \approx C$ (implying that $f \in Pol_1 A$) then $f|_B$ is
an isomorphism between the structures \((B, \text{Pol} A)_{\mid_B}, \theta|_B\) \((\theta \in \text{Con} A)\) and \((C, \text{Pol} A)_{\mid_C}, \theta|_C\) \((\theta \in \text{Con} A)\).

(6) Let \((S, \cdot)\) be a finite semigroup (such as \((\text{Pol}_1 A, \circ)\) for a finite algebra \(A\)—where \(\circ\) denotes composition of functions). Show that if \(x \in S\) then for some integer \(k > 0\), \(e = x^k\) is idempotent; i.e., \(x^{2k} = x^k\). Moreover, there is an integer \(k\) such that \(x^{2k} = x^k\) for all \(x \in S\).

(7) Let \((S, \cdot)\) be a monoid (semigroup with identity element 1 satisfying \(1 \cdot x = x \cdot 1 = x\) for all elements \(x\)). An ideal in \(S\) is a nonvoid set \(I\) such that \(S \cdot I \cdot S = I\) (\(xuy \in I\) whenever \(x, y \in S\) and \(u \in I\)). A quasi-ordering and an equivalence relation are defined on \(S\) by setting \(x \preceq y\) iff \(SxS \subseteq SyS\); and \(x \sim y\) iff \(SxS = SyS\). We put \(x < y\) iff \(x \preceq y\) and \(x \not\sim y\).

Now let \((\alpha, \beta)\) be a tame quotient in the finite algebra \(A\), and let \((S, \cdot) = (\text{Pol}_1 A, \circ)\). Set \(I = \{f \in S : f(\beta) \subseteq \alpha\}\) and \(T = \{f \in S : f(A) \in M_A(\alpha, \beta) \text{ and } f \notin I\}\).

Show that \(I\) is an ideal in \(S\), that
\[
T = \{f \in S : f \notin I, \text{ but } g < f \text{ implies } g \in I\},
\]
and that \(T = \{h \in S : h \sim f\}\) for each \(f \in T\).

(8) Let \(A\) be any algebra and let \((S, \cdot) = (\text{Pol}_1 A, \circ)\). A right ideal in \(S\) is a nonvoid subset \(K\) of \(S\) such that \(K \cdot S = K\). For any right ideal \(K\) of \(S\), define a mapping \(\mu_K\) of \(\text{Con} A\) by
\[
\mu_K(\theta) = \{(x, y) \in A^2 : f(x), f(y)\} \in \theta\text{ for all } f \in K\}.
\]

Then prove:

(i) \(\mu_K(\theta) \in \text{Con} A\), \(\mu_K(\theta) \supseteq \theta\) for all \(\theta \in \text{Con} A\); and \(\mu_K\) is a meet endomorphism of \(\text{Con} A\).

(ii) For right ideals \(K_0, K_1\) in \(S\) and \(\theta \in \text{Con} A\),
\[
\mu_{K_0}(\theta) = \mu_{K_0}(\theta), \mu_{K_0}K_1(\theta) = \mu_{K_1}(\mu_{K_0}(\theta)), \mu_{K_0}K_1(\theta) \supseteq \mu_{K_1}(\theta)\).
\]

The following lemma will be the key for connecting the purely lattice theoretic concept of Definition 1.6 to the concept of tame quotient. This lemma has a precursor in [27].
LEMMA 2.10. Let $A$ be a finite algebra and let $\alpha < \beta$ be congruences of $A$ such that the lattice $I[\alpha, \beta]$ has no strictly increasing, non-constant, meet endomorphism. Then every $<\alpha, \beta>$-minimal set is the range of some member of $E(A)$.

PROOF. Suppose that $U \in M_A(\alpha, \beta)$. Let $K = \{ f \in Pol_1A : f(A) \subseteq U \}$. We wish to prove that for some $e \in E(A)$, $e(A) = U$. Define a mapping $\mu$ of $I[\alpha, \beta]$ by

$$\mu(\theta) = \beta \land \mu_K(\theta) = \{ (x, y) \in \beta : (f(x), f(y)) \in \theta \} \text{ for all } f \in K.$$ 

$K$ is obviously a right ideal of the monoid $Pol_1A$. From the last exercise (it is an easy direct argument) we have that $\mu$ is an increasing meet endomorphism of the lattice $I[\alpha, \beta]$. Since $U \in M_A(\alpha, \beta)$ there exists $h \in K$ such that $h(A) = U$ and $h(\beta) \not\subseteq \alpha$. Consequently, $\mu(\alpha) < \beta$ and $\mu$ is non-constant. Thus, from the hypothesis about $I[\alpha, \beta]$, there must exist $\theta \in I[\alpha, \beta]$, $\theta < \beta$, with $\mu(\theta) = \theta$. Then $\mu(\alpha) \leq \mu(\theta) = \theta < \beta$. Since $\mu(\alpha) = \beta \land \mu_K(\alpha)$, we have that $\beta \not\subseteq \mu_K(\alpha)$; and so there exist $f, g \in K$ and $(x, y) \in \beta$ such that $(f(x), f(y)) \not\in \alpha$. This implies that $g(\beta) \not\subseteq \alpha$ as well as $fg(\beta) \not\subseteq \alpha$; and since $f(A) \subseteq U$, $g(A) \subseteq U$ it follows that $fg(A) = U = g(A)$ by the $<\alpha, \beta>$-minimality of $U$. We now define $e = f^k$ with the integer $k \geq 1$ chosen so that $e^2 = e$. Now $f(U) = fg(A) = U$ so $e(U) = U$. Since $e \in K$ as well, it follows that $e(A) = U$ as desired. 

THEOREM 2.11. If $\alpha$ and $\beta$ are congruences of a finite algebra $A$ such that $\alpha < \beta$ and the lattice $I[\alpha, \beta]$ is tight, then $<\alpha, \beta>$ is tame.

PROOF. We assume that $I[\alpha, \beta]$ is tight and we choose any $<\alpha, \beta>$-minimal set $U$. By the previous lemma, there exists $e \in E(A)$ with $e(A) = U$. Then the restriction $|u|$, considered only as a map on $I[\alpha, \beta]$, is a lattice homomorphism by Lemma 2.3. We have $\alpha|U \neq \beta|_U$ since $U = f(A)$ for an $f \in Pol_1A$ such that $f(\beta) \not\subseteq \alpha$. Thus $|U$ is a non-constant lattice homomorphism on $I[\alpha, \beta]$. It must be 0,1-separating (by Lemma 1.7). Thus $<\alpha, \beta>$ is tame (by Definition 2.6).

Tame congruence theory, based on Definition 2.6 and Theorems 2.8 and 2.11, fills this book. On the other hand, the circle of ideas introduced in the Exercises 2.9(7–8) will play no further role in the book. Before passing on, let us remark that it might be very worthwhile to attempt a systematic exploitation of those ideas. But that is a path we have not explored.

Our next topic is to be a detailed study of the "$<\alpha, \beta>$-minimal algebras" spawned by tame quotients. The operations of several variables in these algebras are quite interesting, and the study will pay rich dividends. In the remainder of this chapter and in Chapter 3, we introduce the necessary concepts to facilitate the study.
DEFINITION 2.12. A finite algebra $C$ will be called minimal relative to its congruence quotient $\langle \delta, \theta \rangle$, or simply $\langle \delta, \theta \rangle$-minimal, iff $C \in M_C(\delta, \theta)$.

LEMMA 2.13. Let $\langle \delta, \theta \rangle$ be a congruence quotient of a finite algebra $A$.

1. $A$ is $\langle \delta, \theta \rangle$-minimal iff for all $f \in \text{Pol}_1 A$, either $f$ is a permutation of $A$ or $f(\theta) \subseteq \delta$.

2. If $A$ is $\langle \delta, \theta \rangle$-minimal then $\langle \delta, \theta \rangle$ is tame.

3. If $\langle \delta, \theta \rangle$ is tame and $U \in M_A(\delta, \theta)$ then the algebra $A_{|U}$ is $\langle \delta_{|U}, \theta_{|U} \rangle$-minimal.

PROOF. This proof is quite easy, and is left as an exercise for the reader. In item (3), “tameness” can be replaced by “there exists $e \in E(A)$ with $U = e(A)$”. \qed

DEFINITION 2.14. A finite algebra $C$ is called minimal iff $C$ is $\langle 0_C, 1_C \rangle$-minimal, equivalently, $|C| > 1$ and every non-constant $f \in \text{Pol}_1 C$ is a permutation of $C$. A finite algebra $C$ is called E-minimal iff $|C| > 1$ and every non-constant $e \in E(C)$ is equal to $id_C$.

Minimal algebras were termed “permutational” by P.P. Pálfy. E-minimal algebras will not enter our work until somewhat later. Here are a few examples of minimal algebras. A set of permutations acting on a finite set constitutes a minimal algebra. Any finite vector space of more than one element is minimal. Every two-element algebra is minimal. Pálfy proved that there are (up to polynomial equivalence) no other minimal algebras than these. This result will be proved in Chapter 4, as Theorem 4.7.

DEFINITION 2.15. Let $C$ be $\langle \delta, \theta \rangle$-minimal, and let $\langle \alpha, \beta \rangle$ be tame in $A$. By a $\langle \delta, \theta \rangle$-trace in $C$ we mean any set $N \subseteq C$ of the form $x/\theta$ such that $x/\theta \neq x/\delta$. By an $\langle \alpha, \beta \rangle$-trace of $A$ we mean any set $N \subseteq A$ such that for some $U \in M_A(\alpha, \beta)$, $N \subseteq U$ and $N$ is an $\langle \alpha_{|U}, \beta_{|U} \rangle$-trace of the $\langle \alpha_{|U}, \beta_{|U} \rangle$-minimal algebra $A_{|U}$ (i.e., $N = x/\beta \cap U$ for some $x \in U$ such that $x/\beta \cap U \not\subseteq x/\alpha$). The body and the tail of $C$ (with respect to $\langle \delta, \theta \rangle$) are defined in this way:

$$\text{body} = \bigcup \{ \langle \delta, \theta \rangle\text{-traces} \},$$

$$\text{tail} = C - \text{body}.$$  

The body and tail of an $\langle \alpha, \beta \rangle$-minimal set $U$ (with respect to $\langle \alpha_{|U}, \beta_{|U} \rangle$) are defined the same way.

In Figure 3, we depict an $\langle \alpha, \beta \rangle$-minimal set with four traces. The vertical strips represent $\beta_{|U}$-classes; while the $\alpha_{|U}$-classes are represented by ellipses.
Here is a lemma relating the concepts introduced in Definitions 2.12, 2.14, and 2.15. For any congruences $\alpha \leq \beta$ in an algebra $A$, by $\beta/\alpha$ we understand the congruence $\theta$ on $A/\alpha$ such that $(x/\alpha, y/\alpha) \in \theta$ iff $(x, y) \in \beta$. For any $f \in \text{Pol } A$, (say $f$ is $n$-ary) by $f_\alpha$ we mean the operation on $A/\alpha$ satisfying $f_\alpha(x_0/\alpha, \ldots, x_{n-1}/\alpha) = f(x_0, \ldots, x_{n-1})/\alpha$.

**Lemma 2.16.** Let $\alpha \leq \delta < \theta$ be congruences of a finite algebra $C$.

1. $\text{Pol } C/\alpha = \{f_\alpha : f \in \text{Pol } C\}$.
2. If $C$ is $(\delta, \theta)$-minimal, then $C/\alpha$ is $(\delta/\alpha, \theta/\alpha)$-minimal.
3. If $C$ is $(\delta, \theta)$-minimal and $N$ is a $(\delta, \theta)$-trace, then $C|_N$ is $(\delta|_N, 1|_N)$-minimal and $(C|_N)/(\delta|_N)$ is a minimal algebra isomorphic to $(C/\delta)|(N/\delta)$.

**Proof.** The set of all operations $f$ on $C$ such that $f$ preserves $\alpha$ and $f_\alpha \in \text{Pol } C/\alpha$ is easily seen to be closed under compositions, and to contain the constant operations, the trivial projection operations, and the basic operations of $C$. Thus this set contains $\text{Pol } C$; and it follows that $f_\alpha \in \text{Pol } C/\alpha$ whenever $f \in \text{Pol } C$. By an analogous argument, $\text{Pol } C/\alpha \subseteq \{f_\alpha : f \in \text{Pol } C\}$. The two sets are equal, and (1) is proved.

To prove (2), suppose that $C$ is $(\delta, \theta)$-minimal and that $f \in \text{Pol}_1 C/\alpha$ and $f(\theta/\alpha) \not\subseteq (\delta/\alpha)$. By (1), we have that $f = g_\alpha$ for a certain $g \in \text{Pol}_1 C$. There are $x/\alpha, y/\alpha \in C/\alpha$ such that $(x/\alpha, y/\alpha) \in \theta/\alpha$ and $(g_\alpha(x/\alpha), g_\alpha(y/\alpha)) \not\in \delta/\alpha$. These facts are equivalent to $(x, y) \in \theta$, $(g(x), g(y)) \not\in \delta$. Since $C$ is $(\delta, \theta)$-minimal, $g$ must be a permutation of $C$. From this it follows that $f$ maps $C/\alpha$ onto itself; since $C/\alpha$ is finite, $f$ is a permutation. This proves that $C/\alpha$ is $(\delta/\alpha, \theta/\alpha)$-minimal.

Now suppose that $C$ is $(\delta, \theta)$-minimal, and let $N$ be a $(\delta, \theta)$-trace. Let $f \in \text{Pol}_1 C|_N$ and suppose that $f(N^2) \not\subseteq \delta|_N$. There exists $g \in \text{Pol}_1 C$ with $g(N) \subseteq N$ and $g|_N = f$ (see Exercise 2.5 (2)). Since $N$ is a $(\delta, \theta)$-trace, $N^2 \subseteq \theta$; and so $g(\theta) \not\subseteq \delta$, implying
that $g$ is a permutation of $C$. Thus $f$ is one-to-one on the finite set $N$, and is itself a permutation of $N$. This argument shows that $C|_N$ is $(\delta_N, 1_N)$-minimal. Using statement (2) we conclude that $(C|_N)/\langle \delta|_N \rangle$ is a minimal algebra. It is easy to see that $(C|_N)/\langle \delta|_N \rangle \cong (C/\delta)|_{N/\delta}$ since $\delta \subseteq \emptyset$ and $N$ is a $\emptyset$-equivalence class.

Here is a simplified picture of an algebra $A$ with a tame quotient $(0_A, \beta)$. We assume that a $(0_A, \beta)$-minimal set $U$ has exactly three traces and a tail composed of two elements. The vertical lines in the picture accomplish the division of $A$ into the blocks of the equivalence relation $\beta$. The $(0_A, \beta)$-traces and tail elements constituting one $(0_A, \beta)$-minimal set are connected by a line in the picture. A number of traces are pictured as white rectangles independently of the $(0_A, \beta)$-minimal set (or sets) which contain them.

![Figure 4](image)

Theorem 2.8 tells us, among other things, that every pair of $(0_A, \beta)$-minimal sets $U_0$ and $U_1$ are isomorphic. The isomorphism, induced by a pair of polynomials of $A$, maps the relation $\beta|_{U_0}$ onto the relation $\beta|_{U_1}$; and thus the traces in $U_0$ are isomorphic (one for one) with those in $U_1$. It follows that each $(0_A, \beta)$-minimal set $U$ contains a full representative set of traces with respect to the equivalence relation $\simeq$. Every trace sits inside one block of $\beta$. Each block of $\beta$ is actually connected by the traces it contains. That is—in the $(0_A, \beta)$ case—every two elements $x$ and $y$ that are $\beta$-equivalent can be connected by a sequence of overlapping traces. (We try to suggest this in Figure 4.) The next lemma formulates this connectivity property more precisely.
LEMMA 2.17. Let \( \langle \alpha, \beta \rangle \) be a tame quotient in a finite algebra \( A \). Define
\[
\rho = \alpha \cup \bigcup \{N^2 : N \text{ is an } \langle \alpha, \beta \rangle\text{-trace} \}.
\]
Then \( \beta \) is the transitive closure of \( \rho \).

PROOF. Choose any \( U \in M_A(\alpha, \beta) \). By 2.8(5), \( \beta \) is the transitive closure of \( \rho' = \alpha \cup \{(gx, gy) : (x, y) \in \beta|U, g \in \text{Pol}_1 A\} \). Suppose that \( (x, y) \in \beta|U, g \in \text{Pol}_1 A \), and \( (g(x), g(y)) \notin \alpha \). Then \( (x, y) \notin \alpha \) and \( x/\beta \cap U = N \) is a trace with \( x, y \in N \). Also \( g : U \cong g(U) \) by 2.8(3), and \( g(U) \in M_A(\alpha, \beta) \). In this situation, \( g(N) = M \) is a trace. Thus \( (g(x), g(y)) \in \rho \). We conclude that \( \rho' \subseteq \rho \), so the transitive closure of \( \rho \) contains that of \( \rho' \), which is \( \beta \). Since obviously \( \rho \subseteq \beta \), then \( \beta \) is the transitive closure of \( \rho \). \( \square \)

Our strategy for discovering what the algebras \( A|U \) (\( U \in M_A(\alpha, \beta) \)) determined by a tame quotient \( \langle \alpha, \beta \rangle \) may be like, will be to first study a special case. That will be when \( \alpha = 0_A \) and \( U \) is equal to its only trace, so \( A|U \) is a minimal algebra. From this we can build toward an understanding of the general case, using Lemma 2.16 and the following lemma.

LEMMA 2.18. Let \( \delta \leq \alpha < \beta \) be congruences of a finite algebra \( A \). Then \( \langle \alpha, \beta \rangle \) is a tame quotient of \( A \) iff \( \langle \alpha/\delta, \beta/\delta \rangle \) is a tame quotient of \( A/\delta \). If \( \langle \alpha, \beta \rangle \) is tame, we have
\[
M_{A/\delta}(\alpha/\delta, \beta/\delta) = \{U/\delta : U \in M_A(\alpha, \beta)\}.
\]
Moreover, for an \( (\alpha, \beta) \)-minimal set \( U \), the \( (\alpha/\delta, \beta/\delta) \) traces in \( U/\delta \) are just the sets \( N/\delta \) where \( N \) is an \( (\alpha, \beta) \) trace in \( U \).

PROOF. Throughout the argument we use the easily proved fact that for any \( f \in \text{Pol}_1 A \), \( f(\beta) \not\subseteq \alpha \) iff \( f_\delta(\beta/\delta) \not\subseteq \alpha/\delta \); and we also use the fact that \( \text{Pol}_1 A/\delta = \{f_\delta : f \in \text{Pol}_1 A\} \).

Let us assume that \( \langle \alpha, \beta \rangle \) is tame and choose any \( U \in M_A(\alpha, \beta) \). By 2.8(2), there is \( e \in E(A) \) with \( U = e(A) \). We must show that \( U/\delta \in M_{A/\delta}(\alpha/\delta, \beta/\delta) \). First, note that \( e_\delta \in E(A/\delta) \) and \( e_\delta(I_A/\delta) = U/\delta \) and \( e_\delta(\beta/\delta) \not\subseteq \alpha/\delta \). Second, let \( f_\delta \) (\( f \in \text{Pol}_1 A \)) be any unary polynomial of \( A/\delta \) such that \( f_\delta(A/\delta) \subseteq U/\delta \) and \( f_\delta(\beta/\delta) \not\subseteq \alpha/\delta \). Notice that \( f_\delta = e_\delta \circ f_\delta = (ef)_\delta \), and so \( ef(\beta) \not\subseteq \alpha \). Thus \( ef(A) = U \), and this implies that \( f_\delta(A/\delta) = (ef)_\delta(A/\delta) = U/\delta \). We can now conclude that \( U/\delta \in M_{A/\delta}(\alpha/\delta, \beta/\delta) \).

In order to see that \( (\alpha/\delta, \beta/\delta) \) is tame, we consider the restriction mapping of \( I[\alpha/\delta, \beta/\delta] \) in \( \text{Con} A/\delta \) into \( \text{Con}((A/\delta)|_{U/\delta}) \). It will be sufficient to show that it is 0,1-separating, since we already know that \( U/\delta \in M_{A/\delta}(\alpha/\delta, \beta/\delta) \) and \( U/\delta \) is the range of an idempotent polynomial of \( A/\delta \). Every congruence of \( A/\delta \) has the form \( \mu/\delta \) for a \( \mu \geq \delta \) in \( \text{Con} A \). Suppose that \( \alpha \leq \mu < \beta \) in \( \text{Con} A \). Since \( \langle \alpha, \beta \rangle \) is tame, there exists \( (x, y) \in \beta|U - \mu|U \). We have \( (x/\delta, y/\delta) \in (\beta/\delta - \mu/\delta) \cap (U/\delta)^2 \). This
shows that the restriction map is 1-separating on \( I[\alpha/\delta, \beta/\delta] \), and the proof that it is 0-separating is essentially the same. We have now proved that \( \langle \alpha/\delta, \beta/\delta \rangle \) is tame and (since \( U \) was arbitrary) that \( U/\delta \in M_{A/\delta}(\alpha/\delta, \beta/\delta) \) whenever \( U \in M_A(\alpha, \beta) \).

To get the other inclusion, let \( W \) be any member of \( M_{A/\delta}(\alpha/\delta, \beta/\delta) \). Choose any \( U \in M_A(\alpha, \beta) \). Applying 2.8(1) to the tame quotient \( \langle \alpha/\delta, \beta/\delta \rangle \), we get that there is \( f_\delta \) such that \( f_\delta : U/\delta \simeq W \). In this situation, \( f_\delta(\beta/\delta) \not\subset \alpha/\delta \), so \( f(\beta) \not\subset \alpha \); and by 2.8(3) \( f(U) \in M_A(\alpha, \beta) \). Furthermore, \( W = f_\delta(U/\delta) = f(U)/\delta \). This concludes our proof that \( M_{A/\delta}(\alpha/\delta, \beta/\delta) = M_A(\alpha, \beta)/\delta \).

We now change our assumptions. Let us suppose that \( \langle \alpha/\delta, \beta/\delta \rangle \) is a tame quotient of \( A/\delta \). We choose an arbitrary \( U \in M_A(\alpha, \beta) \) and \( f \in \Pol_1 A \) such that \( f(A) = U \), \( f(\beta) \not\subset \alpha \). Then \( f_\delta(\beta/\delta) \not\subset \alpha/\delta \), so by 2.8(6) applied to \( \langle \alpha/\delta, \beta/\delta \rangle \) there is \( W \in M_{A/\delta}(\alpha/\delta, \beta/\delta) \) with \( f_\delta : W \simeq f_\delta(W) \). Thus there exists \( g_\delta \) such that \( f_\delta : W \rightarrow f_\delta(W) \) are inverse bijections. Let \( e = (f\delta)^n \) be an idempotent of \( g \). Now \( e_\delta \) equals the identity on \( f_\delta(W) \), hence \( e(\beta) \not\subset \alpha \). We conclude that \( e(A) = f(A) = U \) since \( U \) is \( \langle \alpha, \beta \rangle \)-minimal. To finish the proof that \( \langle \alpha, \beta \rangle \) is tame, we show that \( U/\delta \) is an \( \langle \alpha/\delta, \beta/\delta \rangle \)-minimal set, and using the fact that \( U/\delta \) must then be 0,1-separating on \( I[\alpha/\delta, \beta/\delta] \), we show that \( U \) is \( 0,1 \)-separating on \( I[\alpha, \beta] \). These details are left to the reader.

\[ \square \]

Exercises 2.19

(1) Suppose that \( U \) is an \( \langle \alpha, \beta \rangle \)-minimal set for a congruence quotient \( \langle \alpha, \beta \rangle \) in a finite algebra \( A \), and that \( U = e(A) \) for some \( e \in E(A) \). For every \( \alpha|U \leq \delta' < \gamma' \leq \beta|U \), there exists at least one pair \( \langle \delta, \gamma \rangle \) of congruences of \( A \) such that \( \delta|U = \delta', \gamma|U = \gamma', \alpha \leq \delta \leq \gamma \leq \beta \), and the lattice homomorphism \( |U \) is 0,1-separating on \( I[\delta, \gamma] \). Show that each such quotient \( \langle \delta, \gamma \rangle \) is tame, and that \( U \in M_A(\delta, \gamma) \subseteq M_A(\alpha, \beta) \). (Moreover, \( M_A(\delta, \gamma) = M_A(\alpha, \beta) \) if \( \langle \alpha, \beta \rangle \) is tame.)

(2) Suppose that \( \alpha \leq \delta < \beta \) where \( \delta, \beta \) are congruences of a finite algebra \( A \). Thus \( \langle \delta, \beta \rangle \) is tame, according to Theorem 2.11. Show that if \( \langle \alpha, \beta \rangle \) is tame then \( M_A(\alpha, \beta) = M_A(\delta, \beta) \).

(3) Suppose that \( \langle \alpha_i, \beta_i \rangle \) \( (i = 0, 1) \) are congruence quotients of a finite algebra \( A \) and \( \alpha_0 = \beta_0 \wedge \alpha_1 \), \( \beta_0 = \beta_0 \vee \alpha_1 \). Prove that \( M_A(\alpha_0, \beta_0) = M_A(\alpha_1, \beta_1) \), and \( M_A(\alpha_0, \beta_0) \subseteq M_A(\alpha_0, \beta_0) \cup M_A(\alpha_0, \alpha_1) \). Conclude that if \( \Con A \cong M_3 \) then (all seven congruence quotients are tame and) \( M_A(\alpha, \beta) \) is independent of the quotient \( \langle \alpha, \beta \rangle \).

(4) Prove that when \( \langle \alpha, \beta \rangle \) is tame and \( U \in M_A(\alpha, \beta) \), then \( \alpha < \beta \) if \( \alpha|U < \beta|U \). Thus \( \alpha < \beta \) if \( |U| = 2 \). Show that if \( A \) is a finite lattice and \( \alpha < \beta \) in \( \Con A \), then every \( \langle \alpha, \beta \rangle \)-minimal set has only two elements—and thus \( \langle \alpha, \beta \rangle \) is tame if \( \alpha < \beta \).
(5) Let $M$ be a finite module over a finite ring $R$ with 1, and suppose that $M$ is unitary, i.e., $1 \cdot x = x$ for $x \in M$. We regard $M$ as an algebra

$$\langle M, x + y, 0, r \cdot x (r \in R) \rangle.$$ 

Let $\alpha < \beta$ be congruences of $M$, and let $M_\alpha = 0/\alpha$, $M_\beta = 0/\beta$ be the associated submodules of $M$. Put $\operatorname{ann}(M_\alpha | M_\beta) = \{ r \in R : r \cdot M_\beta \subseteq M_\alpha \}$. Then show that $\langle \alpha, \beta \rangle$ is tame iff $\operatorname{ann}(M_\alpha | M_\beta)$ is a maximal ideal in $R$.

(6) Suppose that $\langle \alpha, \beta \rangle$ is a tame quotient of a finite algebra $A$, that $N$ is an $\langle \alpha, \beta \rangle$-trace (see Definition 2.15), and that $f \in \operatorname{Pol}_1 A$. Prove that either $f(N^2) \subseteq \alpha$, or $f(N)$ is an $\langle \alpha, \beta \rangle$-trace $N'$ and $f : N \simeq N'$. (See Theorem 2.8 (3).)