5. THE TYPES OF TAME QUOTIENTS

We are now ready to define and study the five types of tame congruence quotients. In this chapter, we will delineate the distinct characters of these types, primarily in relation to the “polynomial structure” of an algebra. In the next two chapters, we shall consider the congruence lattice of an algebra as a labeled graph, where all of the prime quotients are labeled with their respective types. We shall be concerned with the ways in which this labeling is influenced by the unlabeled congruence lattice, construed purely as an abstract lattice.

**DEFINITION 5.1.**

1. Let \( \langle \alpha, \beta \rangle \) be a tame quotient of congruences in a finite algebra \( A \). Let \( U \) be any element of \( M_A(\alpha, \beta) \). We define the type of \( \langle \alpha, \beta \rangle \), written \( \text{typ}(\alpha, \beta) \), to be the type of \( A_U \) relative to \( \langle \alpha_U, \beta_U \rangle \).

2. Let \( \langle \gamma, \lambda \rangle \) be any quotient of congruences in a finite algebra \( A \). By \( \text{typ}\{\gamma, \lambda\} \) we denote the set \( \{\text{typ}(\alpha, \beta) : \gamma \leq \alpha \prec \beta \leq \lambda\} \).

3. Let \( A \) be any finite algebra. We call \( A \) tame iff the quotient \( \langle 0_A, 1_A \rangle \) is tame (implying that \( |A| > 1 \)). If \( A \) is tame, we put \( \text{typ}(A) = \text{typ}(0_A, 1_A) \).

4. Let \( A \) be any finite algebra. By \( \text{typ}\{A\} \) we denote the set \( \text{typ}\{0_A, 1_A\} \) of types.

The type of a tame quotient \( \langle \alpha, \beta \rangle \) in a finite algebra \( A \) is well-defined by the above. To verify this, let \( U_0 \) and \( U_1 \) be \( \langle \alpha, \beta \rangle \)-minimal sets. According to Theorem 2.8 (1) and Exercise 2.9 (5), there exists an isomorphism between the structures \( \langle U_0, \text{Pol}_A[\alpha_U, \beta_U] \rangle \) and \( \langle U_1, \text{Pol}_A[\alpha_U, \beta_U] \rangle \). Therefore, the type of \( A_U \), relative to \( \langle \alpha_U, \beta_U \rangle_U \), is the same for \( i = 0 \) and \( i = 1 \).

A first corollary of this definition and of our earlier work is worth noting. Recall that for a tame quotient \( \langle \alpha, \beta \rangle \) in \( A \), an \( \langle \alpha, \beta \rangle \)-trace is simply any set \( N \) such that for some \( U \in M_A(\alpha, \beta) \) and \( x \in U \), we have \( N = (x/\beta) \cap U \neq (x/\alpha) \cap U \).

**COROLLARY 5.2.** Let \( \langle \alpha, \beta \rangle \) be a tame quotient in a finite algebra \( A \).

1. For every \( \langle \alpha, \beta \rangle \)-trace \( N \), the algebra \( M = (A|_N)/(\alpha|_N) \) is a minimal (and therefore tame) algebra, and \( \text{typ}(\alpha, \beta) = \text{typ}(M) \).

2. If \( \text{typ}(\alpha, \beta) \neq 1 \) or if \( \alpha \prec \beta \), then for every pair of \( \langle \alpha, \beta \rangle \)-traces \( N_0 \) and \( N_1 \), we have \( N_0 \simeq N_1 \) and \( M_0 \cong M_1 \), where \( M_i = (A|_{N_i})/(\alpha|_{N_i}) \).
PROOF. Let $U$ be any $\langle \alpha, \beta \rangle$-minimal set and let $N$ be an $\langle \alpha|_U, \beta|_U \rangle$-trace. Put $C = A|_U$ and $\langle \delta, \theta \rangle = \langle \alpha|_U, \beta|_U \rangle$. Note that $C|_N = A|_N$, since $U$ is the range of some unary polynomial $e$ of $A$ with $e = e^2$. Thus $M = (A|_N)/(\langle \alpha|_N \rangle) = (C|_N)/(\langle \delta|_N \rangle)$. The type of $C$ relative to $\langle \delta, \theta \rangle$ (equal to typ($\alpha, \beta$) by Definition 5.1) was defined in Definition 4.21 to be the type of the minimal algebra $M$. (Recall from Chapter 4 that this is independent of $N$.) Since $M$ is the only $\langle 0_M, 1_M \rangle$ trace in $M$, this type is the same as typ($M$). This concludes the proof of (1).

For $i \in \{0, 1\}$, let $N_i$ be an $\langle \alpha|_{U_i}, \beta|_{U_i} \rangle$-trace, where $U_i \in M_A(\alpha, \beta)$. We have $U_0 \simeq U_1$ in $A$, so $N_1 \simeq N'$ in $A$ for some $\langle \alpha|_{U_0}, \beta|_{U_0} \rangle$-trace $N'$. Suppose that typ($\alpha, \beta$) $\neq 1$. Then by Lemmas 4.12 and 4.20 (5) all $\langle \alpha_0, \beta_0 \rangle$-traces are $\simeq$ in $A|_{U_0}$. This implies that $N_0 \simeq N' \simeq N_1$ in $A$. (For the definition of polynomial isomorphism, $\simeq$, see Definition 2.7 and the remarks following it.) If $\alpha \prec \beta$, then Lemma 2.3 implies that $\alpha|_{U_0} \prec \beta|_{U_0}$. Thus $\alpha|_{U_0} \vee \Theta(N_0^2) = \beta|_{U_0}$, and this implies that there is $f \in \text{Pol}_1 A|_{U_0}$ such that $f(N_0) \cap N' \neq \emptyset$ and $f(N_0)^2 \not\subseteq \alpha$. By 2.8(3), $f \in \text{Sym} U_0$, and it follows that $f(N_0) = N'$ and $N_0 \simeq N'$ as before. 

To avoid monotony, we shall sometimes refer to the types by their names, introduced in Definition 4.10. These are, in the order $1, \ldots, 5$: **unary**, **affine**, **Boolean**, **lattice**, and **semilattice** type. The **lattice of types** is pictured below:

```
    3
   /\  \\/
  /   \  /
 4    5
   \   /  \
    \ /  \\
    1
```

**Figure 10**

This lattice is obtained from the lattice pictured below Lemma 4.8 by identifying two-element algebras which have the same type, namely $E_2$ and $E_6$, $E_0$ and $E_1$. (This identification is not a lattice homomorphism.) The ordering of types pictured in Figure 10 is in terms of the richness of the set of binary operations depending on both variables, in the polynomial clone of a two-element algebra of the type. For each of the six proper order ideals in the lattice of types, we shall prove in Chapter 9 an **omitting types theorem**. These theorems will characterize locally finite varieties which have no prime quotients of a type belonging to a given ideal, in terms of several equivalent conditions not involving tame congruence theory. For example, a locally finite variety omits the types $1, 2, 5$ if and only if the congruence lattices of its finite algebras are semi-distributive.
Here is an important corollary of our earlier work.

**COROLLARY 5.3.** Let $\delta \leq \gamma \leq \alpha < \beta \leq \lambda$ be congruences of a finite algebra $A$ and assume that $(\alpha, \beta)$ is tame. Then $\text{typ}(\alpha, \beta) = \text{typ}(\alpha/\delta, \beta/\delta)$ (computed in $A/\delta$) and $\text{typ}(\gamma, \lambda) = \text{typ}(\gamma/\delta, \lambda/\delta)$.

**PROOF.** This is an easy consequence of Lemma 2.18, Definition 4.21, Theorem 4.23, and Definition 5.1. \n
**Remark 5.4.** The structure of an $(\alpha, \beta)$-minimal set, when $(\alpha, \beta)$ is tame of type $5$ in $A$, is largely defined in Lemma 4.15. When the type is $3$ or $4$, Lemma 4.17 defines the situation. When the type is $2$, Lemma 4.20 applies, and Lemmas 4.24–4.27, 4.30, 4.31, 4.34, and 4.36 provide supplementary information for this case. We shall make frequent use of these Lemmas. Notice that when $\text{typ}(\alpha, \beta) = 3$ or $4$, all the $(\alpha, \beta)$-traces are two-element sets (by Lemma 4.17). When $\text{typ}(\alpha, \beta) = 2$, the induced algebras on the traces are Mal’cev (by Lemma 4.20).

The first substantial theorem of this chapter characterizes tame quotients of affine type. It was first proved by the authors, but the argument used here is due to P.P. Pályi.

**THEOREM 5.5.** A tame quotient has affine type if and only if it is Abelian and not strongly Abelian.

**PROOF.** Let $(\alpha, \beta)$ be a tame quotient in a finite algebra $A$. If $\text{typ}(\alpha, \beta) \in \{3, 4, 5\}$ then, choosing any pair of elements $u$ and $v$ in an $(\alpha, \beta)$-trace $N$, such that $(u, v) \in \beta - \alpha$, it follows from Lemma 4.15 or Lemma 4.17 that $A|_{(u,v)}$ is a non-Abelian algebra. (In fact, $A$ has a binary polynomial $f$ such that $f(u, u) = u = f(u, v)$ and $f(v, u) = u$, while $f(v, v) = v$, or the same equations with $u$ and $v$ interchanged.)

If $\text{typ}(\alpha, \beta) = 2$, then $(\alpha, \beta)$ cannot be strongly Abelian because for every $(\alpha, \beta)$-trace $N$, the algebra $A|_N$ is Mal’cev. (See Definition 3.9 and Lemma 4.20.)

In the next theorem, we will prove that $(\alpha, \beta)$ is strongly Abelian if $\text{typ}(\alpha, \beta) = 1$. Assuming this fact, it follows that if $(\alpha, \beta)$ is Abelian and not strongly Abelian, then $\text{typ}(\alpha, \beta) = 2$ (as all other possibilities are ruled out). To prove Theorem 5.5, all we have to do is prove that $\text{typ}(\alpha, \beta) = 2$ implies $(\alpha, \beta)$ is Abelian. Since both properties are invariant under factoring out $\alpha$, we can assume that $\alpha = 0_A$. (See Proposition 3.7 and Corollary 5.3.)

Now assume that $(0_A, \beta)$ is tame in $A$ of affine type, i.e., $\text{typ}(0_A, \beta) = 2$. It must be shown that $\beta$ is an Abelian congruence, i.e., $C(\beta, \beta; 0_A)$ holds (in the notation of Definition 3.3). Letting $f \in \text{Pol}_{n+1}A$ for some $n$, and $(c_1, d_1), \ldots, (c_n, d_n) \in \beta$, our task is to prove that for all $(a, b) \in \beta$, if $f(a, c_1, \ldots, c_n) = f(a, d_1, \ldots, d_n)$ then $f(b, c_1, \ldots, c_n) = f(b, d_1, \ldots, d_n)$ (or more briefly, $f(a, \bar{c}) = f(a, \bar{d}) \Rightarrow f(b, \bar{c}) = f(b, \bar{d})$). Note that if the implication $f(b_0, \bar{c}) = f(b_0, \bar{d}) \Rightarrow f(b_1, \bar{c}) = f(b_1, \bar{d})$ holds
whenever \( \{b_0, b_1\} \) is contained in a \((0_A, \beta)\)-trace, then it must hold whenever \( \langle b_0, b_1 \rangle \in \beta \). (See Lemma 2.17 and the remarks proceeding it.)

Thus, we can assume that \( N \) is a \((0_U, \beta|U)\)-trace for a certain \( U \in M_A(0_A, \beta) \), that \( \{a, b\} \subseteq N \), and that \( f(a, c) = f(a, d) \) while \( f(b, c) \neq f(b, d) \). We shall derive a contradiction from these assumptions, and that will finish the proof.

By Theorem 2.8 (4), there exists \( h \in \text{Pol}_1A \) such that \( h(A) = U \) and \( h(f(b, c)) \neq h(f(b, d)) \). Replacing \( f \) by \( h \circ f \), we now have that \( f(A^{n+1}) \subseteq U \) and the other assumptions are unaltered. The elements \( f(a, c), f(b, c), f(b, d) \) all belong to a \((0_U, \beta|U)\)-trace \( N' \). Note that Lemma 4.20 applies to the algebra \( A|U \) relative to \((\delta, \theta) = (0_U, \beta|U) \). By 4.20 (5), there is a unary polynomial \( g \) of \( A \) such that \( g(U) \subseteq U \) (i.e., \( g|U \in \text{Pol}_1A|U \)) and \( g \) maps \( N' \) bijectively onto \( N \). Replacing \( f \) by \( g \circ f \), we now have that

\[
\{a, b, f(a, c), f(b, c), f(b, d)\} \subseteq N
\]

and the other assumptions are unaltered. Pályi's argument begins at this point.

We define \( T_i = c_i/\beta \) (1 \( \leq \) \( i \) \( \leq \) \( n \)), and observe that

\[
f(N \times T_1 \times \cdots \times T_n) \subseteq U \cap f(a, c)/\beta = N.
\]

Using Lemma 2.17 again, we choose, for each \( i \in \{1, \ldots, n\} \) a sequence \( N(i, 0), \ldots, N(i, k) \) of \((0_A, \beta)\)-traces such that

\[
(5.5.1) \quad c_i \in N(i, 0) \text{ and } d_i \in N(i, k), \quad \text{and } N(i, j) \cap N(i, j + 1) \neq \emptyset
\]

for all \( j < k_i \). Obviously, we can arrange that all of the \( k_i \) have the same value \( k \).

Notice that \( \bigcup N(i, j) : j \leq k \subseteq T_i \) for each \( i \).

We choose, by Corollary 5.2 (2), for each \( i \) and \( j \) a function \( \alpha_{ij} \) in \( \text{Pol}_1A \) such that

\[
(5.5.2) \quad \alpha_{ij}(N) = N(i, j) \quad \text{and} \quad \alpha_{ij}|_N \text{ is one-to-one}.
\]

We now define some polynomials of the algebra \( A|N \) by setting

\[
(5.5.3) \quad f_j(x, x_1, \ldots, x_n) = f(x, \alpha_{1j}(x_1), \ldots, \alpha_{nj}(x_n)) \quad \text{for} \quad 0 \leq j \leq k.
\]

Notice that for \( j \leq k \) and \( x, x_1, \ldots, x_n \in N \), we have \( \alpha_{ij}(x_i) \in N_{ij} \subseteq T_i \) for all \( i \), and hence \( f_j(x, x_1, \ldots, x_n) \in N \). Each operation \( f_j|_N \) is therefore a polynomial operation of \( A|N \).

The algebra \( A|N = (A|U)|_N \) is a minimal algebra of type 2; it is polynomially equivalent to a vector space over a finite field \( F \). Thus there exist elements \( e_0, \ldots, e_k \in N \), and elements \( \mu_{ij} \in F \) (\( i \leq n, j \leq k \)) such that, expressed in terms of the vector space operations of \( A|N \), we have

\[
(5.5.4) \quad \text{for } 0 \leq j \leq k \text{ and for all } x_0, x_1, \ldots, x_n \in N,
\]

\[
f_j(x_0, x_1, \ldots, x_n) = \mu_{0j} \cdot x_0 + \cdots + \mu_{nj} \cdot x_n + e_j.
\]
Claim. For all \( j < k \), \( \mu_{0j} = \mu_{0 \ j+1} \).

To prove this claim, choose elements \( u_1, v_1, \ldots, u_n, v_n \) in \( N \) such that \( \alpha_{ij}(u_i) = \alpha_{i \ j+1}(v_i) \) for \( 1 \leq i \leq n \) (which can be done, by (5.5.1) and (5.5.2)). For all \( x \in N \), we have

\[
f_j(x, u_1, \ldots, u_n) = f_{j+1}(x, v_1, \ldots, v_n)
\]

by (5.5.3), and so

\[
\mu_{0j} \cdot x + \mu_{1j} \cdot u_1 + \cdots + \mu_{nj} \cdot u_n + e_j = \mu_{0 \ j+1} \cdot x + \mu_{1 \ j+1} \cdot v_1 + \cdots + \mu_{n \ j+1} \cdot v_n + e_{j+1}
\]

by (5.5.4). Thus \( (\mu_{0j} - \mu_{0 \ j+1}) \cdot x \) is constant, independent of \( x \in N \), implying that \( \mu_{0j} = \mu_{0 \ j+1} \), as claimed.

We can now bring this proof to a conclusion. Let \( \mu = \mu_{00} = \mu_{ab} \). Choose, by (5.5.1) and (5.5.2), elements \( c'_1, d'_1, \ldots, c'_n, d'_n \) in \( N \) such that \( \alpha_{i0}(c'_i) = c_i \) and \( \alpha_{ik}(d'_i) = d_i \) for \( 1 \leq i \leq n \). By (5.5.3),

\[
\alpha_0(a, c'_1, \ldots, c'_n) = f(a, \bar{c}) = f(a, \bar{d}) = f_k(a, d'_1, \ldots, d'_n).
\]

Written another way,

\[
\mu \cdot a + \mu_{10} \cdot c'_1 + \cdots + \mu_{n0} \cdot c'_n + e_0 = \mu \cdot a + \mu_{1k} \cdot d'_1 + \cdots + \mu_{nk} \cdot d'_n + e_k.
\]

Obviously, this equation must remain valid when we replace \( a \) by \( b \). But that means that \( f(b, \bar{c}) = f(b, \bar{d}) \). This contradicts our starting assumption, and ends the proof of this theorem.

\[\square\]

**Theorem 5.6.** A tame quotient has unary type if and only if it is strongly Abelian.

**Proof.** In the first two paragraphs of the preceding proof, we noted that \( \langle \alpha, \beta \rangle \) tame and strongly Abelian implies \( \text{typ}(\alpha, \beta) = 1 \). We shall now prove the converse. As in the last proof, it suffices to derive the result in the case \( \alpha = 0_A \). We now assume that \( \langle 0_A, \beta \rangle \) is tame in \( A \), of unary type.

Claim 1. If \( N, N_0, N_1 \) are \( \langle 0_A, \beta \rangle \)-traces and \( f \in \text{Pol}_2A \) and \( f(N_0 \times N_1) \subseteq N \), then \( f|_{N_0 \times N_1} \) depends on at most one variable.

To prove it, suppose, to the contrary, that for some \( x_0, x_1, u \in N_0 \) and \( y_0, y_1, v \in N_1 \) we have \( f(x_0, v) \neq f(x_1, v) \) and \( f(u, y_0) \neq f(u, y_1) \). Then by Exercise 2.19 (6), we have that \( \alpha_i : N_i \simeq N \) \( (i = 0, 1) \), where \( \alpha_0(x) = f(x, v) \) and \( \alpha_1(x) = f(u, x) \). There are unary polynomials \( \beta_i \) such that \( \alpha_i \beta_i|N = \text{id}_N \) and \( \beta_i \alpha_i|N_i = \text{id}_{N_i} \) \( (i = 0, 1) \). The polynomial \( h(x, y) = f(\beta_0(x), \beta_1(y)) \), restricted to \( N \), is a polynomial of \( A|N \). Since \( A|N \) is a minimal algebra and \( \text{typ}(A|N) = \text{typ}(0_A, \beta) = 1 \), the operation \( h|N \) can depend on only one variable. But, clearly, like \( f|_{N_0 \times N_1} \) it depends on both, so we have a contradiction, establishing the claim.

Claim 2. If \( N \) is a \( \langle 0_A, \beta \rangle \)-trace, \( T_0 \) and \( T_1 \) are \( \beta \)-equivalence classes, \( f \in \text{Pol}_2A \), and \( f(T_0 \times T_1) \subseteq N \), then \( f|_{T_0 \times T_1} \) depends on at most one variable.
To prove it, suppose, to the contrary, that for some \( u \in T_0 \) and \( v \in T_1 \), the functions \( \alpha_0(x) = f(x, v) \) and \( \alpha_1(x) = f(u, x) \) are non-constant on \( T_0 \) and on \( T_1 \), respectively. By an obvious application of Lemma 2.17, there must exist \( (0_A, \beta) \)-traces \( N_i \subseteq T_i \) such that \( \alpha_i|_{N_i} \) is non-constant \( (i = 0, 1) \). We claim that for any \( v' \in T_1 \), \( f(x, v') = \alpha_0(x) \) on \( N_0 \). Use Lemma 2.17 to get \( (0_A, \beta) \)-traces \( M_0, \ldots, M_k \) such that \( v \in M_0 \), \( v' \in M_k \) and \( M_i \cap M_{i+1} \neq \emptyset \) for all \( i < k \). Let \( v_1 \in M_0 \cap M_1 \). By Claim 1, \( \alpha_0(x) = f(x, v) = f(x, v_1) \) for \( x \in N_0 \). An easy induction along these lines yields \( f(x, v') = \alpha_0(x) \).

A similar argument implies that for any \( u' \in T_0 \), \( f(u', y) = \alpha_1(y) \) for all \( y \) in \( N_1 \). But now if \( (x, y) \in N_0 \times N_1 \), then

\[
\alpha_0(x) = f(x, y) = \alpha_1(y);
\]

and so \( \alpha_0|_{N_0} \) is constant, a contradiction.

Claim 3. If \( N \) is an \( (0_A, \beta) \)-trace, \( f \in \text{Pol}_A \) (for any integer \( n \)), \( T = T_0 \times \cdots \times T_{n-1} \) where \( T_0, \ldots, T_{n-1} \) are \( \beta \)-equivalence classes, and \( f(T) \subseteq N \), then \( f|_T \) depends on at most one variable.

This claim reduces to Claim 2 through obvious applications of Lemma 4.1. If \( f|_T \) depends on two or more variables, then at most \( n - 2 \) applications of the lemma will produce a binary polynomial \( f' \) (which is \( f \) with constants substituted for \( n - 2 \) of its variables) that contradicts Claim 2.

We can now finish the proof of this theorem. By Definitions 3.9 and 3.10, our task is to prove the following. Letting \( f \in \text{Pol}_A \) (for any \( n \)) and \( c_0, d_0, c_1, d_1, e_1, \ldots, e_{n-1}, d_n-1, e_{n-1} \in A \), and assuming that \( c_0 \equiv d_0 \) (mod \( \beta \)) and \( e_i \equiv d_i \equiv e_i \) (mod \( \beta \)) for \( 1 \leq i < n \), and that

\[
f(c_0, e) = f(c_0, e_1, \ldots, e_{n-1}) \neq f(d_0, e_1, \ldots, e_{n-1}) = f(d_0, e),
\]

then we must have that

\[
f(e) = f(c_0, \ldots, c_{n-1}) \neq f(d_0, \ldots, d_{n-1}) = f(d).
\]

To prove this (under the stated assumptions), let \( T_i = c_i/\beta \) (for \( 0 \leq i < n \)). Notice that \( f(c_0, e) \equiv f(d_0, e) \) (mod \( \beta \)). We apply Theorem 2.8(4) and obtain a set \( U \in M_A(0_A, \beta) \) and a polynomial \( h \in \text{Pol}_1A \) such that \( h(A) = U \) and \( hf(c_0, e) \neq hf(d_0, e) \). Let \( N \) be the \( (0_U, \beta|_U) \)-trace containing \( hf(c_0, e) \). Let \( f'(x_0, \ldots, x_{n-1}) = hf(x_0, \ldots, x_{n-1}) \), and observe that, since \( f'(A^n) \subseteq U \) and \( f'(c_0, e) \in N \), we have \( f'(T_0 \times \cdots \times T_{n-1}) \subseteq N \).

By Claim 3, \( f'|_{T_0 \times \cdots \times T_{n-1}} \) depends on at most one variable. Since \( f'(c_0, e) \neq f'(d_0, e) \), it must depend on the first variable, and no other. Therefore \( f'(e) = f'(c_0, e) \).
and \( f'(d) = f'(d_0, \varepsilon) \), implying that \( f'(\varepsilon) \neq f'(d) \); a fortiori, \( f(\varepsilon) \neq f(d) \), as desired. This ends the proof that \( \beta \) is strongly Abelian.

The two theorems above generalize Lemma 4.14 and Exercise 4.19 (3) (which treated the special case where \( A \in M_{\mathbf{A}}(\alpha, \beta) \)). These results can be neatly reformulated as a local reduction principle for the Abelian property—a tame quotient \( (\alpha, \beta) \) is Abelian (or strongly Abelian) iff the corresponding minimal algebras, obtained by factoring the trace algebras, are Abelian (or strongly Abelian). We shall see a similar principle later in this chapter, concerning prime quotients of lattice or semilattice type, which involves an orderability property in place of the Abelian properties.

The Abelian types are 1 and 2; the non-Abelian types are 3, 4 and 5. The next theorem summarizes what we have learned thus far about the types, and contains a new result that has some interesting corollaries. After looking at these corollaries, we shall begin a study of the non-Abelian types. Recall that a quotient \( (\alpha, \beta) \) is called prime iff \( \alpha \prec \beta \), i.e., \( \beta \) covers \( \alpha \).

**THEOREM 5.7.** Let \( A \) be a finite algebra.

1. Every prime congruence quotient of \( A \) is tame.
2. For any quotient \( (\alpha, \beta) \) of \( A \), the following are equivalent:
   1. \( (\alpha, \beta) \) is prime and non-Abelian.
   2. \( (\alpha, \beta) \) is tame and \( \text{typ}(\alpha, \beta) \in \{3, 4, 5\} \).
3. A tame quotient \( (\alpha, \beta) \) has type 1 if it is strongly Abelian, and has type 2 if it is Abelian but not strongly Abelian.
4. For any quotient \( (\alpha, \beta) \) of \( A \) that is not strongly Abelian, the following are equivalent:
   1. \( (\alpha, \beta) \) is tame.
   2. The interval lattice \( I[\alpha, \beta] \) is tight.
   3. \( I[\alpha, \beta] \) is 0,1-simple and complemented.
   4. \( I[\alpha, \beta] \) admits a 0,1-separating homomorphism onto the congruence lattice of a vector space. (This homomorphism is essentially unique.)

**PROOF.** Statement (1) follows from Theorem 2.11 and Definition 1.6. Any two-element lattice is tight.

Statement (3) reiterates Theorems 5.5 and 5.6. The implication "(i) implies (ii)" in statement (2) follows from this and from (1). The implication "(ii) implies (i)" in statement (2) is proved in this way. Let \( (\alpha, \beta) \) be tame and of non-Abelian type, i.e., 3, 4 or 5. Let \( U \in M_{\mathbf{A}}(\alpha, \beta) \). The \( (\alpha|_U, \beta|_U) \)-minimal algebra \( A|_U \) of type 3, 4 or 5 has, by examination of Lemma 4.15 or 4.17, the property that \( \alpha|_U \prec \beta|_U \) in the congruence lattice \( \text{Con} A|_U \). By Definition 2.6 and Theorem 2.8 (2), the restriction map \( |_U \) is a 0,1 separating lattice homomorphism of \( I[\alpha, \beta] \) onto the interval \( I[\alpha|_U, \beta|_U] \) in \( \text{Con} A|_U \). Only a two-element lattice can have a 0,1-separating homomorphism.
onto a two-element lattice. Therefore $\alpha \prec \beta$ in $\text{Con } A$. We know that $\langle \alpha, \beta \rangle$ is non-Abelian, from (3).

To prove (4), observe that if $\alpha \prec \beta$ then statements (ii), (iii) and (iv) are trivially true (take a vector space of dimension 1 over a two-element field to prove (iv)) and (i) is true by (1). Thus, we assume that $I[\alpha, \beta] \geq 3$. Now (iv) implies (iii) by Exercises 1.14 (1, 3); and (iii) implies (ii) by Example 1.11. By Theorem 2.11, we have that (ii) implies (i).

To show (i) implies (iv), let us now assume that $\langle \alpha, \beta \rangle$ is not strongly Abelian (and is not prime) and that (i) holds. Thus $\text{typ}(\alpha, \beta) = 2$; the other possibilities are ruled out by (2) and (3). We choose $U \in M_A(\alpha, \beta)$, and we put $C = A_U$ and $\langle \delta, \theta \rangle = \langle \alpha_U, \beta_U \rangle$. Thus $C$ is minimal of type 2 relative to $\langle \delta, \theta \rangle$. Let $N$ be any $\langle \delta, \theta \rangle$-trace (i.e., a $\langle \alpha_U, \beta_U \rangle$-trace). By Lemma 4.24, $I[\delta, \theta]$ is isomorphic to the congruence lattice of the vector space $(C_{|N})/C_{|N}) = (A_{|N})/(A_{|N})$. By Theorem 2.8 (2), the restriction $|U$ is a 0,1-separating homomorphism of $I[\alpha, \beta]$ onto $I[\delta, \theta]$. This completes the proof that 4(i) implies 4(iv) when $\langle \alpha, \beta \rangle$ is not strongly Abelian. The essential uniqueness comes from Lemma 1.10 (2).

Here is a very easy and noteworthy corollary of Theorem 5.7.

**COROLLARY 5.8.** Let $A$ be any finite algebra having at least three congruences. If $\text{Con } A$ is a tight lattice, then $A$ is Abelian. If $\text{Con } A$ is tight, and does not admit a 0,1-separating homomorphism onto the congruence lattice of a vector space, then $A$ is strongly Abelian.

**Remark 5.9.** The simplest lattices for which Corollary 5.8 is interesting are the height two lattices $M_n$ ($n \geq 3$) with $n$ atoms, pictured in Figure 1. $M_n$ is tight, and has a 0,1-separating homomorphism "onto a vector space" iff $n - 1$ is a power of a prime.

Here is another, not so immediate, corollary of Theorem 5.7. The result is obtained by applying Corollary 5.8 to a finitely generated free algebra of a locally finite variety, augmented by a substitution operation. We omit the proof, since it would take us somewhat out of our way to develop the required concepts at this point. The proof can be found in [22].

**COROLLARY 5.10.** If $V$ is a locally finite variety that possesses more than two subvarieties, its lattice of subvarieties is not a finite tight lattice.

**Exercises 5.11**

1. Let $\langle \alpha, \beta \rangle$ be a tame quotient in a finite algebra $A$. Prove that the following are equivalent.
(i) $\langle \alpha, \beta \rangle$ is Abelian.

(ii) For each $\beta$-equivalence class $T$, the algebra $(\mathbf{A}|_T)/(\alpha|_T)$ is Abelian.

(iii) For each two-element set $\{u, v\} \subseteq A$ with $(u, v) \in \beta - \alpha$, the algebra $\mathbf{A}|_{\{u,v\}}$ is Abelian.

(iv) There does not exist a pair $(u, v) \in \beta - \alpha$ and a binary polynomial $f$ of $\mathbf{A}$ such that $\{u, v\}$ is closed under $f$ and $(\{u, v\}, f)$ is a semilattice.

(Sufficient hints for the solution of this exercise can be found in the proof of Theorem 5.5.)

(2) Let $\langle \alpha, \beta \rangle$ be a tame quotient in a finite algebra $\mathbf{A}$. Prove that the following are equivalent.

(i) $\langle \alpha, \beta \rangle$ is strongly Abelian.

(ii) $(\mathbf{A}|_T)/(\alpha|_T)$ is a strongly Abelian algebra, for each $\beta$-equivalence class $T$.

(iii) There do not exist a pair $(u, v) \in \beta - \alpha$ and a binary polynomial $f$ of $\mathbf{A}$ satisfying $f(u, v) = f(v, u) = u$ and $f(v, v) = v$.

(iv) For every $f \in \text{Pol}_1 \mathbf{A}$ (for any $n$), and for every set $T = T_0 \times \cdots \times T_{n-1}$, where $T_0, \ldots, T_{n-1}$ are $\beta$-equivalence classes, if $f(T)$ is contained in a single $(\alpha, \beta)$-trace then $f|_T$ depends, modulo $\alpha$, on at most one variable.

(See the proof of Theorem 5.6. For the equivalence of (i) and (iii), notice that if a trace algebra $\mathbf{A}|_N$ has a Mal’cev polynomial $d(x, y, z)$ and if $u, v \in N$, then the polynomial $f(x, y) = d(x, v, y)$ satisfies $f(u, v) = f(v, u) = u$ and $f(v, v) = v$.)

(3) Let $\langle \alpha, \beta \rangle$ be a tame quotient in a finite algebra $\mathbf{A}$ with $\text{typ}(\alpha, \beta) \neq 1$. Prove a version of Theorem 2.8 for the traces, modified as follows: Replace $\mathbf{M}_\mathbf{A}(\alpha, \beta)$ by $\{N : N \in \alpha, \beta\text{-trace}\}$. In (2), delete the existence of $e$. In (4), replace “$f(A) = U$” by “$f(x/\beta) = N$”. Change (5) to read: For each $N$, $\beta$ is the transitive closure of $\alpha \cup \{(g(N))^2 : g \in \text{Pol}_1 \mathbf{A}\}$.

(4) Using the result of the last exercise, show that if $\langle \alpha, \beta \rangle$ is tame, of non-unique type, and $T = x/\beta \neq x/\alpha$, then $\langle \alpha|_T, 1_T \rangle$ is tame in $\mathbf{A}|_T$, $\text{typ}(\alpha|_T, 1_T) = \text{typ}(\alpha, \beta)$, and the $\langle \alpha|_T, 1_T \rangle$-minimal sets are precisely the $\langle \alpha, \beta \rangle$-traces contained in $T$.

(5) Show that when $\langle \alpha, \beta \rangle$ is tame in $\mathbf{A}$ and $\text{typ}(\alpha, \beta) \neq 2$, then $\text{typ}(\alpha, \beta) = \{\text{typ}(\alpha, \beta)\}$. (See Definition 5.1 for the notation. This result is trivial unless $\text{typ}(\alpha, \beta) = 1$; but in this case, the result of Exercise 2 above does the trick.) Stronger: if $\alpha \leq \gamma < \lambda \leq \beta$ and $\langle \alpha, \beta \rangle, \langle \gamma, \lambda \rangle$ are tame, then $\text{typ}(\alpha, \beta) = \text{typ}(\gamma, \lambda)$ unless $\text{typ}(\alpha, \beta) = 2$.

Exercises 2.19 (1–2) may be useful for (5), and also for the next exercise.

(6) Suppose that $\alpha \leq \gamma < \lambda \leq \beta$, that $\langle \alpha, \beta \rangle$ and $\langle \gamma, \lambda \rangle$ are tame in $\mathbf{A}$, and that $\text{typ}(\alpha, \beta) = 2$. Prove that $\langle \gamma, \lambda \rangle$ is Abelian; and that $\text{typ}(\gamma, \lambda) = 2$ if, for an $\langle \alpha, \beta \rangle$-trace $N$, we have $\gamma|_N < \lambda|_N$. In particular, $\text{typ}(\gamma, \lambda) = 2$ if $\gamma = \alpha$ or if $\lambda = \beta$. 

(7) This exercise complements the last two by constructing a ten-element tame algebra of type 2 that has a tame quotient of type 1. We take \( A = \{0, 1, \ldots, 9\} \) and put \( U_0 = \{0, 1, 2, 3\} \), \( U_1 = \{3, 4, 5, 6\} \), \( U_2 = \{6, 7, 8, 9\} \). These sets will be the \((0_A, 1_A)\)-minimal sets of our algebra. We define two equivalence relations, \( \gamma \) with equivalence classes \( \{0, 3, 6, 9\} \), \( \{1, 2, 4, 5\} \), \( \{7, 8\} \), and \( \lambda \) with equivalence classes \( \{0, 3, 6, 9\} \), \( \{1, 2, 7, 8\} \), \( \{4, 5\} \).

![Figure 11](image)

We define \( F \) to be the set of all functions \( f \) from \( A \) to \( A \) such that \( f = \text{id} \); or \( f \) is constant; or \( f(A) = U_i \) for some \( i \in \{0, 1, 2\} \), \( f^2 = f \), and \( f \) is one-to-one or constant on each \( U_j \) and \( f \) preserves \( \gamma \) and \( \lambda \). If we think of \( A \) as a ruler in three rigid segments with hinges at corners 3 and 6 (see Figure 11), then we can find three members of \( F \) which consist in “folding” all segments onto one segment without violating the physical integrity of the ruler. Call these functions \( e_0, e_1, e_2 \) (with \( e_i(A) = U_i \)). There are two other obvious members of \( F \), \( e_0' \) which maps 4, 5, 6 to 3, maps 7 to 2, 8 to 1, and 9 to 0, and leaves \( U_0 \) pointwise fixed; and \( e_2' \) which projects \( A \) onto \( U_2 \) in a similar fashion. Define a 3-ary operation \( d_0(x, y, z) \) on \( U_0 \) by the rules: \( d_0(x_0, x_1, x_2) = x_i \) if \( \{i, j, k\} = \{0, 1, 2\} \) and \( x_j = x_k \); \( d_0(x_0, x_1, x_2) = \) “the fourth element of \( U_0 \) if \( x_0, x_1, x_2 \) are distinct.” Note that \( U_0 \) is the universe of a four-element vector space over the two-element field, in which \( d_0(x, y, z) = x + y + z \).

Define \( d(x, y, z) \) on \( A \) so that \( d(x, y, z) = d_0(e_0(x), e_0(y), e_0(z)) \). Define \( A = (A, d, e_0, e_1, e_2, e_0', e_2') \). Now prove that \((0_A, 1_A)\) and \((\gamma, \lambda)\) are tame quotients of \( A \), and that \( \text{typ}(0_A, 1_A) = 2 \) while \( \text{typ}(\gamma, \lambda) = 1 \).

(8) Let \( 0 \prec \beta \) in \( A \) with \( \text{typ}(0, \beta) = 2 \). Using Lemma 4.27, Corollary 4.34, and Lemma 4.36, show that if \( U \in M_A(0, \beta) \) and \( B \) is the body of \( U \) then \( C(\Theta(B^2), \beta; 0) \)—i.e., the congruence generated by collapsing \( B \) centralizes \( \beta \).

The next two lemmas reveal interesting and useful properties of non-Abelian prime quotients. In the first lemma we do not require that the algebra be finite.
LEMMA 5.12. Let \( (\alpha, \beta) \) be a non-Abelian prime quotient of an algebra \( A \). There exists a (unique) congruence \( \delta \) such that for all congruences \( \mu \) of \( A \), \( \mu \wedge \beta = \alpha \) iff \( \alpha \leq \mu \leq \delta \).

PROOF. We can prove this using not much more than the definition of an Abelian quotient (Definitions 3.3 and 3.6). Suppose that the conclusion stated in this lemma fails. Then \( \delta = \bigvee \{ \mu : \mu \wedge \beta = \alpha \} \) satisfies \( \delta \wedge \beta \neq \alpha \). Obviously \( \delta \geq \alpha \) and \( \delta \wedge \beta \in I[\alpha, \beta] \), and so \( \delta \wedge \beta = \beta \) since \( \alpha < \beta \). Thus we have that \( \beta \leq \delta \). Now by Proposition 3.4 (4), we have \( C(\mu, \beta; \alpha) \) holding whenever \( \mu \wedge \beta = \alpha \). Thus, by 3.4 (2), we have \( C(\delta, \beta; \alpha) \). Since \( \beta \leq \delta \), it follows by 3.4 (1) that \( C(\beta, \beta; \alpha) \), i.e., that \( (\alpha, \beta) \) is Abelian.

\( \square \)

Remark 5.13. The largest congruence \( \delta \) such that \( \delta \wedge \beta = \alpha \) is called the pseudo-complement of \( \beta \) over \( \alpha \). This is a purely lattice-theoretic concept. Such pseudo-complements need not exist, in general. It is easily seen that when \( (\alpha, \beta) \) is prime and non-Abelian, the pseudo-complement of \( \beta \) over \( \alpha \) is identical to \( \text{ann}(\alpha | \beta) \), defined in Exercise 3.8 (2).

Remark 5.14. Let \( A \) be finite and \( (\alpha, \beta) \) be a non-Abelian prime quotient of \( A \). Let \( \delta \) be the pseudo-complement of \( \beta \) over \( \alpha \). Choose \( U \in M_A(\alpha, \beta) \) and let \( 1 \in U \) be the isolated element of the \( \langle \alpha | U, \beta | U \rangle \)-trace, as defined by Lemma 4.15 or 4.17 (whichever applies). Thus \( 1/(\alpha | U) = \{1\} \). Clearly, \( A | U \) possesses a largest congruence \( \lambda \) such that \( 1/\lambda = \{1\} \); and we have \( \alpha | U \leq \lambda \). Since \( | U \) is a lattice homomorphism, \( A \) has a largest congruence \( \lambda' \) satisfying \( \lambda'| U = \lambda \). It can easily be shown, using the information in Lemma 4.15 or Lemma 4.17, that \( \lambda' \) is identical with \( \delta \), the pseudo-complement of \( \beta \) over \( \alpha \).

LEMMA 5.15. Let \( (\alpha, \beta) \) be a non-Abelian prime quotient of a finite algebra \( A \), and let \( \gamma \in \text{Con} A \).

1. If \( \alpha \vee \gamma = \beta \), then there exists a smallest congruence \( \delta \) such that \( \delta \geq \alpha \wedge \gamma \) and \( \alpha \vee \delta = \beta \).

2. If \( (\alpha, \beta) \) is of Boolean or lattice type, then there exists a smallest congruence \( \delta \) such that \( \alpha \vee \delta = \beta \).

PROOF. Choose any \( (\alpha, \beta)\)-trace \( N \). According to Lemma 4.15 or Lemma 4.17, we have \( N = I \cup O \) (disjoint union) where \( I = \{1\} \) and \( \alpha|_N = I^2 \cup O^2 \). By Lemma 2.4, the map \( |_N \) is a lattice homomorphism on \( I[0, \beta] \). Hence, since \( \alpha < \beta \), the condition \( \alpha \vee \delta = \beta \) is equivalent to \( \delta \leq \beta \) and \( \delta \cap (I \times O) \neq \emptyset \). Now if the type of \( (\alpha, \beta) \) is 3 or 4, then Lemma 4.17 tells us that \( O = \{0\} \) (for some element 0) and thus \( \alpha \vee \delta = \beta \) is equivalent to \( (0, 1) \in \delta \leq \beta \). Thus in this case, \( \Theta(0, 1) \) is the smallest congruence that joins with \( \alpha \) to give \( \beta \). This proves (2).
To prove (1), assume that $\alpha \lor \gamma = \beta$. Choosing $(1, u) \in \gamma \cap (I \times O)$, we put $\delta = (\alpha \land \gamma) \lor \Theta(u, 1)$. Now let $\lambda \geq \alpha \land \gamma$ and $\alpha \lor \lambda = \beta$. Notice that from our definition of $\delta$ we have $\alpha \land \gamma = \alpha \land \delta$, and so $\alpha \land \delta \leq \lambda$. Choose $(1, v) \in \lambda \cap (I \times O)$. Let $p$ be the pseudo-meet operation on $N$ supplied by Lemma 4.15 or 4.17. Since $p(u, 1) = u$ and $p(1, v) = v$, we have $u \leq p(u, v) \leq v$. Since $p(u, v) \in O$, we actually have $p(u, v) = u \lor v$. Since $\alpha \land \delta \leq \lambda$, then $(u, v) \in \lambda \lor (\alpha \land \delta) = \lambda$ and $(u, 1) \in \lambda$; and so $\lambda \geq (\alpha \land \gamma) \lor \Theta(u, 1) = \delta$. This finishes the proof of (1). \qed

**Remark 5.16.** The smallest congruence $\delta$ such that $\delta \lor \alpha = \beta$, when it exists, is called the **pseudo complement of $\alpha$ under $\beta$**.

Every finite simple algebra is tame, and its type is the same as that of its unique prime congruence quotient (see Definition 5.1(3)). Here is our first result on finite simple algebras. We shall study them at length in Chapter 14.

**Corollary 5.17.** Let $A, B_1, \ldots, B_n$ be finite algebras such that $A$ is simple and of Boolean or lattice type. If $A$ belongs to the variety generated by $\{B_1, \ldots, B_n\}$, then $A$ is a homomorphic image of a subalgebra of one of the $B_i$.

**Proof.** Let $A$ belong to the variety generated by $B_1, \ldots, B_n$. By Theorem 0.2, $A$ must be a homomorphic image of a subalgebra of $(B_1 \times \cdots \times B_n)^k$ for a finite integer $k$. Therefore, for some $m$, we have an algebra $S \subseteq \prod_{j=1}^m C_j$ where $\{C_1, \ldots, C_m\} \subseteq \{B_1, \ldots, B_n\}$, and a congruence $\alpha$ of $S$ with $S/\alpha \cong A$. Since $A$ is simple, $\alpha \not\leq 1_S$ in $\cong S$, and $\alpha = \alpha \lor 1_S$. Let $\delta$ be the pseudo-complement of $\alpha$ under $1_S$, which exists by Lemma 5.15. For $1 \leq j \leq m$, let $\eta_j$ be the kernel of the projection of $S$ into $C_j$. Since $\delta \not\leq \alpha$, and $\bigwedge\{\eta_j : j = 1, \ldots, m\} = 0_S$, then for some $j$ we have $\delta \not\leq \eta_j$. By the definition of $\delta$, it follows that $\eta_j \not\leq \alpha$, implying that $\eta_j \leq \alpha$ since $\alpha \not\leq 1_S$. Let $i$ be such that $C_i = B_i$. Then $S/\alpha (\cong A)$ is a homomorphic image of $S/\eta_j$, which is isomorphic to a subalgebra of $B_i$. This concludes our proof. \qed

**Definition 5.18.** A lattice $L$ is said to be **meet semi-distributive**, or to satisfy $SD(\land)$, iff whenever elements $x, y, z$ in $L$ satisfy $x \land y = x \land z$, they also satisfy $x \land y = x \land (y \lor z)$. The property $SD(\lor)$, **join semi-distributivity**, is the dual of $SD(\land)$. The conjunction of these two properties, denoted $SD$, is called semi-distributivity.

**Lemma 5.19.** Let $\theta, \psi_0, \psi_1$ be congruences of a finite algebra $A$.

1. If $\theta \land \psi_0 = \theta \land \psi_1 = \alpha$, and if $\beta$ is a congruence such that $\alpha \land \beta \leq \theta \land (\psi_0 \lor \psi_1)$, then $\text{typ}(\alpha, \beta) \in \{1, 2\}$.
2. If $\theta \lor \psi_0 = \theta \lor \psi_1 = \beta$, and if $\alpha$ is a congruence such that $\theta \lor (\psi_0 \land \psi_1) \leq \alpha \land \beta$, then $\text{typ}(\alpha, \beta) \in \{1, 2, 5\}$. 

PROOF. Under the hypotheses of statement (1), $\beta \wedge \psi_0 = \beta \wedge \psi_1 = \alpha$, and
$\beta \leq \psi_0 \vee \psi_1$. Thus the pseudo-complement of $\beta$ over $\alpha$ does not exist. By Lemma
5.12, it follows that $<\alpha, \beta>$ is Abelian. This proves (1).

The proof of (2) is similar, using Lemma 5.15.

COROLLARY 5.20. Suppose that $\alpha$ and $\beta$ are congruences of a finite algebra $A$
such that $\alpha < \beta$ and $\text{typ}\{\alpha, \beta\} \subseteq \{3, 4, 5\}$. Then the interval lattice $I[\alpha, \beta]$ satisfies
$\text{SD}(\wedge)$. If $\text{typ}\{\alpha, \beta\} \subseteq \{3, 4\}$, then $I[\alpha, \beta]$ satisfies $\text{SD}(\lor)$.

PROOF. This is an immediate consequence of Lemma 5.19.

Shortly, we shall prove that the presence of a prime quotient of semilattice type
in the congruence lattice of an algebra $A$ forces a failure of join semi-distributivity
in the congruence lattice of a certain subalgebra of $A^2$. Similar results concerning
prime quotients of unary or affine type will be proved in Chapter 6. These results, in
combination with Corollary 5.20, will show that when we consider in place of a finite
algebra $A$ the class of all subalgebras of finite direct powers of $A$, then congruence
meet semi-distributivity is equivalent to the absence of Abelian prime quotients in
all algebras of the class, while congruence join semi-distributivity is equivalent to the
absence of Abelian prime quotients as well as those of semilattice type.

Remark 5.21. The smallest lattices satisfying one, but not both, semi-distributivity
conditions are pictured below.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{Figure 12}
\end{figure}

The lattice $D_1$ fails $\text{SD}(\wedge)$, and $D_2$, its dual, fails $\text{SD}(\lor)$. These lattices have an
interesting role in our classification scheme for locally finite varieties (presented in
Chapter 9), along with the smallest non-modular lattice, $N_5$, and the smallest modular,
non-distributive lattice, $M_3$. We remark that $D_2$ is isomorphic to the congruence
lattice of $S^2$, where $S$ is a two-element semilattice. $D_1$ is isomorphic to the lattice of
convex subsets of a three-element linearly ordered set.

To conclude this chapter, we shall find characterizations of each of the three non-
Abelian types of prime quotients, in language that does not involve tame congruence
theory.
Recall from Chapter 0 that an admissible n-ary relation on an algebra \( A \) is a subset of \( A^n \) closed under the operations of \( A^n \). A tolerance of \( A \) is a binary relation over \( A \) that is admissible, reflexive and symmetric. Note that an admissible n-ary relation \( \rho \) on \( A \) is closed under all polynomial operations of \( A \) (acting coordinatewise in \( A^n \)) iff it contains the diagonal of \( A^n \), i.e., \( \langle x, \ldots, x \rangle \in \rho \) for all \( x \in A \). (If \( \rho \) is binary, this means that \( \rho \) is reflexive.)

**Lemma 5.22.** If an algebra \( A \) has a Mal'cev polynomial, then every admissible binary relation of \( A \), reflexive over \( A \), is a congruence relation.

**Proof.** Let \( \rho \) be admissible and reflexive. Let \( q(x, y, z) \) be a Mal'cev polynomial operation of \( A \) (Definition 4.5). Since \( \rho \) is reflexive, it is closed under \( q \). Suppose that \( \langle x, y \rangle, \langle y, z \rangle \in \rho \). Then \( q(\langle x, x \rangle, \langle x, y \rangle, \langle y, y \rangle) = q(x, x, y), q(x, y, y) = \langle y, x \rangle \) belongs to \( \rho \). (In this calculation, we are using the operation on \( A^2 \) which is \( q \) acting at each coordinate.) Similarly, \( q(\langle x, y \rangle, \langle y, y \rangle, \langle y, z \rangle) = \langle x, z \rangle \) belongs to \( \rho \). Thus \( \rho \) is reflexive, symmetric, and transitive, which makes it a congruence relation. \( \square \)

We recall some more definitions from Chapter 0. The converse of a binary relation \( \rho \) is the relation \( \rho^c = \{ \langle y, x \rangle : \langle x, y \rangle \in \rho \} \) (read "\( \rho \) converse"). The relation product of binary relations \( \rho \) and \( \sigma \) is the relation \( \rho \circ \sigma = \{ \langle x, z \rangle : \langle x, y \rangle \in \rho \text{ and } \langle y, z \rangle \in \sigma \} \) for some \( y \). When \( \sigma \) is an equivalence relation, we say that \( \rho \) is \( \sigma \)-closed if \( \rho = \sigma \circ \rho \circ \sigma \); the \( \sigma \)-closure of \( \rho \) is the relation \( \sigma \circ \rho \circ \sigma \).

**Definition 5.23.** If \( \langle \alpha, \beta \rangle \) is a prime quotient of a finite algebra \( A \), then by the basic tolerance for \( \langle \alpha, \beta \rangle \) we mean the intersection of all \( \alpha \)-closed tolerances \( \tau \) of \( A \) satisfying \( \alpha \neq \tau \subseteq \beta \).

**Lemma 5.24.** Let \( \langle \alpha, \beta \rangle \) be a prime quotient of a finite algebra \( A \) with \( \text{typ}(\alpha, \beta) \neq 1 \), and let \( \rho \) be the basic tolerance for \( \langle \alpha, \beta \rangle \).

1. If \( N \) is an \( \langle \alpha, \beta \rangle \)-trace and \( \langle x, y \rangle \in N^2 - \alpha \), then \( \rho \) is the smallest \( \alpha \)-closed tolerance containing \( \langle x, y \rangle \). The transitive closure of \( \rho \) is \( \beta \).

2. If \( \text{typ}(\alpha, \beta) \in \{2, 3\} \), then \( \rho \) is the \( \alpha \)-closure of \( \alpha \cup \{N^2 : N \text{ is an } \langle \alpha, \beta \rangle \text{-trace}\} \); and \( \rho \) is the smallest \( \alpha \)-closed admissible reflexive relation \( \tau \) satisfying \( \alpha \neq \tau \subseteq \beta \).

3. If \( \text{typ}(\alpha, \beta) \in \{4, 5\} \), then there are precisely two minimal \( \alpha \)-closed, admissible, reflexive relations \( \tau \) such that \( \alpha \neq \tau \subseteq \beta \). These relations, \( \rho_0 \) and \( \rho_1 \), satisfy:

   i. \( \rho_0 = \rho_1^c \) and \( \rho_0 \cap \rho_1 = \alpha \);

   ii. \( \rho_0 \cup \rho_1 \) is the \( \alpha \)-closure of \( \alpha \cup \{N^2 : N \text{ is an } \langle \alpha, \beta \rangle \text{-trace}\} \);

   iii. \( \rho \) is the \( \alpha \)-closure of the admissible relation generated by \( \rho_0 \cup \rho_1 \).
Proof. Our proof begins with a claim.

Claim 1. Let $\tau$ be an $\alpha$-closed admissible reflexive relation on $A$ with $\alpha \neq \tau \subseteq \beta$, and let $N$ be any $(\alpha, \beta)$-trace. Then $N^2 \subseteq \tau \cup \tau^\cup$, and $\alpha \subseteq \tau$. If $\text{typ}(\alpha, \beta) \in \{2, 3\}$, then $N^2 \subseteq \tau$.

To begin the proof of this claim, we note that every $\alpha$-closed reflexive relation trivially contains $\alpha$. Now choosing any $(a, b) \in \tau - \alpha$, an easy application of Theorem 2.8 (4) and Corollary 5.2 (2) yields the existence of a unary polynomial $f(x)$ with $\langle f(a), f(b) \rangle \in N^2 - \alpha$; therefore $\tau|_N \subseteq \alpha|_N$. Since $\tau$ is reflexive, it is preserved by all of the polynomial operations of $A$, and so $\tau|_N$ is an admissible, reflexive, $\alpha|_N$-closed relation of the algebra $A|_N$.

If $\text{typ}(\alpha, \beta) = 2$, then $A|_N$ is Mal'cev (by Lemma 4.20); and if $\text{typ}(\alpha, \beta) = 3$, then $A|_N$ is polynomially equivalent to a two-element Boolean algebra (by Lemma 4.17), and again is Mal'cev. Hence if $\text{typ}(\alpha, \beta) \in \{2, 3\}$ then it follows from Lemma 5.22 that $\tau|_N$ is a congruence of $A|_N$. Now since $\alpha < \beta$ and $|_N$ is a homomorphism of $I[\alpha, \beta]$ onto $I[\alpha|_N, \beta|_N]$, we have $\alpha|_N < \beta|_N = \tau|_N$ in $\text{Con} A|_N$. Consequently, if $\text{typ}(\alpha, \beta) \in \{2, 3\}$, then $\tau|_N$ can only be identical with $1_N$, and so $\tau \supseteq N^2$ in this case. If $\text{typ}(\alpha, \beta) \in \{4, 5\}$ then Lemma 4.15 or 4.17 implies that $N$ is the union of $c/(\alpha|_N)$ and $d/(\alpha|_N)$, for any $(c, d) \in 1_N - \alpha|_N$; so it is obvious in this case that $\tau \cup \tau^\cup \supseteq N^2$, since $\alpha$ is $\alpha$-closed and $\tau|_N \not\subseteq \alpha|_N$.

Claim 2. If $\text{typ}(\alpha, \beta) \in \{2, 3\}$, then the $\alpha$-closure of $\alpha \cup \{N^2 : N$ is an $(\alpha, \beta)$-trace$\}$ is a tolerance.

To prove this claim, let $\tau$ denote this relation which we wish to prove is a tolerance. Then $\tau$ is obviously symmetric and reflexive. To prove that it is admissible, let $f$ be any $n$-ary polynomial of $A$ and let $(u_i, v_i) \in \tau$ for $0 \leq i < n$. In showing that $\langle f(u), f(v) \rangle \in \tau$, we can assume that for a certain $k \leq n$ we have $\langle u_i, v_i \rangle \in \alpha$ for $k \leq i < n$, and for each $i < k$ we have a trace $N_i$ and elements $c_i, d_i \in N_i$ such that $\langle u_i, c_i \rangle \in \alpha$, $\langle u_i, d_i \rangle \in \alpha$, and $\langle c_i, d_i \rangle \not\in \alpha$. Since $\tau$ is $\alpha$-closed, it suffices to prove that $\langle f(c_0, \ldots, c_{k-1}, u_k, \ldots, u_{n-1}), f(d_0, \ldots, d_{k-1}, u_k, \ldots, u_{n-1}) \rangle$ belongs to $\tau$. Thus, changing notation, we can assume that $k = n$ and that $\langle u_i, v_i \rangle \in N_i^2 - \alpha$ for $0 \leq i < n$.

Since we've assumed that $\text{typ}(\alpha, \beta) \in \{2, 3\}$, by Corollary 5.2 (2) there exist unary polynomials $f_i(x)$ and elements $u'_i, v'_i \in N_0$ such that $\langle u'_i, v'_i \rangle \in N_0^2 - \alpha$ and $f_i(u'_i) = u_i$, $f_i(v'_i) = v_i$ for $0 \leq i < n$. Replacing $f$ by the polynomial $f'$ defined by $f'(x_0, \ldots, x_{n-1}) = f(f_0(x_0), \ldots, f_{n-1}(x_{n-1}))$ and writing $N$ for $N_0$, and changing notation once again, we can now assume that $u_i, v_i \in N$ for all $i$. The argument breaks into two cases.

Case 1: Let $\text{typ}(\alpha, \beta) = 2$. Thus $(A|_N)/(\alpha|_N)$ is a vector space of dimension $1$. (Since $\alpha < \beta$, as before we have $\alpha|_N < 1_N$, so the vector space is a simple
algebra.) Because \( u_0 / \alpha, v_0 / \alpha \) are unequal elements of this vector space, there exists, for all \( i, 1 \leq i < n \), a vector space polynomial \( h_i \) such that \( h_i(u_0 / \alpha) = u_i / \alpha \) and \( h_i(v_0 / \alpha) = v_i / \alpha \). Therefore there exist polynomials \( h'_i \) of \( A \) such that \( h'_i(u_0) \equiv u_i \pmod{\alpha} \) and \( h'_i(v_0) \equiv v_i \pmod{\alpha} \). We define \( h(x) = f(x, h'_1(x), \ldots, h'_{n-1}(x)) \), and observe that \( f(\bar{u}) \equiv h(u_0) \pmod{\alpha} \) and \( f(\bar{v}) \equiv h(v_0) \pmod{\alpha} \). Therefore, we are reduced to proving that \( \langle h(u_0), h(v_0) \rangle \in \tau \). This follows from the result of Exercise 2.19(6).

Case 2: Let \( \text{typ}(\alpha, \beta) = 3 \). Thus, by Lemma 4.17, \( N = \{u_0, v_0\} \), and \( A \) has a polynomial \( h' \) such that \( h'(u_0) = v_0 \) and \( h'(v_0) = u_0 \). The argument used for Case 1 works equally well in this case.

The proof of Claim 2 is finished. Assertions (1) and (2) of this lemma are easy consequences of Claim 1 and Claim 2. For (1), note that by Claim 1, every \( \alpha \)-closed tolerance \( \tau \) such that \( \alpha \neq \tau \subseteq \beta \) must contain \( \alpha \cup N^2 \) for all traces \( N \) (since \( \tau \) is symmetric). Thus \( \rho \) (the basic tolerance) must contain \( \alpha \cup N^2 \) for all traces \( N \). Now if \( N \) is a trace and \( \langle x, y \rangle \in N^2 - \alpha \), then taking \( \tau \) to be the \( \alpha \)-closure of the tolerance generated by \( \langle x, y \rangle \), we have that \( \tau \subseteq \rho \) by the last sentence, but also \( \rho \subseteq \tau \) by the definition of \( \rho \); consequently \( \rho = \tau \). That \( \beta \) equals the transitive closure of \( \rho \) is a consequence of Lemma 2.17. This finishes the proof of (1). The proof of (2) is somewhat easier, and we omit it.

**Claim 3.** The admissible reflexive relation generated by a pair \( \langle a, b \rangle \in A^2 \) is
\[
r(a, b) = \{ (f(a), f(b)) : f \in \text{Pol}_1 A \};
\]
and the tolerance generated by \( \langle a, b \rangle \) is
\[
t(a, b) = \{ (f(a, b), f(b, a)) : f \in \text{Pol}_2 A \}.
\]

The proof of this claim is a simple matter of showing that these relations are admissible; that \( r(a, b) \) is reflexive and contains \( \langle a, b \rangle \); that every admissible reflexive relation containing \( \langle a, b \rangle \) contains \( r(a, b) \) as a subset; and similar facts for \( t(a, b) \). The proofs are left to the reader.

We now assume that \( \text{typ}(\alpha, \beta) \in \{4, 5\} \), in order to prove assertion (3) of this lemma. We choose a trace \( N \), and choose \( \langle a, b \rangle \in N^2 - \alpha \). We define \( \rho_0 \) and \( \rho_1 \) to be the \( \alpha \)-closures of \( r(a, b) \) and \( r(b, a) \) respectively. By Claim 1, every \( \alpha \)-closed, admissible, reflexive relation \( \tau \) such that \( \alpha \neq \tau \subseteq \beta \) contains either \( \rho_0 \) or \( \rho_1 \) (i.e., contains \( \langle a, b \rangle \) or \( \langle b, a \rangle \)). Clearly \( \alpha \neq \rho_i \subseteq \beta \) (\( i = 0, 1 \)). We now will show that \( \rho_0 \cap \rho_1 = \alpha \), from which it should be clear that \( \rho_0 \) and \( \rho_1 \) are the unique minimal members of the family of relations under consideration. Suppose that there is a pair \( \langle c, d \rangle \in \rho_0 \cap \rho_1 \), \( \langle c, d \rangle \neq \alpha \). By Exercise 5.11 (3), we can choose \( h \in \text{Pol}_1 A \) with \( \langle h(c), h(d) \rangle \in N^2 - \alpha \) and \( h(c/\beta) = N \). By Claim 3, there are \( f_0, f_1 \in \text{Pol}_1 A \) with
$f_0(a) \equiv c \equiv f_1(b) \pmod{\alpha}$ and $f_0(b) \equiv d \equiv f_1(a) \pmod{\alpha}$. Now $N$ is the disjoint union of $a/(\alpha|_N)$ and $b/(\alpha|_N)$. If $h(c)$ is in the first class and $h(d)$ in the second, then $q = h f_0$ satisfies $q(a) \equiv b$, $q(b) \equiv a \pmod{\alpha}$. If the order of $h(c)$ and $h(d)$ is reversed, then $q = h f_0$ satisfies the same congruences. Choosing an $(\alpha, \beta)$-minimal set $U$ such that $N$ is a trace in $U$, and an $e \in E(A)$ with $e(A) = U$, it follows from $(q(a), q(b)) \in N^2 - \alpha$ that $N$ is closed under $e$ and $(e q)_{|N}$ is a permutation which exchanges the two $\alpha|_N$-classes. This contradicts the fact that $(A|_N)/(\alpha|_N)$ is polynomially equivalent to a lattice or semilattice. The contradiction finishes our proof that $\rho_0 \cap \rho_1 = \alpha$. It follows easily from our definition of $\rho_0$ and $\rho_1$ that $(\rho_1)^\cup = \rho_0$.

To prove 3(ii), recall that by Claim 1, $\rho_0 \cup \rho_1 (= \rho_0 \cup \rho_1)$ contains $\alpha \cup \{N^2 : N \text{ a trace} \}$. On the other hand, by Claim 3 and Exercise 2.19 (6),

$$r(a, b) \cup r(b, a) \subseteq \alpha \cup \bigcup\{N^2 : N \text{ a trace}\}.$$ 

These facts easily yield 3(ii).

To prove 3(iii), notice that $\rho_0 \cup \rho_1$ is symmetric and reflexive. Therefore the $\alpha$-closure of the admissible relation it generates is a tolerance. Thus, by definition of $\rho_1$, we have that this tolerance includes $\rho$. On the other hand, by (1), $\rho$ is the $\alpha$-closure of $t(a, b)$; and therefore $\rho$ contains $\rho_0$ and $\rho_1$, by their definitions.

**DEFINITION 5.25.** Let $(\alpha, \beta)$ be any congruence quotient of an algebra $A$. By an $(\alpha, \beta)$-pre-order we mean an admissible binary relation $\tau$ of $A$ such that $\tau$ is a pre-ordering of $A$ (i.e., it is reflexive and transitive over $A$), $\alpha = \tau \cap \tau^\cup$, and the transitive closure of $\tau \cup \tau^\cup$ is $\beta$. We say that $(\alpha, \beta)$ is **orderable** iff there exists an $(\alpha, \beta)$-pre-order.

**THEOREM 5.26.** Let $(\alpha, \beta)$ be a tame quotient of a finite algebra $A$, with $\text{typ}(\alpha, \beta) \neq 1$.

1. $(\alpha, \beta)$ is orderable iff $\text{typ}(\alpha, \beta) \in \{4, 5\}$.

2. Assume that $\text{typ}(\alpha, \beta) \in \{4, 5\}$. There exist precisely two minimal $(\alpha, \beta)$-pre-orders, $\zeta_0$ and $\zeta_1$, and two maximal $(\alpha, \beta)$-pre-orders, $\xi_0$ and $\xi_1$, such that every $(\alpha, \beta)$-pre-order $\tau$ satisfies $\zeta_0 \subseteq \tau \subseteq \xi_0$ or $\zeta_1 \subseteq \tau \subseteq \xi_1$. We have $\zeta_1 = \xi_0^\cup$, $\xi_1 = \xi_0^\cup$, and $\zeta_0$ and $\zeta_1$ are the transitive closures of the relations $\rho_0$ and $\rho_1$ of Lemma 5.24 (3).

**PROOF.** We begin by assuming that $\text{typ}(\alpha, \beta) = 2$ or 3 and there exists an $(\alpha, \beta)$-pre-order $\tau$, and derive a contradiction from these assumptions. It is easily seen that $\tau \neq \alpha$. Since $\sigma \circ \tau = \tau \supseteq \alpha$, $\tau$ is $\alpha$-closed. By Lemma 5.24 (2), $\tau \supseteq N^2$ for every trace $N$. This contradicts the condition that $\tau \cap \tau^\cup = \alpha$.

We now assume that $\text{typ}(\alpha, \beta) = 4$ or 5. Letting $\rho_0$ and $\rho_1$ be the two minimal, $\alpha$-closed, admissible, reflexive relations (see Lemma 5.24 (3)), we define $\zeta_i$ to be the
transitive closure of $\rho_i$ $(i = 0, 1)$. We choose an $(\alpha, \beta)$-trace $N$ and an element 1 in $N$. If \( \text{typ}(\alpha, \beta) = 5 \), then we arrange that \( \{1\} \subseteq 1/(\alpha|N) \) is the neutral element of the semilattice \((A|N)/(\alpha|N)\) (using Lemma 4.15). Thus \( \{1\} = 1/(\alpha|N) \) whether the type is 4 or 5. We choose any \( b \in N - \{1\} \). Thus we have \( \langle 1, b \rangle \in N^2 - \alpha \), and \( N = \{1\} \cup b/(\alpha|N) \). By Lemma 5.24 (3), the pair \( \langle b, 1 \rangle \) belongs to precisely one of \( \rho_0 \) and \( \rho_1 \); we shall assume that \( \langle b, 1 \rangle \in \rho_0 \).

Since \( \rho_1 = \rho_0^\gamma \), we have \( \zeta_1 = \zeta_0^\gamma \). The relations \( \zeta_0 \) and \( \zeta_1 \) are pre-orders, by their construction. It is easily seen that the transitive closure of any admissible reflexive relation is admissible. The relations \( \zeta_0 \) and \( \zeta_1 \) are therefore admissible pre-orders. Obviously, \( \zeta_0 \cup \zeta_1 \subseteq \beta \).

We now define

\[
\xi_0 = \{(x, y) \in \beta : \text{ for all } f \in \text{Pol}_1 A \text{ such that } f(x/\beta) \subseteq N \text{ and } f(x) = 1, \text{ we have } f(y) = 1\}
\]

and we define \( \xi_1 = \zeta_0^\gamma \). We proceed to prove several claims.

**Claim 1.** \( \zeta_i \subseteq \xi_i \subseteq \beta \) and \( \xi_i \) is an admissible pre-order (for \( i = 0, 1 \)).

It suffices to prove this claim for \( i = 0 \). Since it is obvious that \( \xi_0 \) is a pre-order, our first task will be to show that it is admissible. By Exercise 5.28 (2), this amounts to showing that \( \xi_0 \) is closed under the unary polynomials of \( A \). So let \( h \in \text{Pol}_1 A \) and \( \langle u, v \rangle \in \xi_0 \). To see that \( \langle h(u), h(v) \rangle \in \xi_0 \), let \( f \in \text{Pol}_1 A \) be such that \( f(h(u)/\beta) \subseteq N \) and \( f(h(u)) = 1 \). Then \( f(h(u)/\beta) \subseteq N \) and \( f(h(u)) = 1 \), so \( f(h(v)) = 1 \). We conclude that \( \langle h(u), h(v) \rangle \in \xi_0 \), and that \( \xi_0 \) is admissible.

Because \( \xi_0 \) is a pre-order, to show that \( \zeta_0 \subseteq \xi_0 \), we need only show that \( \rho_0 \subseteq \xi_0 \). To do this, let \( \langle u, v \rangle \in \rho_0 \) and \( f \in \text{Pol}_1 A \) with \( f(u/\beta) \subseteq N \) and \( f(u) = 1 \). Thus \( \langle 1, f(v) \rangle \in \rho_0 \cap N^2 \). If \( f(v) \neq 1 \), then \( f(v) \in b/(\alpha|N) \), and since \( \rho_0 \) is $\alpha$-closed, we have \( \langle b, 1 \rangle \in \rho_0 \) as well as \( \langle b, 1 \rangle \in \rho_0 \); but this contradicts Lemma 5.24 (3(i)). Therefore \( f(v) = 1 \), and we conclude that \( \langle u, v \rangle \in \xi_0 \), and consequently that \( \zeta_0 \subseteq \xi_0 \). This finishes our proof of Claim 1.

**Claim 2.** \( \xi_0 \cap \xi_1 = \alpha \) and the transitive closure of \( \zeta_0 \cup \zeta_1 \) is \( \beta \). Therefore \( \zeta_i, \xi_i \) are \( (\alpha, \beta) \)-pre-orders (\( i = 0, 1 \)).

The second sentence in this claim follows from the first, from Claim 1, from the definition of \( (\alpha, \beta) \)-pre-order, and from the facts that \( \zeta_0 \cap \zeta_1 \supseteq \rho_0 \cap \rho_1 = \alpha \) and \( \xi_0 \cup \xi_1 \subseteq \beta \). That the transitive closure of \( \zeta_0 \cup \zeta_1 \) equals \( \beta \) is a consequence of Lemma 5.24 (3(ii)) and Lemma 2.17.

Let us prove that \( \xi_0 \cap \xi_1 = \alpha \). Suppose that \( \langle u, v \rangle \in \xi_0 \cap \xi_1 \). Then for every \( f \in \text{Pol}_1 A \) satisfying \( f(u/\beta) \subseteq N \) we have \( f(u) = 1 \iff f(v) = 1 \), and therefore \( \langle f(u), f(v) \rangle \in \alpha \) since \((N - \{1\})^2 \subseteq \alpha \). Using Exercise 5.11 (3), we conclude that \( \langle u, v \rangle \in \alpha \).
Clairn 3. If $\tau$ is any $\langle \alpha, \beta \rangle$-pre-order, then $\zeta_0 \subseteq \tau \subseteq \xi_0$ or $\zeta_0 \subseteq \tau^{\cup} \subseteq \xi_0$.

Let $\tau$ be an $\langle \alpha, \beta \rangle$-pre-order. We recall from earlier in the proof that this implies $\alpha \neq \tau$, and that $\tau$ is $\alpha$-closed. By Lemma 5.24 (3) we have $\rho_i \subseteq \tau$ for some $i \in \{0, 1\}$, giving that $\rho_0 \subseteq \tau \lor \rho_0 \subseteq \tau^{\cup}$. We may as well assume that $\rho_0 \subseteq \tau$. Thus, obviously, $\zeta_0 \subseteq \tau$. Finally, since $\tau$ is an $\alpha$-closed subset of $\beta$, the assumption $\tau \not\subseteq \xi_0$ would imply directly that $\langle 1, b \rangle \in \tau$. (Use the definition of $\xi_0$.) This, in turn, would put $\langle 1, b \rangle \in (\tau \cap \tau^{\cup}) - \alpha$, contradicting that $\tau$ is an $\langle \alpha, \beta \rangle$-pre-order. Therefore $\tau \subseteq \xi_0$. This finishes the proof of Claim 3.

The minimality of $\zeta_i$ and maximality of $\xi_i$ follows from Claim 3 and the obvious fact that any $\langle \alpha, \beta \rangle$-pre-order must be incomparable to its converse. \hfill \Box

We have characterized the unary type of tame quotient as strongly Abelian, the affine type as Abelian but not strongly Abelian, and the Boolean type (in Theorem 5.26) as non-Abelian and non-orderable. The next theorem exhibits a subtle difference between prime quotients of Boolean or lattice type, and those of semilattice type.

**Theorem 5.27.** Let $\langle \alpha, \beta \rangle$ be a non-Abelian prime quotient of a finite algebra $A$. Let $\rho$ be the basic tolerance for $\langle \alpha, \beta \rangle$, $R$ be the subalgebra of $A^2$ with universe $R = \rho$, and $L$ be the interval sublattice $I([\alpha \times \alpha]|_R, (\beta \times \beta)|_R)$ in $\text{Con} R$.

1. If $\text{typ}(\alpha, \beta) \in \{3, 4\}$ then $L \cong M_2$.
2. If $\text{typ}(\alpha, \beta) = 5$ then $L \cong D_2$.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (2,0); \draw (0,0) -- (0,2); \draw (0,2) -- (2,2); \draw (2,0) -- (2,2); \draw (0,1) -- (1,0); \draw (1,0) -- (0,1); \draw (1,0) -- (2,1); \draw (2,1) -- (1,0);
\end{tikzpicture}
\end{center}

**Proof.** Let $\pi : A^2 \rightarrow (A/\alpha)^2$ be the natural mapping. From the characterization of $R (= \rho)$ as the least $\alpha$-closed tolerance $\tau$ such that $\alpha \neq \tau \subseteq \beta$, it is easy to prove that $\pi(R)$ is the basic tolerance for the quotient $\langle \alpha/\alpha, \beta/\alpha \rangle$ in $A/\alpha$. The kernel of $\pi|_R$ is $\langle \alpha \times \alpha \rangle|_R$, and we have $\langle \beta \times \beta \rangle|_R = \pi^{-1}(\langle (\beta/\alpha) \times (\beta/\alpha) \rangle|_{\pi(R)})$. Therefore, the lattice $L$ is isomorphic to the lattice derived in the same way from the quotient $\langle \alpha/\alpha, \beta/\alpha \rangle$ which, as we know, has the same type as $\langle \alpha, \beta \rangle$. These considerations show that it suffices to prove the theorem under the assumption $\alpha = 0_A$.

So let $\langle 0_A, \beta \rangle$ be a prime non-Abelian quotient of $A$. Choose $U_0 \in M_A(0_A, \beta)$ and a binary polynomial $p(x, y)$ of $A$ such that $p$ is a pseudo-meet operation for $A|_{U_0}$ relative to $\langle 0_{U_0}, \beta|_{U_0} \rangle$. (See Definitions 4.16 and 4.18.) Let $N_0 = \{0, 1\}$ be the unique trace in $U_0$, with $1$ the neutral element for $p$. 


Note that, by Lemma 5.24, $R$ is the tolerance generated by $(0, 1)$. By various parts of 5.24 (including “Claim 3” in its proof), we have

\[(5.27.1) \quad N^2 \subseteq R \subseteq \beta \text{ for every } (0_{A}, \beta)\text{-trace } N; \text{ and} \]
\[R = \{ (f(0, 1), f(1, 0)) : f \in \text{Pol}_2 A \} . \]

Since $R \supseteq 0_A$, it is easily checked that the polynomials of $R$ are precisely all the operations of the form

\[f((x_0, y_0), \ldots, (x_{n-1}, y_{n-1})) = (g(0, 1, \bar{x}), g(1, 0, \bar{y}))\]

with $g \in \text{Pol}_{n+2} A$ if $f \in \text{Pol}_n R$. We have, in particular, that every polynomial of $A$, acting coordinatewise on $n$-tuples of pairs in $R$, is a polynomial of $R$.

**Claim.** If $\theta \in \text{Con } R$ and $\theta \leq \beta \times \beta$, then $\theta$ is generated by $\theta|_{N^2}$.

To prove this claim, we let $\theta$ be any congruence in the interval $I[0_R, (\beta \times \beta)|_{I_R}]$ of $\text{Con } R$, and we set $\theta_0$ equal to the congruence of $R$ generated by $\theta|_{N^2}$. Let $((a, b), (c, d))$ be any element of $\theta$. We wish to show that $((a, b), (c, d)) \in \theta_0$. We shall assume that $a \neq c$. (The proof in the case $b \neq d$ is similar, and if $a = c$ and $b = d$, then there is nothing to prove.)

Since $a, c \in \beta - 0_A$, there is $f \in \text{Pol}_1 R$ with $f(a/\beta) = N_0 = \{0, 1\}$, and $f(a) \neq f(c)$. Note that $\{b, c, d\} \subseteq a/\beta$, and therefore

\[((f(a), f(b)), (f(c), f(d))) \in \theta|_{N^2} . \]

We form the “meets” of $(f(a), f(b))$ and $(f(c), f(d))$ with the element $(1, 0)$ (using the operation $p$), and obtain that

\[f(a), 0 \equiv f(c), 0 \pmod{\theta} . \]

Since $\{f(a), f(c)\} = \{0, 1\}$, we thus have $(0, 0) \equiv (1, 0) \pmod{\theta_0}$.

Now $(a, b) = (f_0(0, 1), f_0(1, 0))$ and $(c, d) = (f_1(0, 1), f_1(1, 0))$ for some binary polynomials $f_0$ and $f_1$ of $A$; i.e.,

\[a, b) = f'_0((0, 1), (1, 0)) \text{ and } (c, d) = f'_1((0, 1), (1, 0))\]

where $f'_0, f'_1 \in \text{Pol}_2 R$ are $f_0, f_1$ acting coordinatewise. Therefore, we have

\[(5.27.2) \quad (a, b) \equiv f'_0((0, 1), (0, 0)) = (f_0(0, 0), f_0(1, 0)) \pmod{\theta_0} , \]
\[(c, d) \equiv f'_1((0, 1), (0, 0)) = (f_1(0, 0), f_1(1, 0)) \pmod{\theta_0} . \]

Case 1: If $b \neq d$, then by the same argument, $(0, 0) \equiv (0, 1) \pmod{\theta_0}$, and we find

\[a, b) \equiv (f_0(0, 0), f_0(0, 0)) = (c_0, c_0) \pmod{\theta_0} , \]
(c, d) \equiv (f_1(0, 0), f_1(0, 0)) = \langle e_1, e_1 \rangle \pmod{\theta_0}

for some \(e_0\) and \(e_1\). Since \(\theta_0\) is \(\alpha\), the restriction of \(\theta\) to the diagonal subalgebra \(0_A = \{(x, x) : x \in A\}\) must be either the identity relation on this subset, or \((\beta \times \beta)|_{0_A}\). Thus if \(e_0 \neq e_1\), then \(\langle e_0, e_0 \rangle \equiv \langle e_1, e_1 \rangle \pmod{\theta}\) implies that \(\langle (0, 0), (1, 1) \rangle \in \theta|_{N_2^2} \subseteq \theta_0\). This gives that \(\theta_0 \supseteq (\beta \times \beta)|_{0_A}\), and hence \(\langle e_0, e_0 \rangle \equiv \langle e_1, e_1 \rangle \pmod{\theta_0}\). Thus, \(e_0 \neq e_1\) implies \(\langle a, b \rangle \equiv \langle c, d \rangle \pmod{\theta_0}\) in Case 1, and \(e_0 = e_1\) obviously gives the same conclusion.

**Case 2:** Assume that \(b = d\). If \(\text{typ}(0_A, \beta) = 3\) or \(4\), then there is also a pseudo-join operation for \(N_0\). Then from \(\langle 0, 0 \rangle \equiv (1, 0) \pmod{\theta_0}\) follows \(\langle 0, 1 \rangle \equiv (1, 1) \pmod{\theta_0}\) (by taking the join with \(0, 1\)). From this, and formulas (5.27.2), we obtain that

\[
(a, b) \equiv f_0((1, 1), (0, 0)) = \langle f_0(1, 0), f_0(1, 0) \rangle \equiv (b, b) \pmod{\theta_0}
\]

and likewise \(\langle c, d \rangle \equiv (d, d) \equiv (b, b) \pmod{\theta_0}\). Therefore the proof of the claim is finished if \(\text{typ}(0_A, \beta) = 5\). So we assume now that \(\text{typ}(0_A, \beta) = 5\). We can also assume that \(f_0(0, 0) \neq f_1(0, 0)\) (else we are done, by (5.27.2)).

To conclude the proof of the claim, we choose a \(g \in \text{Pol}_1 A\) such that \(g(b/\beta) \subseteq N_0\) and \(g f_0(0, 0) \neq g f_1(0, 0)\). Without losing generality, we assume that \(g f_0(0, 0) = 0\), \(g f_1(0, 0) = 1\). We must have

\[
g f_1(1, 0) = g(d) = g(b) = g f_1(0, 0) = 1,
\]

else \(g f_1(1, 0) = 0\) and there is an obvious polynomial \(h\) satisfying \(h(0) = 1\), \(h(1) = 0\), contradicting that \(A|_{N_0}\) is a semilattice. From all the equalities collected in this paragraph (and \(f_0(1, 0) = b = f_1(1, 0)\) as well), we have that \(g\), acting coordinatewise, transforms the two ordered pairs on the far right of (5.27.2) onto \(0, 1\) and \(1, 1\), respectively. Thus \(\langle 0, 1 \rangle \equiv (1, 1) \pmod{\theta_0}\); and it follows that \(\langle a, b \rangle \equiv (b, b) \equiv \langle c, d \rangle \pmod{\theta_0}\) just as we saw for types 3 and 4 above. This finishes our proof of the claim.

To finish the proof of the theorem, we apply Lemma 2.4. Let \(U_0^* = U_0^2 \cap R\), and let \(e = e^2 \in \text{Pol}_1 A\) with \(e(A) = U_0\), and define \(e'\) by \(e'((x, y)) = (e(x), e(y))\). Thus \(e' \in \text{Pol}_1 R\), \((e')^2 = e'\), and \(e'(R) = U_0^*\). We have that \(N_2^2\) is an equivalence class of \((\beta \times \beta)|_{U_0^2} = ((\beta \times \beta)|_{U_0^2}|_{U_0^2}^2\). Applying Lemma 2.4, we have that

\[
|N_2^2 : I[R, (\beta \times \beta)|_{R}] | \rightarrow \text{Con R}|_{N_2^2}.
\]

The claim proved above implies that this lattice homomorphism is one-to-one. Our lattice \(L\) is therefore isomorphic to \(\text{Con R}|_{N_2^2}\).

It follows from our previous description of the polynomials of \(R\), and from the fact that \(R|_{N_2^2} = (R|_{U_0^2})|_{N_2^2}\) (because \(e'\) exists), and from our knowledge of \(U_0\) and \(N_0\), that \(R|_{N_2^2}\) is polynomially equivalent to \((A|_{N_0})^2\). (The proof of this whopper is an exercise.) Assertions (1) and (2) of this theorem can now be validated by computing
the congruence lattices of the direct squares of a two-element Boolean algebra, lattice, and semilattice.

\qed

Exercises 5.28

(1) Prove these additions to Lemma 5.24. If typ(\(\alpha, \beta\)) = 3, there exists a smallest admissible, reflexive \(\tau\) satisfying \(\tau \subseteq \beta, \tau \not\subseteq \alpha\); and this relation is a tolerance. If typ(\(\alpha, \beta\)) = 4, there exist precisely two minimal relations having these properties. (These assertions omit the requirement that \(\tau\) be \(\alpha\)-closed.)

(2) Prove that a pre-order on the universe of an algebra is admissible iff it is closed under unary polynomials.

(3) Prove the unproved assertion in the last paragraph of the proof of Theorem 5.27.

(4) Suppose that \(\langle \alpha, \beta \rangle\) is a prime quotient of \(A\) of type 2, with basic tolerance \(R\), and that \(R \subseteq A^2\) is the algebra whose universe is \(R\). Show that the interval \(I[\langle \alpha \times \alpha \rangle_R, \langle \beta \times \beta \rangle_R]\) contains a sublattice isomorphic to \(M_3\).

(5) Let \(\alpha, \beta, R\) be as above, except that typ(\(\alpha, \beta\)) = 5. Let \(\delta_0, \delta_1, \delta_2\) be the three co-atoms of \(I[\langle \alpha \times \alpha \rangle_R, \langle \beta \times \beta \rangle_R]\) as pictured in Theorem 5.27. Prove that typ(\(\delta_i, \langle \beta \times \beta \rangle_R\)) = 5 and that \(R/\delta_i \cong A/\alpha\), for \(i = 0, 1, 2\).

(6) Let \(A\) be an algebra such that \(A|_A\) is Mal'cev. Prove that for any \(\alpha, \beta \in \text{Con } A\), \(\alpha \circ \beta = \beta \circ \alpha = \alpha \vee \beta\). (\(A\) has permuting congruences.) From this, derive that \(\text{Con } A\) is a modular lattice.