6. LABELED CONGRUENCE LATTICES

A finite lattice is completely described by its Hasse diagram, a directed graph whose vertices are the elements of the lattice, and whose edges are the prime quotients. When $L$ is the congruence lattice of a finite algebra $A$, the Hasse diagram of $L$ has a natural labeling, namely, label every edge by its type as a prime congruence quotient of $A$. This mapping from the set of edges to the set $\{1, \ldots, 5\}$ is just the function typ, defined in Definition 5.1; we call it the type labeling of $L$. The type labeling of $L = \text{Con } A$ is determined by the polynomials of $A$; in fact, it is determined by $\text{Pol}_2 A$. Notice that if we replace $A$ by $(A, \text{Pol}_1 A)$, then $L$ remains unchanged and the sets of $(\alpha, \beta)$-minimal sets and $(\alpha, \beta)$-traces are unchanged, for every prime quotient $(\alpha, \beta)$, but the type labeling of $L$ changes to a trivial labeling using only the type label 1.

We recommend trying to visualize the labels as colors, rather than numbers (although officially we shall stick with the numbers). Use the color chart: 1 = orange, 2 = red, 3 = blue, 4 = green, 5 = yellow. The colored graphs thus obtained from finite algebras have many regularities, following from results proved in this chapter and the next one. Each graph is divided into disjoint regions (convex sublattices) within which only the colors red and orange appear. Edges joining separate regions are colored blue, green or yellow. Moreover, each red-orange region is divided into subregions colored entirely with orange, while edges between adjoining subregions are red. The regions and subregions constitute the blocks of two congruences on the congruence lattice which we shall study in the next chapter: the solvability congruence and the strong solvability congruence. Modulo the solvability congruence, the lattice is meet semi-distributive. Modulo strong solvability, the red-orange regions are modular lattices. An edge $(\alpha, \beta)$ between separate red-orange regions must be painted yellow, unless $\alpha$ has a pseudo-complement under $\beta$. (This follows from Lemma 5.19.)

Our purpose in this chapter is to collect results relating the omission of type labels to the non-occurrence of certain lattices as sublattices of congruence lattices. The results proved here will enable us to make smooth progress in subsequent chapters. This chapter is divided into two parts. In the first, we primarily study the labeled congruence lattice of one algebra, and in the second, we compare the variety generated by a finite algebra to the varieties generated by certain of its induced algebras. Our first result doesn't really belong to either part, but fits here as comfortably as anywhere.
LEMMA 6.1. Let \( A \) be a finite algebra, and let \( \langle \alpha_1, \beta_1 \rangle, \ldots, \langle \alpha_n, \beta_n \rangle \) be a list of prime quotients of \( A \) such that every congruence \( \Theta \) on \( L = \text{Con} A \), except \( \Theta = 0_L \), satisfies \( \langle \alpha_i, \beta_i \rangle \in \Theta \) for some \( 1 \leq i \leq n \). Choose \( U_i \in M_A(\alpha_i, \beta_i) \) for each \( i \). The function defined on \( L \) by \( f(\delta) = \{ \delta|_{U_i} : 1 \leq i \leq n \} \) is an isomorphism of \( L \) with a subdirect product of the \( \text{Con} A|_{U_i} \). In symbols

\[
f : L \overset{\sim}{\rightarrow} \prod_{ad} \{ \text{Con} A|_{U_i} : 1 \leq i \leq n \}.
\]

PROOF. By Lemma 2.3 and Theorem 2.8, \( |U_i| \) is an onto lattice homomorphism for all \( i \), and \( \alpha_i|_{U_i} \neq \beta_i|_{U_i} \). Therefore \( f \) is a lattice homomorphism, and \( (\ker f) \cap \{ \langle \alpha_i, \beta_i \rangle : 1 \leq i \leq n \} = \emptyset \). Hence ker \( f = 0_L \), and \( f \) is one-to-one.

\[ \square \]

LEMMA 6.2. Let \( \langle \alpha_i, \beta_i \rangle (i = 0, 1) \) be prime quotients of a finite algebra \( A \) such that \( \beta_0 \land \alpha_1 = \alpha_0 \) and \( \beta_0 \lor \alpha_1 = \beta_1 \). Then \( M_A(\alpha_0, \beta_0) = M_A(\alpha_1, \beta_1) \) and \( \text{typ}(\alpha_0, \beta_0) = \text{typ}(\alpha_1, \beta_1) \).

PROOF. That \( M_A(\alpha_0, \beta_0) = M_A(\alpha_1, \beta_1) \) when \( \langle \alpha_0, \beta_0 \rangle \) and \( \langle \alpha_1, \beta_1 \rangle \) are projective quotients is the result of Exercise 2.19(3).

To prove the equality of types, let \( U \) be an \( \langle \alpha_0, \beta_0 \rangle \)-minimal set. Suppose first that \( \langle \alpha_1, \beta_1 \rangle \) is of non-Abelian type. Let \( N_1 \) be the unique \( \langle \alpha_1, \beta_1 \rangle \)-trace in \( U \). (Note that \( U \) is also an \( \langle \alpha_1, \beta_1 \rangle \)-minimal set.) By Lemma 2.4, \( \beta_0|_{N_1} \lor \alpha_1|_{N_1} = \beta_1|_{N_1} \) \( (= 1_{N_1}) \) and \( \beta_0|_{N_1} \land \alpha_1|_{N_1} = \alpha_0|_{N_1} \). From these equations and the picture of \( N_1 \) supplied by Lemmas 4.15 and 4.17, it follows that \( N_1 \) contains precisely one \( \langle \alpha_0, \beta_0 \rangle \)-trace \( N_0 \) of \( U \), and we have this picture of \( N_1 \):

![Figure 13](image)

In the picture, \( \alpha_0|_{N_0} = \alpha_1|_{N_0} \); \( \{ 1 \} \) and \( N_1 - \{ 1 \} \) are the two \( \alpha_1|_{N_1} \)-classes; \( \{ 1 \} \) and \( N_0 - \{ 1 \} \) are two of the \( \alpha_0|_{N_1} \)-classes; and \( N_1 - N_0 \) is partitioned in some way into \( \alpha_0|_{N_1} \)-classes.

Now for any \( a \in N_0 - \{ 1 \} \), a pseudo-meet operation with respect to \( \langle \alpha_1, \beta_1 \rangle \), restricted to \( \{ 1, a \} \), is a semilattice operation, showing that \( \langle \alpha_0, \beta_0 \rangle \) is non-Abelian.

Thus Lemmas 4.15, 4.17 also apply to \( U \) with respect to \( \langle \alpha_0, \beta_0 \rangle \), and \( N_0 \) is the unique \( \langle \alpha_0, \beta_0 \rangle \)-trace in \( U \). If \( \text{typ}(\alpha_1, \beta_1) \neq 5 \), then \( N_1 = N_0 \), a two-element set, and then \( \text{typ}(\alpha_0, \beta_0) = \text{typ}(\alpha_1, \beta_1) = \text{typ}(A|_{N_1}) \). If \( \text{typ}(\alpha_1, \beta_1) = 5 \), then it is very easy.
to see that $A|_{N_0}$ cannot have a pseudo-join operation; hence $\text{typ}(\alpha_0, \beta_0) = 5$ also. This finishes our proof for equality of types in the case that $\langle \alpha_1, \beta_1 \rangle$ is not Abelian.

Let us suppose now that $\langle \alpha_1, \beta_1 \rangle$ is Abelian. By Proposition 3.7 (2), $\langle \alpha_0, \beta_0 \rangle$ is also Abelian. One can easily show, directly from the definition, that if $\langle \alpha_1, \beta_1 \rangle$ is strongly Abelian then $\langle \alpha_0, \beta_0 \rangle = \langle \beta_0 \land \alpha_1, \beta_0 \land \beta_1 \rangle$ is strongly Abelian. Therefore, the only thing remaining to be proved is that $\text{typ}(\alpha_1, \beta_1) = 2$ implies $\text{typ}(\alpha_0, \beta_0) \neq 1$. Assume that $\text{typ}(\alpha_1, \beta_1) = 2$, and let $B_i$ be the $\langle \alpha_i, \beta_i \rangle$-body of $U$ ($i = 0, 1$). Let $d(x, y, z)$ be a pseudo-Mal'cev operation for $U$ relative to $\langle \alpha_1, \beta_1 \rangle$ (Lemma 4.20). Then $d|_{B_i}$ is Mal'cev, and obviously $B_0 \subseteq B_1$. For any $\langle \alpha_0, \beta_0 \rangle$-trace $N_0 \subseteq B_0$, $N_0$ is closed under $d$ since $d(x, x, x) = x$; therefore $(A|_{N_0})/(\alpha|_{N_0})$ is Mal'cev, and cannot be a minimal algebra of type 1.

Exercises 6.23 (1–7) give examples showing that the type labeling need not be so well-behaved with regard to projective quotients of which one is prime and the other is not. It is not true, for instance, that if $\beta_0 \land \alpha_1 = \alpha_0$, $\beta_0 \lor \alpha_1 = \beta_1$, and $\alpha_0 \prec \beta_0$, then the equality $\text{typ}(\alpha_1, \beta_1) = \text{typ}(\alpha_0, \beta_0)$ must hold.

**Lemma 6.3.** Suppose that $\delta_0, \delta_1, \delta_2, \gamma_0, \gamma_1, \psi, \theta$ are congruences of a finite algebra $A$ constituting a sublattice of $\text{Con} A$ isomorphic to $D_2$, as pictured below. If $\delta_1 \leq \alpha \prec \theta$, then $\text{typ}(\alpha, \theta) \in \{1, 5\}$. 

![Diagram](image)

**Proof.** Since $\delta_1 \lor \delta_0 = \delta_1 \lor \delta_2 = \theta$ and $\delta_1 \lor (\delta_0 \land \delta_2) = \delta_1$, it follows from Lemma 5.19 (2) that $\text{typ}(\alpha, \theta) \in \{1, 2, 5\}$. Assume that $\text{typ}(\alpha, \theta) = 2$, in order to derive a contradiction. Let $N$ be any $\langle \alpha, \theta \rangle$-trace. By Lemma 2.4, $|_N$ is a homomorphism of $I[\theta, \alpha]$ onto $\text{Con} A|_N$; and by Lemma 4.20, $A|_N$ is a Mal'cev algebra. According to Exercise 5.28 (6), $\text{Con} A|_N$ is a modular lattice. Therefore,

$$\delta_2|_N = (\delta_2 \land (\gamma_1 \lor \delta_0)|_N = \delta_2|_N \land (\gamma_1|_N \lor \delta_0|_N) = \gamma_1|_N \lor (\delta_2|_N \land \delta_0|_N) = \gamma_1|_N.$$  

Consequently, $\theta|_N = (\delta_1 \lor \delta_2)|_N = \delta_1|_N \lor \delta_2|_N = \delta_1|_N \lor \gamma_1|_N = \delta_1|_N$. This gives $\alpha|_N = \theta|_N$, since $\delta_1 \leq \alpha \prec \theta$; but that is absurd since $N$ is an $\langle \alpha, \theta \rangle$-trace.
LEMMA 6.4. Suppose that $\delta_0, \delta_1, \delta_2, \gamma_0, \gamma_1, \psi, \theta$ are congruences of a finite algebra $A$ constituting a sublattice of $\text{Con } A$ isomorphic to $D_1$, as pictured below. If $\theta < \alpha \leq \delta_1$, then $\text{typ}(\theta, \alpha) = 1$.

\[ \begin{array}{c}
\delta_0 \quad \delta_1 \quad \delta_2 \\
\gamma_0 \quad \gamma_1 \\
\theta \quad \psi \\
0 \\
\end{array} \quad \text{implies} \quad \begin{array}{c}
\delta_1 \\
\alpha \\
\theta \\
0 \\
\end{array} \]

PROOF. Since $\delta_1 \wedge \delta_0 = \delta_1 \wedge \delta_2 = \theta$ and $\delta_1 \wedge (\delta_0 \vee \delta_2) = \delta_1$, it follows from Lemma 5.19(1) that $\text{typ}(\theta, \alpha) \in \{1, 2\}$. Assume that $\text{typ}(\theta, \alpha) = 2$, in order to derive a contradiction. Let $U$ be any $(\theta, \alpha)$-minimal set and let $B$ be the body of $U$, and $T$ be the tail. Since $\delta_1 \wedge \delta_1 = \theta$ ($i = 0, 2$), we have $\delta_i|_U = \delta_i|_U \vee \theta|_U \nleq \alpha|_U$ ($i = 0, 2$). So by Lemma 4.27(4), $\delta_0|_U \cup \delta_2|_U \subseteq B^2 \cup T^2$. From this, and from the fact that $\delta_0|_U \vee \delta_2|_U = (\delta_0 \vee \delta_2)|_U$, it is obvious that $\delta_0|_B \vee \delta_2|_B = (\delta_0 \vee \delta_2)|_B$, a fact which will be needed below.

We repeat the calculation used in proving the last lemma, working with congruences over the Mal'cev algebra $A|_B$. It is obvious that $|_B$ preserves lattice meets; and the one non-trivial join we need is preserved (see the last paragraph). We have $\delta_0|_B \wedge \gamma_1|_B = \theta|_B \leq \delta_2|_B$, and $\delta_0|_B \vee \delta_2|_B = \psi|_B \geq \gamma_1|_B$. Since $\text{Con } A|_B$ is modular, this implies that $\gamma_1|_B = \delta_0|_B \vee \delta_2|_B$. Thus $\delta_1|_B = \delta_1|_B \wedge \gamma_1|_B = \delta_1|_B \wedge \delta_2|_B = \theta|_B$. Since $\theta < \alpha \leq \delta_1$, this contradicts the fact that $\theta|_B \neq \alpha|_B$.

\[ \Box \]

LEMMA 6.5. Let $\theta_0, \theta_1, \theta_2, \delta_1, \delta_0$ be congruences of a finite algebra constituting a pentagon; i.e., $\theta_0 = \delta_1 \wedge \theta_1 < \delta_0 < \delta_1 < \delta_0 \vee \theta_1 = \theta_2$. If $\text{typ}(\theta_0, \theta_1) \subseteq \{1, 2\}$, then $\text{typ}(\delta_0, \delta_1) = \{1\}$.

\[ \begin{array}{c}
\delta_0 \quad \delta_1 \quad \delta_2 \\
\theta_0 \quad \theta_2 \quad \theta_1 \\
\{1, 2\} \\
\end{array} \quad \text{implies} \quad \begin{array}{c}
\delta_1 \\
\delta_0 \\
\{1\} \\
\end{array} \]

PROOF. The truth of the lemma will follow if we can prove it under the assumption that $\delta_0 < \delta_1$. We make that assumption, and assume that $\text{typ}(\theta_0, \theta_1) \subseteq \{1, 2\}$, and $\text{typ}(\delta_0, \delta_1) \neq \{1\}$. We will derive a contradiction. Let $U$ be a $(\delta_0, \delta_1)$-minimal set, and let $B$ and $T$ be its $(\delta_0, \delta_1)$-body and tail.
It is impossible to have \( \delta_0|_B \lor \theta_1|_B \geq \delta_1|_B \). For we would then have a pentagon \( \delta_0|_B, \theta_1|_B, \delta_0|_B \lor \theta_1|_B, \delta_1|_B, \delta_0|_B \) in \( \text{Con} \, A|_B \), forcing \( \text{typ}(\delta_0, \delta_1) \neq 2 \); but if \( \text{typ}(\delta_0, \delta_1) \in \{3, 4, 5\} \), then \( \delta_1|_B = B^2 \), and there is no such pentagon. Therefore \( \delta_0|_B \lor \theta_1|_B \neq \delta_1|_B \); but, of course, \( \delta_0|_U \lor \theta_1|_U \geq \delta_1|_U \). From this, we draw the conclusion that \( \theta_1 \cap (B \times T) \neq \emptyset \). (Recall that \( B \) is a union of \( \delta_1|_U \)-classes and \( \delta_1 \) equals \( \delta_0 \) on \( T \).) Notice that \( \theta_0 \cap (B \times T) = \emptyset \), since \( \theta_0 \subseteq \delta_0 \).

Clearly, there exists a prime quotient \( \langle \gamma, \lambda \rangle \) in the interval \( I[\theta_0, \theta_1] \) with

\[
\gamma|_U \cap (B \times T) = \emptyset \neq \lambda|_U \cap (B \times T).
\]

Now we assumed that \( \text{typ}(\theta_0, \delta_1) \subseteq \{1, 2\} \); thus \( \langle \gamma, \lambda \rangle \) and \( \langle \gamma|_U, \lambda|_U \rangle \) are Abelian. It follows by an application of Lemma 4.27(4ii), to the algebra \( C = A|_U \) and its quotient \( \langle \delta_0|_U, \delta_1|_U \rangle \), that \( \text{typ}(\delta_0, \delta_1) \neq 2 \).

Thus we must have \( \text{typ}(\delta_0, \delta_1) \in \{3, 4, 5\} \). Let \( p(x, y) \) be a pseudo-meet operation of \( A|_U \) relative to \( \langle \delta_0|_U, \delta_1|_U \rangle \). Choose \( \langle b, u \rangle \in \lambda \cap (B \times T) \) and let \( v = \langle b, u \rangle \). By Lemma 4.15 or Lemma 4.17, \( p(b, v) = v = p(v, v) \), and \( v \equiv p(b, b) = b \mod \lambda \). Using that \( \langle \gamma, \lambda \rangle \) is Abelian, we have \( p(b, b) = b \equiv p(v, b) \mod \gamma \). But \( b \in B \) and \( u, v, p(v, b) \in T \). This contradicts the fact that \( \gamma \cap B \times T = \emptyset \); and so our proof is finished. (Incidentally, we have just worked the second half of Exercise 4.37(5).)

**Lemma 6.6.** Let \( \delta_0, \delta_1, \delta_2, \psi, \theta \) be congruences of a finite algebra constituting a diamond; i.e., \( \delta_i \lor \delta_j = \theta \) and \( \delta_i \land \delta_j = \psi \) for \( 0 \leq i < j \leq 2 \). If \( \psi \prec \alpha \leq \delta_1 \) and \( \delta_1 \leq \beta \prec \theta \), then \( \{\text{typ}(\psi, \alpha), \text{typ}(\beta, \theta)\} \subseteq \{1, 2\} \).

**Proof.** By Exercise 3.8(6), each quotient \( \langle \psi, \delta_1 \rangle \) is Abelian. Thus \( \langle \psi, \alpha \rangle \) is Abelian and must have type 1 or 2. By Lemma 5.19(2), we have \( \text{typ}(\beta, \theta) \in \{1, 2, 5\} \). If \( \text{typ}(\beta, \theta) = 5 \) and \( N \) is a \( (\beta, \theta) \)-trace, then \( \delta_1|_N, \theta|_N, \psi|_N \) form a diamond with \( \delta_1|_N \leq \beta|_N < \theta|_N \). We must have \( (1, u) \in \delta_0|_N - \beta \) for some \( u \), where 1 is the neutral element for the pseudo-meet operation \( p(x, y) \) (since \( \delta_0|_N \lor \beta|_N = 1_N \)). But \( \langle 1, u \rangle, p \rangle \) is a semilattice, contradicting that \( \langle \psi, \delta_0 \rangle \) is Abelian and \( (1, u) \in \delta_0 - \psi \).}

**Remark 6.7.** With the aid of the theory of solvability presented in Chapter 7, it can be proved that when \( D_1 \) occurs as a sublattice of \( \text{Con} \, A \) (\( A \) finite) as in
Lemma 6.4, then \( \text{typ}\{\theta, \delta_1\} \subseteq \{1, 2\} \); and when \( M_3 \) appears, as in Lemma 6.6, then \( \text{typ}\{\psi, \theta\} \subseteq \{1, 2\} \).

The preceding four lemmas give restrictions governing the type labels that can appear in certain positions relative to an occurrence of \( D_2, D_1, N_5 \) or \( M_3 \) as a sublattice of \( \text{Con } A \). As a corollary to these results, we can conclude that if the unary type does not occur in the type labeling of \( \text{Con } A \), then \( D_1 \) is not embedded in \( \text{Con } A \) (and analogous results for \( D_2 \) and \( M_3 \)).

Our purpose in the second half of this chapter will be to produce an approximate converse to each of these corollaries. A result of the kind we will be proving is already to be found in Theorem 5.27: If \( 5 \in \text{typ}\{A\} \), then the congruence lattice of a certain subalgebra of \( A^2 \) has a copy of \( D_2 \). Before beginning this task, we present two more lemmas, and this notable corollary of Lemma 6.5.

**COROLLARY 6.8.** Suppose that \( \alpha, \beta \) are congruences of a finite algebra \( A \) with \( \alpha < \beta \) and \( \text{typ}\{\alpha, \beta\} = \{2\} \). Then \( I[\alpha, \beta] \) is a modular lattice.

**LEMMA 6.9.** Let \( \langle \alpha, \beta \rangle \) be a prime quotient in a finite algebra \( A \). Suppose that \( \theta_0, \ldots, \theta_n \in I[\alpha,1_A] \) satisfy \( \beta \leq \bigvee \{\theta_i : i \leq n\} \). If \( \text{typ}(\alpha, \beta) \neq 1 \), then for some \( i \leq n \), \( \text{typ}(\alpha, \beta) \subseteq \text{typ}(\alpha, \theta_i) \).

**PROOF.** If \( \langle \alpha, \beta \rangle \) is non-Abelian, then \( \beta \leq \theta_i \) for some \( i \) (by Lemma 5.12), and the desired conclusion is trivial. Thus, we assume that \( \text{typ}(\alpha, \beta) = 2 \). Let \( U \) be an \( \langle \alpha, \beta \rangle \)-minimal set, let \( B \) be the body of \( U \), and let \( T \) be the tail. By Lemma 4.27 (4i), if \( \theta_i|_U \not\subseteq B^2 \cup T^2 \) for some \( i \), then \( \theta_i|_U \geq \beta|_U \); but then \( \theta_i \geq \beta \), and we are done. Thus, assume that \( B \) is a union of \( \theta_i|_U \)-classes, for all \( i \). Then \( \beta|_U \leq \bigvee \{\theta_i|_U : i \leq n\} \); and so we can pick an \( \theta_i \) with \( \theta_i|_U \geq \alpha|_U \). Let \( \langle \tau, \lambda \rangle \) be a prime quotient in \( I[\alpha, \theta_i] \) satisfying \( \tau|_U = \alpha|_U \) and \( \alpha|_U < \lambda|_U \).

We claim that \( \text{typ}(\tau, \lambda) = 2 \). To prove it, we choose any \( (u, v) \in \lambda|_U - \tau|_U \), and \( e \in E(A) \) with \( e(A) = U \). Since \( (u,v) \in \lambda \), there is a sequence \( u = u_0, \ldots, u_m = v \) with

\[
\langle u_i, u_{i+1} \rangle \in \tau \cup \bigcup \{N^2 : N \text{ a } \langle \tau, \lambda \rangle\text{-trace } \} \text{ for all } i < m.
\]

Now for all \( \langle \tau, \lambda \rangle \)-traces \( N \), either \( e(N)^2 \subseteq \tau \), or \( e(N) \) is a \( \langle \tau, \lambda \rangle \)-trace. Since \( \langle u,v \rangle \not\subseteq \tau \), there exists a \( \langle \tau, \lambda \rangle \)-trace \( N \subseteq U \) such that \( N \cap u/(\tau|_U) \neq \emptyset \). But \( u/(\tau|_U) \subseteq B \), and \( N^2 \subseteq \theta_i|_U \subseteq B^2 \cup T^2 \); hence \( N \subseteq B \).

The function \( e \), restricted to a \( \langle \tau, \lambda \rangle \)-minimal set including \( N \) as trace, must be a polynomial isomorphism of this set onto a \( \langle \tau, \lambda \rangle \)-minimal set \( V \subseteq U \). We can choose \( e' \in E(A) \) with \( e'(A) = V \); and then we have \( e'|_U \in E(A|_U) \) and \( N \subseteq e'(U) \cap B \). Applying Lemma 4.30, we find that \( e'|_U = \text{id}|_U \), and so \( U = V \). Since \( U = e(A) \), and \( B \) and \( N \) are congruence blocks in \( U = V \) (see Lemma 4.27 (3)), we have that \( A|_N = (A|_B)|_N \) and this algebra is Mal'cev and nilpotent. (By Theorem 4.31 and Lemma 4.36, \( A|_B \) is nilpotent, and its induced algebras must inherit this property.) Thus the minimal algebra \( (A|_N)/(\tau|_N) \) can only have type 2. \( \Box \)
LEMMA 6.10. Let \( \langle \alpha, \beta \rangle \) be a prime quotient in a finite algebra \( A \). Suppose that \( \theta_0, \ldots, \theta_n \in I[0_A, \beta] \) satisfy \( \alpha \geq \bigwedge \{ \theta_i : \ i \leq n \} \). If \( \text{typ}(\alpha, \beta) \notin \{1, 5\} \), then for some \( i \leq n \), \( \text{typ}(\alpha, \beta) \in \text{typ}(\theta_i, \beta) \).

PROOF. If \( \text{typ}(\alpha, \beta) \in \{3, 4\} \), then by Lemma 5.15, \( \theta_i \leq \alpha \) for some \( i \) and we are done. Thus, let \( \text{typ}(\alpha, \beta) = 2 \), and let \( U \) be an \( \langle \alpha, \beta \rangle \)-minimal set, and \( B \) and \( T \) be its body and tail. We assume that \( \theta_i \vee \beta = \beta \) for all \( i \) (else \( \theta_i \leq \alpha \)). We shall use several times the fact that \( |B| \) is a homomorphism of \( I[0_A, \beta] \) onto the modular lattice \( I[0_B, \beta] \subseteq \text{Con} \ A|_B \), which follows easily from Lemma 2.4 and Lemma 4.27 (3). (There exists a congruence \( \theta \) of \( A \), with \( \theta \geq \beta \), such that \( B \) is a \( \theta \)-equivalence class.)

To begin, we have \( \theta_i|_B \vee \alpha|_B = \beta|_B \) for all \( i \), and \( \bigwedge \{ \theta_i|_B \} \leq \alpha|_B \). Also, \( \alpha|_B \prec \beta|_B \), since \( \alpha|_B < \beta|_B \) and \( \alpha \prec \beta \) and \( |B| \) is an onto homomorphism. Note also that when \( \tau \leq \beta \) we have \( \tau \leq \alpha \iff \tau|_B \leq \alpha|_B \). We can obviously assume that \( \bigwedge \{ \theta_i : i \neq j \} \not\leq \alpha \) for each \( j \leq n \), and in fact, that whenever \( \theta_j < \tau \leq \beta \) then \( \tau \wedge \bigwedge \{ \theta_i : i \neq j \} \not\leq \alpha \). These facts then hold also for the \( \theta_i|_B, \alpha|_B, \beta|_B \) in \( \text{Con} \ A|_B \).

Let \( \chi_0 \) denote \( \theta_0|_B, \chi_0 \) be a cover of \( \chi_0 \) in the interval \( I[\chi_0, \beta|_B] \) (i.e., \( \chi_0 < \chi_0 \leq \beta|_B \)), \( \chi_1 = \bigwedge \{ \theta_i|_B : 1 \leq i \leq n \} \wedge \chi_0 \), and \( \chi_1 = \chi_1 \wedge \alpha|_B \). Our assumptions imply that \( \langle \alpha|_B, \beta|_B \rangle \setminus \langle \chi_1, \chi'_1 \rangle \) and \( \langle \chi_0 \wedge \chi_1, \chi'_0 \rangle \setminus \langle \chi_0, \chi'_0 \rangle \), in the notation of Exercise 6.23 (13). The proof of this is an exercise for the reader. Now \( \alpha|_B \prec \beta|_B \) and \( \chi_0 < \chi'_0 \); so by Exercise 6.23 (13), \( \chi_1 \prec \chi'_1 \) and \( \chi_0 \wedge \chi'_1 \prec \chi'_1 \). Note that \( \chi_0 \wedge \chi'_1 \leq \alpha|_B \), so \( \chi_0 \wedge \chi'_1 \leq \chi'_1 \wedge \alpha|_B = \chi_1 \), and thus \( \chi_0 \wedge \chi'_1 = \chi_1 \).

Now we define \( \psi_1 \) to be the largest congruence of \( A \) such that \( \psi_1 \leq \beta \) and \( \psi_1|_B = \chi_1 \), and let \( \psi'_1 \) be the smallest congruence \( \geq \psi_1 \) such that \( \psi'_1|_B = \chi'_1 \). By previous remarks, \( \psi_1 \leq \alpha \) and we have \( \psi'_1 \leq \beta \) and \( \psi'_1 \prec \psi_1 \). It is easy to see that \( I[\alpha, \beta] \setminus I[\psi_1, \psi'_1] \). We put \( \psi_0 = \psi'_1 \vee \theta_0 \), so that \( \psi_0 \leq \beta \) and \( \psi_0|_B = \chi'_0 \), and we choose \( \psi_0 \) so that \( \theta_0 \vee \psi_1 \leq \psi_0 < \psi'_0 \). Thus \( \text{typ}(\psi_0, \psi_0') \in \text{typ}(\theta_0, \beta) \). Now it is easy to see that \( I[\psi_0, \psi_0'] \setminus I[\psi_0, \psi'_0] \). By Lemma 6.2, \( \text{typ}(\psi_0, \psi'_0) = \text{typ}(\psi_1, \psi'_1) = \text{typ}(\alpha, \beta) \), and this concludes our proof. \( \square \)
We now change our focus. Here begins a study of locally finite varieties, to which we devote the remainder of this book. Recall from Chapter 0 that a variety is a nonvoid class \( \mathcal{V} \) of similar indexed algebras such that

\[
H\mathcal{V} = S\mathcal{V} = P\mathcal{V} = \mathcal{V},
\]

meaning that \( \mathcal{V} \) is closed under the formation of homomorphic images, subalgebras, and algebras isomorphic to Cartesian products of algebras in \( \mathcal{V} \). A variety is **locally finite** iff all of its finitely generated algebras are finite. The variety generated by a class \( \mathcal{K} \) of similar algebras is denoted \( \mathcal{V}(\mathcal{K}) \); thus \( \mathcal{V}(\mathcal{K}) = HSP\mathcal{K} \). If \( \mathcal{K} = \{ A \} \), we write \( \mathcal{V}(A) \) in place of \( \mathcal{V}(\mathcal{K}) \). A variety generated by a finite set of finite algebras is said to be **finitely generated**. Recall that finitely generated varieties are locally finite.

**Definition 6.11.** Let \( \mathcal{K} \) be any class of algebras. By \( CON\mathcal{K} \) we denote the class of all congruence lattices of algebras in \( \mathcal{K} \). By \( \mathcal{K}_{fin} \) we denote the class of all finite members of \( \mathcal{K} \). We denote the set \( \bigcup\{ ttyp(A) : A \in \mathcal{K}_{fin} \} \) by \( ttyp(\mathcal{K}) \), and call it the **type set** of \( \mathcal{K} \).

The type set of a locally finite variety determines a surprising collection of varietal properties, as we shall see in Chapter 9. We remark that

\[
\begin{align*}
ttyp(\mathbf{G}-\mathit{Sets}) & = \{ \text{unary type} \}, \quad \mathbf{G} \text{ a finite non-trivial group;} \\
ttyp(\mathbf{F}-\mathit{Vector}\ \mathit{Spaces}) & = \{ \text{affine type} \}, \quad \mathbf{F} \text{ a finite field;} \\
ttyp(\mathbf{B}-\mathit{Boolean}\ \mathit{Algebras}) & = \{ \text{boolean type} \}; \\
ttyp(\mathbf{D}-\mathit{Distributive}\ \mathit{Lattices}) & = \{ \text{lattice type} \}; \\
ttyp(\mathbf{S}-\mathit{Semilattices}) & = \{ \text{semilattice type} \}.
\end{align*}
\]

The underlying theme of this second half of Chapter 6 will be that when \( A \) is a finite algebra and \( N \) is a trace set for one of the prime congruence quotients of \( A \), then \( A|_N \) (after conversion to an indexed algebra) generates a variety which is interpretable in a very well-behaved fashion into the variety generated by \( A \). Through these interpretations, we will find that every lattice in \( CON\mathcal{V}(A|_N) \) is a complete homomorphic image of an interval in some member of \( CON\mathcal{V}(A) \). Every locally finite variety \( \mathcal{V} \) “contains”, in the sense of our interpretations, one of the varieties in the above list corresponding to each type-label appearing in \( ttyp(\mathcal{V}) \).
DEFINITION 6.12. Let \( A \) be any algebra and let \( U \) be any nonvoid subset of the universe of \( A \). By \( A\{U\} \) we mean the indexed algebra \( \langle U, f : f \in (\text{Pol } A)|_U \rangle \). This indexed algebra \( A\{U\} \) is polynomially equivalent to \( A|_U \), and will be called \( A|_U \) with the normal indexing.

DEFINITION 6.13. Let \( A \) be any indexed algebra and \( T \) be any set. By a diagonal subalgebra of \( A^T \) we mean any subalgebra of \( A^T \) containing the diagonal \( \Delta \), the set of all constant functions from \( T \) into \( A \). For every nonvoid subset \( S \) of \( A^T \), we denote by \( A(S) \) the subalgebra of \( A^T \) generated by \( S \cup \Delta \). This algebra will be called the extension of \( A \) by \( S \). For any operation \( f \) on \( A \), we write \( f(T) \) for the operation on \( A^T \) which is \( f \) acting coordinatewise. For any equivalence relation \( \beta \) on \( A \), we write \( \beta(T) \) for the equivalence relation defined by \( (x, y) \in \beta(T) \) iff for all \( t \in T \), \( (x(t), y(t)) \in \beta \).

Notice that \( A(\emptyset) \cong A \), and that \( A(S) \) is, up to isomorphism, an extension of \( A \). We are interested in this construction mainly in the case where \( S \subseteq N^T \) for some trace \( N \) of \( A \), but we shall prove our basic results in a more general setting.

LEMMA 6.14. Suppose that \( A \) is an algebra; that \( e \in E(A) \), \( U = e(A) \) and \( \beta \in \text{Con } A \); and that \( S \) is a \( \beta|_U \)-equivalence class. Let \( T \) be any set and let \( S' = \langle S', \ldots \rangle \) be a subalgebra of \( (A|_S)^T \). Then define \( A' = A(S') \); \( e' = e(T)|_{A'} \); \( U' = e'(A') \); and \( \beta' = \beta(T)|_{A'} \).

1. The universe of \( A' \) is closed under \( f(T) \) for all \( f \in \text{Pol } A \). We have that \( \text{Pol } A' \supseteq \{f(T)|_{A'} : f \in \text{Pol } A \} \), and
   \[
   A' = \{ f(T)(s_0, \ldots, s_{n-1}) : f \in \text{Pol}_n A \text{ for some } n \text{ and } \{s_0, \ldots, s_{n-1}\} \subseteq S' \}.
   \]

2. We have that \( e' \in E(A') \); \( U' = e'(A') = A' \cap U^T \); \( \beta' \in \text{Con } A' \); \( S' \) is a \( \beta'|_U \)-equivalence class; and \( S' = A' \cap S^T \).

3. \( A'|_{S'} = S'|_{S'} \).

PROOF. Statement (1) depends on nothing more than the fact that \( A' \) is the diagonal subalgebra of \( A^T \) generated by \( S' \). To prove it, let \( f \in \text{Pol } A \). For some \( m \) and \( n \), we have an \( m+n \)-ary term operation \( g \in \text{Clo}_{m+n} A \) and elements \( a_0, \ldots, a_{m-1} \) in \( A \) such that \( f(x_0, \ldots, x_{n-1}) = g(a_0, \ldots, a_{m-1}, x_0, \ldots, x_{n-1}) \). We write \( (a_i) \) for the member of \( \Delta \subseteq A^T \) whose constant value is \( a_i \). Now \( g(T) \) is a term operation of \( A^T \), and \( f(T)(x_0, \ldots, x_{n-1}) = g(T)((a_0), \ldots, (a_{m-1}), x_0, \ldots, x_{n-1}) \). Obviously, \( A' \) is closed under \( f(T) \), since it is closed under \( g(T) \) and \( (a_0), \ldots, (a_{m-1}) \in A' \); and since \( g(T)|_{A'} \) is a term operation of \( A' \), it follows that \( f(T)|_{A'} \) is a polynomial operation of \( A' \). The description of \( A' \) follows easily from these considerations.
To prove (2), we note that $e' \in \text{Pol}_1A'$ by (1), and clearly $e'e' = e'$; so $e' \in E(A')$. For $x \in A'$, we have $x \in e'(A')$ iff $e'(x) = x$ iff $e(x(t)) = x(t)$ for all $t \in T$ iff $x \in UT$. Obviously, $\beta' \in \text{Con} A'$, and it is easy to see that $A' \cap S^T$ is a $\beta'[U]$-equivalence class which includes $S'$. To prove that $A' \cap S^T \subseteq S'$, let $a \in A' \cap S^T$. By (1), we can write $a = f^{(T)}(s_0, \ldots, s_{n-1})$, $f \in \text{Pol}_nA$, $s_0, \ldots, s_{n-1} \in S'$. We can assume that $f = ef'$, since $e^{(T)}(a) = a$. For any $t \in T$, $f(s_0(t), \ldots, s_{n-1}(t)) = a(t) \in S$, implying that $f(S^T) \subseteq S$. This holds because $f(U^n) \subseteq U$, $S$ is a $\beta[U]$-equivalence class, and $a(t), s_0(t), \ldots, s_{n-1}(t) \in S$. (Note that if $T = \emptyset$ then our lemma holds trivially; $A^T$ is then a one-element algebra.) Therefore $f|_S$ is an operation of $A1_S$. It follows that $a \in S'$, as claimed, since $a = f^{(T)}(s_0, \ldots, s_{n-1}) = (f|_S)^{(T)}(s_0, \ldots, s_{n-1})$ and $S'$ is a subuniverse of $(A1_S)^T$. This finishes the proof of (2).

In order to prove (3), we need these characterizations of Pol $S'$ and Pol $A'$.

(6.14.1) \[ \text{Pol}_mS' = \{ f^{(T)}|_{S'}(s_0, \ldots, s_{m-1}, x_0, \ldots, x_{n-1}) : f \in \text{Pol}_{m+n}A \] for some $m$, $f(S^{m+n}) \subseteq S$, and $s_0, \ldots, s_{m-1} \in S'$.

(6.14.2) \[ \text{Pol}_mA' = \{ f^{(T)}|_{A'}(s_0, \ldots, s_{m-1}, x_0, \ldots, x_{n-1}) : f \in \text{Pol}_{m+n}A \] for some $m$, and $s_0, \ldots, s_{m-1} \in S'$.

To prove (6.14.1), notice that the term operations of $A1_S$ are precisely the $f|_S$ such that $f \in \text{Pol}_kA$ and $f(S^k) \subseteq S$ for some $k$. Consequently, those of $S'$ are precisely the $(f|_S)^{(T)}|_{S'} = f^{(T)}|_{S'}$ such that $f \in \text{Pol}_kA$ and $f(S^k) \subseteq S$ for some $k$. The characterization of Pol $S'$ follows immediately from this.

To prove (6.14.2), recall that by (1), $f^{(T)}|_{A'}$ is a polynomial of $A'$, for all $f \in \text{Pol} A$. Replacing some variables by “constants” (i.e., by $s_0, \ldots, s_{m-1}$) in a polynomial creates a new polynomial. Therefore we have $\subseteq$ in (6.14.2). To get the reverse inclusion, let $g^{(T)}|_{A'} (g \in \text{Clo} A)$ be any $k+n$-ary term operation of $A'$, and let $w_0, \ldots, w_{k-1} \in A'$. It must be shown that where

\[ h(x_0, \ldots, x_{n-1}) = g^{(T)}|_{A'}(w_0, \ldots, w_{k-1}, x_0, \ldots, x_{n-1}), \]

$h(x_0, \ldots, x_{n-1})$ can be expressed in the form claimed in (6.14.2). By an obvious extension of (1), there exist (for some $m$), $s_0, \ldots, s_{m-1} \in S'$ and $g_0, \ldots, g_{k-1} \in \text{Pol}_mA$ with $w_i = g_i^{(T)}(s_0, \ldots, s_{m-1})$ for $0 \leq i < k$. We define

\[ f(y_0, \ldots, y_{m-1}, x_0, \ldots, x_{n-1}) = g_0(y_0), \ldots, g_{k-1}(y_{k-1}, x_0, \ldots, x_{n-1}). \]

Then $f \in \text{Pol}_{m+n}A$, and it is easy to check that

\[ f^{(T)}|_{A'}(s_0, \ldots, s_{m-1}, x_0, \ldots, x_{n-1}) = h(x_0, \ldots, x_{n-1}). \]
This finishes our proof of (6.14.2).

The proof of (3) is immediate from (6.14.1) and (6.14.2), using the trick of composing a polynomial with $e$, that was employed in the proof of (2).

**LEMMA 6.15.** Assume that $A$ is an algebra; that $e \in E(A)$, $U = e(A)$, $\beta \in \text{Con } A$; and that $S$ is a $\beta |U$-equivalence class. Let $\theta$ be any congruence of $A$ satisfying $\theta \leq \beta$. Define $A' = A/\theta$; $e' = e/\theta$; $U' = e'(A')$; $\beta' = \beta/\theta$; and $S' = S/\theta$. Then $e' \in E(A')$; $\beta' \in \text{Con } A'$; $S'$ is a $\beta' |U'$-equivalence class; and $A'[S'] \cong (A[S])/(\theta[S]).$

**PROOF.** This lemma looks formidable, but is actually quite trivial to prove. The proof is left up to the reader.

**LEMMA 6.16.** Suppose that $A$ is a finite algebra; that $e \in E(A)$, $U = e(A)$, $\beta \in \text{Con } A$; and that $S$ is a $\beta |U$-equivalence class. Then $\text{typ}(A[S]) \subseteq \text{typ}(0_A, \beta)$.

**PROOF.** Let $\langle \delta, \theta \rangle$ be any prime quotient of $A[S]$. By Lemma 2.4, restriction is a homomorphism of $I[0_A, \beta]$ onto $\text{Con } A[S]$. We can choose $\delta', \theta' \in \text{Con } A$ such that $\delta' \prec \delta \leq \beta$ and $\delta[S] = \delta$, $\theta[S] = \theta$. We then choose $\langle c, \delta \rangle \in S \cap (\theta - \delta)$. Notice that $e(c/\theta) \subseteq c/\theta$ and $e(c/\delta) \not\subseteq c/\delta$. Thus by Exercise 5.11 (3) (the version of Theorem 2.8 (6) adapted to traces) there is a $\langle \delta', \delta \rangle$-trace $N \subseteq c/\theta$ such that $e(N) = N_1$ is a $\langle \delta, \delta \rangle$-trace. Now $N_1 \subseteq U \cap c/\beta = S$.

We can assume that $c, d \in N_1$ (or choose new elements). There is $e_1 \in E(A)$ with $e_1(A) \subseteq c(A) = U$, $e_1(A) \in M_A(\delta, \delta')$, and $N_1$ a $\langle \delta, \delta' \rangle$-trace in $e_1(A)$. Clearly $e_1(N_1) = N_1$ and $e_1(S) \subseteq S$. Since $e_1(\theta) \not\subseteq \delta$, the range of the polynomial $e_1[S]$ contains a $\langle \delta, \delta' \rangle$-minimal set $V$. We have $V = e'(S)$ for some $e' \in E(A[S])$; and of course there exists $e_2 \in E(A)$ with $e_2(S) = e'$. We may assume that $e_2|\theta = e_2$. Clearly $e_2(\delta) \not\subseteq \delta$, and $e_2(A) \subseteq e_1(A)$, hence $e_2(A) = e_1(A)$. It follows from this that all elements of $N_1$ are fixed by $e_2$, and thus $N_1 \subseteq e_2(S) = V$. Now $c/\theta \cap V = M$ is a $\langle \delta, \delta \rangle$-trace in $V$, and $V \subseteq N_1$. But since $V \subseteq e_1(A)$, we have $M \subseteq c/\theta \cap e_1(A) = N_1$. So the set $M = N_1$ is both a $\langle \delta, \delta \rangle$-trace and a $\langle \delta, \delta \rangle$-trace. It is easy to prove that $A|[M] = (A[S])|[M]$; and clearly $\delta|M = \delta|[M]$. Hence $\text{typ}(\delta, \delta) = \text{typ}(A[M]/(\delta[M])) = \text{typ}(\delta, \delta)$.

**THEOREM 6.17.** Suppose that $A$ is an algebra; that $e \in E(A)$, $U = e(A)$, $\beta \in \text{Con } A$; and that $S$ is a $\beta |U$-equivalence class.

1. For every algebra $C \in V(A[S])$, there exist $A' \in V(A)$, $e' \in E(A')$, $U' = e'(A')$, $\beta' \in \text{Con } A'$, and a $\beta' |U'$-equivalence class $S'$ satisfying:
   (i) $A'[S'] \cong C[C]$.
   (ii) There exists a complete lattice homomorphism of $I[0_A', \beta']$ onto $\text{Con } C$.
   (iii) If $A$ and $C$ are finite, then $A'$ is finite.

2. If $A$ is finite, then $\text{typ}(V(A[S])) \subseteq \text{typ}(V(A))$. 
PROOF. When (1i) holds, then (1ii) follows, by Lemma 2.4 and Exercise 2.5 (1). To prove (1), let C ∈ V(A1 S). There exists S" ⊆ (A1 S)T for some set T, and a congruence θ on S" with C ≅ S"/θ. If A and C are finite, then we can take a finite T, by Theorem 0.2. Applying Lemma 6.14, we obtain A" ⊆ AT, c" ∈ E(A") etc. By Lemma 2.4, there exists θ" ∈ I[0, A", β"] with θ"|S" = θ. Applying Lemma 6.15 to the system (A", e", U", β", S", θ"), we obtain a system (A', e', U', β', S'). We have

A'|S" ≅ (A"|S")/(θ"|S") = (S"|S")/θ ≅ C|C.

This proves (1). Statement (2) follows from (1), by Lemma 6.16.

Theorem 6.17 has some interesting corollaries. In order to introduce them, we need one more lemma.

LEMMA 6.18. Let M be a finite minimal algebra of unary type. The variety V(IM) generated by M with the normal indexing contains, for every nonvoid set S, an algebra S = (S,...) such that S has only trivial polynomials, i.e., S|S = (S)|S.

PROOF. Let Λ = (Sym M) ∩ (Pol1 M). Every polynomial f(x0, ..., xn-1) of M is constant, or of the form f(x0, ..., xn-1) = σ(x_i) for some i < n and for some σ ∈ Λ. The set Λ is a subuniverse of the group Sym M.

Let S be any set of at least two elements. Choosing elements u ≠ v in M, let D be the subset of MS consisting of the constant functions and all the functions p(σ, s) (σ ∈ Λ, s ∈ S) defined by

p(σ, s)(s') = \begin{cases} σ(u) & \text{if } s' ∈ S - \{s\} \\ σ(v) & \text{if } s' = s \end{cases}

It is easy to check that D is a subuniverse of (IM)^S. The algebra D ⊆ (IM)^S whose universe is D has a congruence θ defined by

(x, y) ∈ θ ↔ (∃σ ∈ Λ)(σ(x(s)) = y(s) for all s ∈ S).

The algebra E = D/θ has at least |S| elements, and every term operation of E is either constant or a projection. This lemma follows from these considerations.

The variety of bounded distributive lattices is the variety generated by the algebra \{0, 1\}, V, ∧, 0, 1. The variety of bounded semilattices is the variety generated by the algebra \{0, 1\}, ∧, 0, 1. Recall that Πκ denotes the lattice of all equivalence relations on the cardinal κ (and that κ is a set of ordinal numbers). We define
Labeled Congruence Lattices

\[ \mathcal{L}_1 = \{ \Pi_\kappa : \kappa \text{ is any cardinal} \} ; \]
\[ \mathcal{L}_F = \{ \text{Con } V : V \text{ is an } F\text{-vector space} \} , \quad F \text{ is a field} ; \]
\[ \mathcal{L}_3 = \{ \text{Con } B : B \text{ is a Boolean algebra} \} ; \]
\[ \mathcal{L}_4 = \{ \text{Con } L : L \text{ is a bounded distributive lattice} \} ; \]
\[ \mathcal{L}_5 = \{ \text{Con } S : S \text{ is a bounded semilattice} \} . \]

**Theorem 6.19.** Let \( A \) be any finite indexed algebra.

1. If \( 1 \in \text{typ}(A) \), then for every \( L \in \mathcal{L}_1 \) there exists \( B \in V(A) \), and \( \theta \in \text{Con } B \), and a complete 0,1-separating homomorphism of \( I[0_B, \theta] \) onto \( L \). If \( L = \Pi_n \) for an integer \( n \geq 4 \), then \( B \) can be chosen to be finite and \( (0_B, \theta) \) to be tame of unary type.

2. If \( (\alpha, \beta) \) is tame in \( A \) of affine type with associated field \( F \), then for every \( L \in \mathcal{L}_F \) there exists \( B \in V(A) \), and \( \theta \in \text{Con } B \), and a complete 0,1-separating homomorphism of \( I[0_B, \theta] \) onto \( L \). If \( L = \text{Con } V \) for a finite vector space \( V \) of dimension \( \geq 1 \), then \( B \) can be chosen to be finite, and \( \theta \) can be chosen so that \( (0_B, \theta) \) is tame of affine type, and the associated minimal algebras \( B|_{N'} \) satisfy \( B|_{N'} \cong V|_{V'} \).

3. If \( 3, 4 \) or 5 belong to \( \text{typ}(A) \), then for every \( L \in \mathcal{L}_i \) (\( i = 3, 4 \) or 5, respectively), there exists \( B \in V(A) \), finite if \( L \) is finite, and \( \theta \in \text{Con } B \), and a complete 0,1-separating homomorphism of \( I[0_B, \theta] \) onto \( L \).

**Proof.** We begin with (1). Suppose that \( 1 \in \text{typ}(A) \). Replacing \( A \) by a homomorphic image, we can assume that \( 0_A < \beta \) in \( \text{Con } A, \text{typ}(0_A, \beta) = 1 \), \( U \in M_A(0_A, \beta), e \in E(A), e(A) = U \), and \( N \) is a \( (0_A, \beta) \)-trace in \( U \). The algebra \( N = A|_N = N|_N \) is minimal of unary type.

Let \( L = \Pi_\kappa \), the lattice of all equivalence relations over the cardinal number \( \kappa \).

By Lemma 6.18, there exists \( C \in V(N) \) with \( \text{Con } C \cong L \), and \( C \) is finite if \( \kappa \) is finite. By Theorem 6.17, there exists \( A' \in V(A) \) (with \( A' \) finite if \( C \) is finite), and \( \beta' \in \text{Con } A' \), and a complete lattice homomorphism \( \pi' : I[0_{A'}, \beta'] \to L \). For any such homomorphism, there exists a smallest \( \theta' \leq \beta' \) such that \( \pi'((0_{A'}) = \pi'((\beta')) \), and a largest \( \delta' \leq \theta' \) such that \( \pi'(\delta') = \pi'(0_{A'}) \). Replacing \( A' \) by \( B = A'/\delta' \), and using that \( I[\delta', \theta'] \cong I[0_B, \theta] \), where \( \theta = \theta'/\delta' \), we have a complete 0,1-separating homomorphism \( \pi : I[0_B, \theta] \to L \).

Now if \( L = \Pi_\kappa \), \( n \geq 4 \), then \( B \) is finite by construction. The interval \( I[0_B, \theta] \) is a tight lattice, by Lemma 1.10 and Example 1.12. By Theorem 2.11, \( (0_B, \theta) \) is tame. Now \( \Pi_\kappa \) is simple and nonmodular. Hence by Lemma 1.10, \( I[0_B, \theta] \) cannot
admit a 0,1-separating homomorphism onto the congruence lattice of a vector space. Therefore by Theorem 5.7, \((0_B, \theta)\) is of unary type (strongly Abelian). This concludes the proof of (1).

The proof of (2) is similar. We can assume to start with that \((0_A, \beta)\) is a prime quotient of affine type, and that \(N = A|_N = W|_W\) where \(W\) is a vector space over \(F\) of dimension 1. The steps followed before lead to \(B \in \mathcal{V}(A), \theta \in \text{Con } B, \) and \(S \subseteq B\) such that \(\pi = |_S : I[0_B, \theta] \rightarrow \text{Con } B|_S\) is complete and 0,1-separating, \(B|_S \cong V|_V,\) and \(L \cong \text{Con } V.\) Assuming that \(L\) is finite and \(\dim V > 1,\) then \(B\) is finite; and as before (by Example 1.13), \(I[0_B, \theta]\) is tight and \(L\) is its simple homomorphic image, unique up to isomorphism. By Theorem 5.7, the type of \((0_B, \theta)\) can only be 1 or 2. It is not 1, because \(B|_S\) is Mal’cev and \(S\) is contained in a \(\theta\)-class. Thus \((0_B, \theta)\) is of type 2. Letting \(N'\) be any \((0_B, \theta)\)-trace, we have that \(L' = \text{Con } B|_{N'}\) is a 0,1-separating simple homomorphic image of \(I[0_B, \theta];\) and so \(L \cong L'.\) Now, via coordinatization in projective geometry, the field of scalars and the dimension of a vector space can be recovered from its congruence lattice (or, as is more usual, from the lattice of subspaces). Therefore, we can conclude that \(B|_{N'} \cong V|_V.\)

The proof of (3) follows the same pattern, and is easier. Note that if \(L = \text{Con } Q\) where \(Q\) is a Boolean algebra, bounded distributive lattice, or bounded semilattice, and if \(L\) is finite, then \(Q\) is finite. \(\Box\)

We shall draw further corollaries from Theorem 6.17 after the next definition and lemma.

**DEFINITION 6.20.** A lattice \(L\) will be called **finitely projective** if \(L\) is finite and for each onto lattice homomorphism \(\varphi : L' \rightarrow L\) with \(L'\) finite, there exists a homomorphism \(\sigma : L \rightarrow L'\) satisfying \(\varphi \sigma = \text{id}_L.\)

Note that if \(L\) is finitely projective, and if \(\varphi : L' \rightarrow L\) with \(L'\) finite, then \(L'\) has a sublattice isomorphic to \(L.\) Finitely projective lattices are characterized in Exercise 6.23(14).

**LEMMA 6.21.** Each of the lattices \(N_3, D_1, D_2\) and \(M_n (n \geq 1)\) is finitely projective.

**Proof.** We prove this fact for \(D_1,\) and leave the remaining proofs to the reader. \(D_1\) is pictured in Figure 12. (See Remark 5.21.) Suppose that \(\varphi : L' \rightarrow D_1\) and \(L'\) is finite. Let 0 and 1 denote the least and the largest elements of \(D_1\) respectively. Let \(a = \bigvee \varphi^{-1}\{0\}\) and \(b = \bigwedge \varphi^{-1}\{1\}\). Let \(L''\) be the interval \(I[a, a \vee b]\) in \(L'\) and put \(\varphi'' = \varphi|_{L''}.\) If \(x \in L',\) then \(x'' = (a \vee x) \wedge (a \vee b)\) belongs to \(L''\) and \(\varphi''(x'') = \varphi(x).\) Therefore \(\varphi''\) maps \(L''\) onto \(L.\) Moreover, \(\varphi'' : L'' \rightarrow L\) is 0,1-separating.

Now let \(u, v, w\) be the three atoms of \(D_1,\) satisfying \(u \vee w = 1, (u \vee v) \wedge (u \vee w) = v.\) (See Figure 12.) Choose any \(u'', v'', w'' \in L''\) with
\[
\varphi''(u'') = u, \varphi''(v'') = v, \varphi''(w'') = w.
\]
Replacing \( u'' \) by \( (u'' \lor u'') \land (v'' \lor w'') \), we can be sure that

\[
(u'' \lor u'') \land (v'' \lor w'') = u''.
\]

Since \( \varphi((u'' \lor w'') = 1 \), we have \( u'' \lor w'' = a \lor b \). Similarly,

\[
u'' \land w'' = u'' \land w'' = v'' \land w'' = a.
\]

The relations

\[
(u'' \lor v'') \land w'' = a = u'' \land (v'' \lor w'')
\]

can be demonstrated in the same manner. One can now check that \( \{a, u'', v'', w'', u'' \lor v'', v'' \lor w'', a \lor b\} \) is a sublattice of \( L' \) isomorphic to \( D_1 \), and that the isomorphism provides the required map \( \sigma : L \to L' \).

By a finite subdirect power of an algebra \( A \) is meant an algebra \( B \) such that for some integer \( n \geq 1 \), \( B \subseteq A^n \) and the image of \( B \) under each of the coordinate projections from \( A^n \) is \( A \).

**THEOREM 6.22.** Let \( A \) be a finite indexed algebra, let \( \mathcal{K} \) be the class of all finite subdirect powers of \( A \), and let \( S(\text{CON} \mathcal{K}) \) be the class of all lattices isomorphic to a sublattice of \( \text{Con} B \) for some \( B \in \mathcal{K} \).

1. These statements are equivalent:
   1. \( 1 \in \text{typ} (\mathcal{K}) \);
   2. \( D_1 \in S(\text{CON} \mathcal{K}) \);
   3. Every finitely projective lattice belongs to \( S(\text{CON} \mathcal{K}) \).

2. \( D_2 \in S(\text{CON} \mathcal{K}) \) iff \( \text{typ} (\mathcal{K}) \cap \{1, 5\} \neq \emptyset \).

3. \( M_2 \in S(\text{CON} \mathcal{K}) \) iff \( \text{typ} (\mathcal{K}) \cap \{1, 2\} \neq \emptyset \).

**PROOF.** That (iiii) implies (ii) follows from the last lemma. That (iiii) implies (ii) follows from Lemma 6.4. Now suppose that (ii) holds; let \( L \) be any finitely projective lattice. It was proved by P. Pudlák and J. Tuma in [30] that every finite lattice is isomorphic to a sublattice of \( \Pi_n \) for some finite \( n \). We can thus assume that \( L \subseteq \Pi_n \) and \( n \geq 4 \). By Theorem 6.19 (1), there is \( B \in \mathcal{K} \), and an interval \( I[\alpha, \beta] \) in \( \text{Con} B \), and a homomorphism \( \varphi : I[\alpha, \beta] \to \Pi_n \). [Theorem 6.19 only gives a finite \( B \in V(A) \), but the proof, via Lemmas 6.14 and 6.15, actually produces an algebra which is a homomorphic image of a diagonal subalgebra of \( A^k \) for a finite \( k \). Thus, if we don't require \( \alpha = 0_B \), we can have \( B \in \mathcal{K} \).] Letting \( L' = \varphi^{-1}(L) \), since \( \varphi|_{L'} : L' \to L \) and \( L \) is finitely projective, there exists \( L'' \cong L \) with \( L'' \subseteq L' \subseteq \text{Con} B \). This finishes the proof that \( L \in S(\text{CON} \mathcal{K}) \), and the proof that (ii) implies (iiii).

That \( \text{typ} (\mathcal{K}) \cap \{1, 5\} \neq \emptyset \) if \( D_2 \in S(\text{CON} \mathcal{K}) \) follows directly from Lemma 6.3. For the other implication in statement (2), Theorem 5.27 (2) implies that \( D_2 \in \).
$S(\text{CON } \mathcal{K})$ if $5 \in \text{typ}\{\mathcal{K}\}$; and (1), just proved (combined with Lemma 6.21) implies that $D_2 \in S(\text{CON } \mathcal{K})$ if $1 \in \text{typ}\{\mathcal{K}\}$.

Statement (3) follows from Lemma 6.6 if $M_3 \subseteq S(\text{CON } \mathcal{K})$, and follows from (1) and Lemma 6.21 if $1 \in \text{typ}\{\mathcal{K}\}$. So assume that $2 \in \text{typ}\{\mathcal{K}\}$. Just as in the proof of (1), Theorem 6.19(2) gives the existence of $B \in \mathcal{K}$ and an interval $I[\alpha, \beta]$ in $\text{Con } B$ which has $M_n$ for some $n \geq 3$ (has, actually, the congruence lattice of a two-dimensional vector space over some finite field) as a homomorphic image. Since $M_3 \subseteq M_n$ and $M_3$ is finitely projective, we have that $M_3 \in S(\text{Con } B)$. 

A class of finite, similar, indexed algebras closed under the formation of subalgebras, homomorphic images, and products of two algebras at a time, is called a pseudo-variety. Statements (1), (2), and (3) of Theorem 6.22 are obviously valid for any pseudo-variety. (This follows from Theorem 6.22.) The subsequent chapters are focused mainly on locally finite varieties, not pseudo-varieties, but most of the results obtained will be valid for pseudo-varieties. Theorem 6.22 is a precursor of the full-scale type-omission theorems for locally finite varieties proved in Chapter 9.

**Exercises 6.23.** We construct a collection of examples which illustrate our theorems and destroy some plausible conjectures. We begin with several four-element algebras having the universe $A = \{0, 1, 0', 1'\}$ and the congruences and congruence lattice pictured in Figure 15.

![Figure 15](image)

The following operations will be used.

\[
\begin{array}{c|ccc}
 & 0 & 1 & 0' & 1' \\
\hline
s_1 & 0 & 1 & 0' & 1' \\
0 & 0 & 1 & 0' & 1' \\
1 & 1 & 0 & 1' & 0' \\
0' & 0' & 1' & 0' & 0' \\
1' & 1' & 0' & 1' & 0' \\
\end{array}
\quad
\begin{array}{c|ccc}
 & 0 & 1 & 0' & 1' \\
\hline
s_2 & 0 & 1 & 0' & 1' \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0' & 0 & 0 & 0' & 0' \\
1' & 0 & 0 & 0' & 0' \\
\end{array}
\quad
\begin{array}{c|ccc}
 & 0 & 1 & 0' & 1' \\
\hline
s_3 & 0 & 1 & 0' & 1' \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0' & 0 & 0 & 0' & 0' \\
1' & 0 & 0 & 0' & 0' \\
\end{array}
\]

![Figure 16](image)
Labeled Congruence Lattices

1. Assume that $\mathbf{A} = \langle \{0, 1, 0', 1'\}, \ldots \rangle$ (operations unknown) and that $\text{Con A} = \{0_\mathbf{A}, 1_\mathbf{A}, \alpha, \beta, \delta\}$ (as displayed in Figure 15). Prove:

   (i) $\mathbf{A}$ is $\langle \alpha, \beta \rangle$-minimal and $\{0, 1\}$ is the only $\langle \alpha, \beta \rangle$-trace.

   (ii) Let $t(x) = s_1(x, 0')$. Either $\text{E}(\mathbf{A}) = \{\text{constants}\} \cup \{\text{id}, t\}$ or $\text{E}(\mathbf{A}) = \{\text{constants}\} \cup \{\text{id}, t, u_t^1, u_t^2\}$.

   (iii) $M_{\mathbf{A}}(0, \alpha) = \{0', 1'\}$; and $\{0, 0', 1, 1'\}$ are the $\langle \alpha, \delta \rangle$-traces.

   (iv) Since $A/\delta = 0/\delta \cup 1/\delta$, $\text{Pol A}/\delta$ is at least as rich as $(\text{Pol A})_{\{0, 1\}}$; hence $\text{typ}(\alpha, \beta) \leq \text{typ}(\delta, 1)$ in the ordering of types pictured in Figure 10 (preceding Theorem 5.5).

   (v) If $\text{typ}(\alpha, \beta) = 2$ and $d(x, y, z)$ is pseudo-Mal'cev for $\langle \alpha, \beta \rangle$, then we must have $s_1(x, y) = d(x, 0, y)$.

2. Show that each of the operations $s_1, s_2, s_3, u_1, u_2$ preserves $\alpha, \beta$, and $\delta$. Show that $\langle A, s_1 \rangle = \langle 0_\mathbf{A}, 1_\mathbf{A}, \alpha, \beta, \delta \rangle$.

3. Let $A_1 = \langle A, s_1 \rangle$, $A_2 = \langle A, s_1, s_3 \rangle$, $A_3 = \langle A, s_1, u_1, u_2 \rangle$, and let $A_{i+3}$ ($i = 1, 2, 3$) be $A_i$ with the operation $s_2$ adjoined. By the last exercise, each of $A_1, \ldots, A_6$ has the pentagon of Figure 15 as congruence lattice. Prove that the type labelings for these algebras are:

   ![Figure 17](image)

   [Note: Using Lemma 6.2, all the type labels except $\text{typ}(\alpha, \beta)$ can be determined by computing the two-element algebras $A/\beta$ and $A/\delta$, and determining their types. In each of the algebras $A = A_i$, the $\langle \alpha, \beta \rangle$-trace algebra $A_{\{0, 1\}}$ has at least the operation of a group; thus $\text{typ}(\alpha, \beta) \in \{2, 3\}$]. The work is finished by proving that in the richest of these algebras, $A_6$, $\langle \alpha, \beta \rangle$ is Abelian. Define $\rho \subseteq A^3$ as $0', 1'}^3 \cup \{(x, y, z) \in \{0, 1\}^3 : s_1(x, y) = z\}$. Show that $\rho$ is admissible for $A_6$. From this, derive that $A_{6_{\{0, 1\}}}$ is Abelian.]
15, and in which
\[ \text{typ}(\alpha, \beta) = 1, \text{typ}(\beta, 1) = u, \text{typ}(\delta, 1) = v. \]

We construct several algebras on the base set \( B = \{0, 1, 0', 1', 1\} \), having the congruences and congruence lattice pictured in Figure 18.

The following operations will be used.

\[
\begin{array}{c|cccc}
 p_1 & 0 & 1 & 0' & 1' \\
\hline
 0 & 0 & 1 & 0' & 1' \\
 1 & 1 & 0 & 1' & 0' \\
 0' & 0' & 1' & 0' & 1' \\
 1' & 1' & 0' & 0' & 1' \\
 \hline
 Q_1 & 0 & 1 & 0' & 1' \\
\end{array}
\]

\[
\begin{array}{c|cccc}
 p_2 & 0 & 1 & 0' & 1' \\
\hline
 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 0 & 1 \\
 0' & 0' & 0 & 0 & 1 \\
 1' & 1' & 0 & 0 & 1 \\
 \hline
 Q_2 & 0 & 1 & 0' & 1' \\
\end{array}
\]

\[
\begin{array}{c|cccc}
 p_3 & 0 & 1 & 0' & 1' \\
\hline
 0 & 0 & 0' & 0' & 0' \\
 1 & 1 & 0' & 0' & 0' \\
 0' & 0' & 0' & 0' & 1 \\
 1' & 1' & 0' & 0' & 1 \\
 \hline
 Q_3 & 0 & 1 & 0' & 1' \\
\end{array}
\]

(5) Show that each operation \( p_1, \ldots, \sigma, \tau \) preserves the equivalence relations \( \alpha, \beta, \delta \) of Figure 18.

(6) For every \( \{u, v, w\} \subseteq \{2, 3, 4, 5\} \), such that \( \{v, w\} \subseteq \{3, 4, 5\} \), construct an algebra \( B = (B, \ldots) \), using a subset of the above defined set of operations, such that \( \text{Con} B = \{0_B, 1_B, \alpha, \beta, \delta\} \) and the type labeling of \( \text{Con} B \) is:
(7) Let $A = \langle \{0, 1\}, f_0, f_1, f_2 \rangle$, $B = \langle \{0, 1\}, g_0, g_1, g_2 \rangle$, where

\[
\begin{align*}
    f_0(x, y) &= g_0(x, y) = x \lor y, \\
    f_1(x, y) &= g_2(x, y) = x \land y, \\
    f_2(x, y) &= g_1(x, y) = 1.
\end{align*}
\]

Let $\eta_0$ and $\eta_1$ be the kernels of the two coordinate projections from $A \times B$, and let $\alpha$ be the equivalence relation on $A \times B$ with blocks $A \times B - \{(0, 0)\}$ and $\{(0, 0)\}$. Show that the labeled congruence lattice of $A \times B$ is

![Diagram](image)

**Figure 20**

Prove that if we replace $f_1, f_2, g_1, g_2$ by $f'_1(x) = 1 - x = g'_2(x)$ and $f'_2(x) = 1 = g'_1(x)$, then the congruence lattice of $A \times B$ is unchanged, and the type labels 5 remain unchanged, but the 4's become 3's.

The above exercises make it obvious that types of prime quotients not present in $A$ or in $B$ can appear in $A \times B$, or in subdirect products of $A$ and $B$. The same situation prevails with regard to the formation of subalgebras. It seems that the only valid type-conservation theorems involve homomorphic images, and the type set $\{1, 2\}$. [We have that $\text{typ}(A/\alpha) \subseteq \text{typ}(A)$ if $A$ is finite. Subalgebras and finite products of finite solvable algebras are solvable, as will be proved in Chapter 7.]

(8) Let $S$ be a finite simple algebra of at least three elements whose minimal sets are two-element sets, and such that $\text{typ}(S) \neq 1$. (Such an $S$ exists having any prescribed type.) Define $A = \langle S, F \rangle$, where $F$ consists of all $f \in \text{Pol} S$ such that $f$ is essentially unary or its range is a two-element set. Then $A$ is simple, $\text{typ}(A) = \text{typ}(S)$, and $A$ has no proper subalgebras. Let $n = |A|$, and define an equivalence relation $\theta$ on $A^n$ by: $(x, y) \in \theta \iff x = y$ or range $(x) \neq A \neq$.
range \( (y) \). Show that \( \theta \in \text{Con } A^n \) and \( A^n/\theta \) is a minimal algebra of unary type.

(9) Let \( B = \langle \{0,1\}, \ldots \rangle \) be any algebra with base set \( \{0,1\} \). Let \( F \) be the set of all operations \( f \) on \( A = \{0,1,2\} \) such that \( \{0,1\} \) is closed under \( f \) and \( f|_{\{0,1\}} \in \text{Pol } B \). Define \( A = \langle A, f(f \in F) \rangle \), show that \( A \) is a simple algebra of type \( 3 \), and that \( \langle \{0,1\}, f|_{\{0,1\}}(f \in F) \rangle \) is a subalgebra of \( A \) whose type equals the type of \( B \).

(10) Prove the claims in the paragraph following Definition 6.11 that concern the type set of a variety generated by a minimal algebra.

(11) Prove that if \( A \) is a finite group or ring, or any finite Mal’cev algebra, then \( \text{typ } \{A\} \subseteq \{2,3\} \).

(12) Let \( A \) be an indexed algebra. Show that \( (1) \Rightarrow (2) \Rightarrow (3) \):

1. For every subalgebra \( B \) of \( A^2 \), \( \text{Con } B \) satisfies \( \text{SD}(\vee) \).

2. For all \( \theta, \lambda \in \text{Con } A \), \( [\theta, \lambda] = \theta \land \lambda \).

3. \( \text{Con } A \) satisfies \( \text{SD}(\land) \).

(The semi-distributive laws \( \text{SD}(\vee) \) and \( \text{SD}(\land) \) are defined in Definition 5.18. The commutator \( [\theta, \lambda] \) is defined in Exercise 3.8\{3\}. The proof of \( (1) \Rightarrow (2) \) involves looking at an algebra \( B = A(\beta) \subseteq A^2 \) whose universe is \( \beta \), and three congruences, \( \eta_0 = \{(x,y) \in B^2 : x(0) = y(0)\} \), \( \eta_1 = \{(x,y) \in B^2 : x(1) = y(1)\} \), and \( \delta \), the congruence generated by \( \{(a,a), (\beta,b) : (a,b) \in \beta\} \). Here, \( \beta \) is any congruence of \( A \) such that \( [\beta,\beta] < \beta \).)

(13) For two quotients \( \langle x_i, y_i \rangle \) (\( i = 0,1 \)) of a lattice \( L \), we write \( \langle x_0, y_0 \rangle \nearrow \langle x_1, y_1 \rangle \) (or \( \langle x_1, y_1 \rangle \searrow \langle x_0, y_0 \rangle \)) iff \( y_0 \land x_1 = x_0 \land y_0 \lor x_1 = y_1 \). When this holds, we say that \( \langle x_1, y_1 \rangle \) is \textbf{projective} to \( \langle x_0, y_0 \rangle \) \textbf{in one step}. Prove that if \( L \) is modular and \( \langle x_0, y_0 \rangle \nearrow \langle x_1, y_1 \rangle \), then the maps \( x \mapsto x \land x_1 \) and \( y \mapsto y \lor y_0 \) are mutually inverse isomorphisms between the interval sublattices \( I[x_0, y_0] \) and \( I[x_1, y_1] \).

(14) It is known that a finite lattice \( L \) is \textbf{finitely projective} (Definition 6.20) iff \( L \) satisfies this condition (\( W \)): for all \( x, y, u, v, a, b \in L \), if \( x \land y = a \leq b = u \lor v \) then \( \{x, y, u, v\} \cap I[a,b] \neq \emptyset \). Try to prove this result of B.A. Davey and B. Sands [8]. (It is also known that a finite lattice is projective in the class of all lattices iff it satisfies (\( W \)) and is semi-distributive. This deep result is due to J. B. Nation [24].)