CHAPTER I  
INTRODUCTION TO GENERAL TOPOLOGY

The scope of the chapter is sufficiently clear from its title. Particular attention has been paid to compactness and there is also a thoroughgoing treatment of inverse mapping systems which come strongly to the fore in (II), and also in (VI, VII) in connection with the homology theory of topological spaces.

General references: The standard treatises and in addition: Alexandroff-Urysohn [a], Čech [g], Steenrod [a], Tukey [T], Wallace [a], Wallman [a].

§1. PRIMITIVE CONCEPTS

1. We introduce a few formal abbreviations:

\[ A \rightarrow B \text{ means "A implies B";} \]
\[ A \leftrightarrow B \text{ means "A is equivalent to B";} \]
\[ A \cong B \text{ means "A is isomorphic with B."} \]

We shall assume that the reader is familiar with the basic concepts of point sets. The null-set is designated by \( \emptyset \) and if \( X \) is a set, \( X = \emptyset \) signifies that \( X \) is empty. If \( X, Y \) are sets we write \( X \subset Y \) or \( Y \supset X \) for: "every element of \( X \) is an element of \( Y \)"; or: "\( X \) is a subset of \( Y \)". We shall also say of three sets \( X, Y, Z \) that "\( Y \text{ is between } X \text{ and } Z \)" whenever \( X \subset Y \subset Z \) or else \( X \supset Y \supset Z \).

The statement "\( x \) is an element of the set \( X \)" is written symbolically \( x \in X \) or \( X \ni x \). Frequently the different elements of a set \( X \) are denoted by the same letter \( x \) with additional affixes as: \( x^1, x_1, \ldots \), or say by \( x_a \) with complementary affixes as: \( x_{a1}, \ldots \). In that case the set will sometimes be designated by \( \{x\}, \{x_a\}, \ldots \). We shall also write \( X = \{x\}, X = \{x_a\}, \ldots \) whenever it is the intention to designate the different elements in the manner just stated. However, the symbol \( [ \] \) is too convenient to be reserved strictly for the preceding usage; deviations will be allowed but their meaning will generally be clear from the context.

Let \( \{X_a\} \) be a collection of sets which may or may not be distinct. Let particularly \( X_a = \{x\} \). Then the set of all the \( x_a \) for all \( a \) is called the union of the \( X_a \) designated by \( U_a X_a \). In this and similar symbols the subscript \( a \) will often be omitted, and we shall write \( U \) in place of \( U_a \), wherever the "\( a \)" is clear from the context. Similarly the set of all the elements which are in every \( X_a \) (i.e., common to all the \( X_a \)) is called the intersection of the \( X_a \) and denoted by \( \bigcap_a X_a \). If the number of \( X_a \) is finite, say consisting of the collection \( X_1, \ldots, X_r \) (\( r \) an integer), we also designate the union and intersection, respectively, by \( X_1 \cup X_2 \cup \cdots \cup X_r \) and \( X_1 \cap X_2 \cap \cdots \cap X_r \).

Given two sets \( X, Y \), the set of all the elements of \( X \) which are not in \( Y \) is called the complement of \( Y \) in \( X \), also the difference of \( X \) and \( Y \), and is denoted by \( X - Y \).
If $P$ is a property and $X = \{x\}$, the totality of all the elements $x$ which satisfy $P$ is denoted by $\{x \mid x$ has the property $P\}$. As an example of this notation, if $|x|$ is the set of all real numbers, the set of all those between $0, 1$ is denoted by $\{x \mid 0 < x < 1\}$.

Negation of any relation shall be indicated by a bar drawn through its symbol as in $Y \not\subseteq X$ ($Y$ not contained in $X$), $Y \neq X$ ($Y$ different from $X$), etc.

The sets $X_a$ are said to be disjoint whenever any two are disjoint ($X_a \cap X_b = \emptyset$, for $a \neq b$).

2. Transformations or functions. Let $X = \{x\}$, $Y = \{y\}$ be two sets and let $G$ be a subset of the set whose elements are the ordered pairs $(x, y)$. We suppose that $G$ has the following property: every element $x$ is found in precisely one pair $(x, y_x) \in G$. There results then an assignment to each $x \in X$ of a definite element $y_x \in Y$ and this assignment is known as a transformation of $X$ into $Y$ or function on $X$ to $Y$. The statement “$T$ is a transformation of $X$ into $Y$” will be generally written in one of the symbolic forms: “$T : X \rightarrow Y$,” “$T : x \rightarrow y_x$,” “$x \rightarrow y_x$ defines $T$.” The set $X$ is the range of $T$ and $y_x$ is the value of $T$ at $x$. The element $y_x$ is frequently designated by $Tx$, and called the transform or image of $x$ under $T$. The set $Y'$ of all the values $Tx$ for all $x \in X$ is a subset of $Y$ called the transform or image of $X$ under $T$, and we write $Y' = TX$. It may happen that $Y' = Y$, i.e., that every element $y$ occurs in some pair $(x, y) \in G$ (every $y$ is a $Tx$), in which case $T$ is said to transform $X$ onto $Y$.

The transformation $T$ is said to be univalent whenever $x \neq x' \rightarrow Tx \neq Tx'$. It is said to be one-one when it is both univalent and a transformation “onto.” That is to say, every $y$ occurs in one and only one pair $(x, y)$.

The set $G$ of the pairs $(x, y)$ serving to define $T$ is known as the graph of $T$.

Example. $x$ is a real variable and $X = Y = \{x\}$, while $T : x \rightarrow x^2$. Then $X' = TX \neq X$, so $T$ is a transformation of $X$ into $X$ but not onto $X$. Suppose now the same situation except that $x$ is a complex variable. This time $T$ is a transformation of $X$ onto $X$.

Let $X, Y, T$ be as before and let $Z \subseteq X$. Then if $x \in Z$ the assignment of $Tx$ to $x$ defines a transformation $T_z : Z \rightarrow Y$ denoted by $T \mid Z$. We also say that $T$ is an extension of $T \mid Z$ to $X$.

(2.1) Multi-valued transformations. Let the sets $X, Y, G$ be as before, except that this time $G$ is not subjected to any restrictions. The elements $y$ in any pair $(x, y) \in G$ in which $x$ occurs make up a set $Y_x \subseteq Y$, which may be $\emptyset$ (an automatic subset of every set). The assignment $T$ to any $x$ of the set $Y_x$ is called a multi-valued transformation of $X$ into $Y$. The terms “value, image, transform” and designations “$Tx, TX,$” are carried over to multi-valued transformations. If it is known that every $Tx$ consists of $n$ elements, $T$ is sometimes said to be $n$-valued. The earlier transformations correspond to $n = 1$, and are sometimes designated as single-valued.

A multi-valued transformation $X \rightarrow Y$ may be considered as a (single-valued) transformation of $X$ into the set $Y'$ of all the subsets of $Y$.

Example. If $X$ is the set of all complex numbers then $x \rightarrow z^x$ is a multi-valued transformation $X \rightarrow X$, the values $Y_z$ being sets of $n$ complex numbers.
(2.2) Indexed system of sets. A multi-valued transformation \( T : X \rightarrow Y \) with \( Tx = Y_x \) is also called a system of sets indexed by \( X \), or more simply an indexed system, and denoted by \( \{Y_x\} \). It may be said that this designation will be used chiefly whenever \( X \) plays a minor role. As an example of an indexed system we may mention set sequences. We have then \( X = \{1, 2, \cdots, n, \cdots\} \), and the set sequence is here a system \( \{Y_n\} \) indexed by \( \{n\} \). Whenever every \( Y_n \) is a single point the set sequence becomes a point sequence or more simply a sequence.

(2.3) Inverse transformations, one-one transformations. Let \( T : X \rightarrow Y \) be single-valued or multi-valued, and set \( X_y = \{x \mid Tx = y\} \). Then \( y \rightarrow X_y \) defines a multi-valued transformation known as the inverse of \( T \) and denoted by \( T^{-1} \). Thus if \( x \) is a complex variable then \( T : x \rightarrow x'' \) is a transformation \( X \rightarrow X \) whose inverse is \( T^{-1} : x \rightarrow x'\prime, \) already considered above.

If both \( T \) and \( T^{-1} \) are single-valued \( T \) is one-one. In terms of the set \( G \) the transformation \( T \) is one-one whenever every \( x \) and every \( y \) each occur in a single pair \((x, y) \in G\).

(2.4) Identification. Let \( R \) be a relation of equivalence between the elements of a set \( X = \{x\} \) and let the resulting equivalence classes be taken as elements of a new set \( Y = \{y\} \). The set \( Y \) is said to be obtained from \( X \) by identification of the elements in each class \( y \).

There is an obvious connection between "identification" and "transformation." Indeed if we define \( T \) by \( Tx = \{y \mid x \sim y\} \) then \( T \) is a transformation \( X \rightarrow Y \). Conversely, if \( T \) is a transformation \( X \rightarrow Y \) and we define the relation \( R \) by "\( x \) and \( x' \) are in the relation \( R \) whenever \( x \) and \( x' \) are elements of the same set \( T^{-1} y\)," then \( R \) is a relation of equivalence and \( Y \) is derived from \( X \) by identification of the elements in each class.

Examples. (2.5) A "book" may be obtained from a collection of rectangles \( \{R_a\} \) by identification of points on a set of edges \( \{E_a\} \), one in each rectangle. Each equivalence class consists of the points at a specified distance from one vertex in each \( E_a \) or of a single point not on an \( E_a \).

(2.6) Let \( X \) consist of a circular region with its boundary circumference \( Z \) and let the relation \( R \) be defined as follows: each interior point is in the relation \( R \) with itself and itself alone; two end points \( x, x' \) of the same diameter are in the relation \( R \) with one another and with no other points. The resulting identification yields the projective plane. Similarly the Euclidean set \( x_1 + \cdots + x_n \leq 1 \) gives rise to projective \( n \)-space.

(2.7) Imbedding. Let \( T \) be a univalent transformation \( X \rightarrow Y \) and let \( X' = TX \). Then the process of replacing \( Y \) by \( (Y - X') \cup X \) is known as imbedding \( X \) in \( Y \).

3. Cartesian products.

(3.1) Definition. Let \( \{X_a\} \) be a system of sets indexed by \( A = \{a\} \), with \( X_a = \{x_a\} \). The cartesian product, or merely product of the \( X_a \) is the set of all the single-valued functions \( \xi(a) \) on \( A \) to \( \bigcup X_a \) such that \( \xi(a) \in X_a \) for every \( a \).

The product is denoted by \( \prod X_a \) or also by \( X_1 \times \cdots \times X_n \) when \( A = \{1, 2, \cdots, n\} \).
If the sets $X_a$ are merely the same set $X$ repeated we also write the product as a power: $X^a$.

**Examples.** (3.2) Take two disjoint sets $X_1 = \{x_1\}$, $X_2 = \{x_2\}$. Then $X_1 \times X_2$ is in one-one correspondence with the collection of all the pairs $(x_1, x_2)$, $x_i \in X_i$. Similarly if $X_i = \{x_i\}$, $i = 1, 2, \ldots, n$, are disjoint sets then $X_1 \times \cdots \times X_n$ is the collection of all the sets $(x_1, \cdots, x_i)$, $x_i \in X_i$.

(3.3) Let $X_1 = X_2 = X = \{x\}$. Then the product $X \times X$, also written $X^2$, is in one-one correspondence with the set of all ordered pairs $(x', x'')$, $x'$ and $x'' \in X$. Here then $(x', x'') = (x'', x')$ when and only when $x' = x''$. Similarly $X \times \cdots \times X$ ($r$ factors), written also $X^r$, is in one-one correspondence with the set of all ordered $r$-uples $(x^1, \cdots, x^r)$ of elements of $X$.

(3.4) **Application to functions.** By a function $f$ on the sets $X_a$, or of the variables $x_a$, to a set $Y$, is meant a function on $PX_a$ to $Y$. If there are only a finite number of $x_a$, say $x_1, \cdots, x_t$, $f$ is often designated by $f(x_1, \cdots, x_t)$.

(3.5) **Graphs.** Let $X$, $Y$ designate the sets of points $x$, $y$ on two cartesian axes $x'Ox$, $y'Oy$ in an Euclidean plane $\pi$. Then the points of $\pi$ are in one-one correspondence with the pairs of coordinates $x$, $y$, i.e., with the elements of $X \times Y$. With a function $f$ on $X$ to $Y$ there is associated the set $G$ of all points $(x, f(x))$, in which we recognize the graph of $f$ in the sense of (2).

By interchanging $x$, $y$ and $X$, $Y$ throughout, $G$ may be viewed as the graph of $f^{-1}$. If $f$ is single-valued every vertical meets $G$ in a single point.

This simple configuration is so effective that its terminology has been increasingly borrowed. Explicitly, given the product $PX_a$ and an element $x$ in the product, we call projection of $x$ on $X_a$, or $a$th coordinate of $x$, its value $x(a)$ at $a$. In the case of a product of two factors $X \times Y$, we call $x$ and $y$ the horizontal and vertical projections of the point $(x, y)$ of the product. The horizontal, vertical, $a$th projection of a set in $X \times Y$ or $PX_a$ is the aggregate of those of its elements.

(3.6) **Other products.** Interesting generalizations of the cartesian product may be obtained. For example, we may take as elements all the unordered pairs $(x^1, x^2)$ of elements of $X_a$, thus obtaining the symmetric product of $X$ by itself. Similarly the unordered sets of $r$ elements give the symmetric product of $X$ by itself $r$ times.

4. **Partially ordered and directed systems.** A set $A$ is said to be partially ordered or merely ordered, if certain pairs of elements $(a, b)$ of $A$ satisfy an ordering relation denoted by $a < b$ and subjected to the sole condition of transitivity: $a < b$ and $b < c \Rightarrow a < c$. Instead of $a < b$ we also write $b > a$. The ordering is said to be: reflexive if $a < a$ for every $a \in A$, proper if $a < a'$ and $a' < a \Rightarrow a = a'$. The set $A$ is said to be simply ordered whenever every pair of elements $a, b$ are ordered: one of $a < b$ or $b < a$ or both must hold.

Let $A$ be ordered by $\leq$. Then $A$ is said to be directed by $>$ [by $<$] whenever given any two elements $a, b$ of $A$ there exists a third $c$ such that $c > a$ and $c > b$ [$c < a$ and $c < b$]. We also write accordingly $A = \{a; >\}$ [$A = \{a; <\}$].

**Examples.** (4.1) $A$ is the set of all real numbers and $a < b \iff a \leq b$. This set is simply ordered and directed both by $<$ and $>$. (4.2) $A$ is the Euclidean plane referred to the coordinates $(x, y)$ and $(x, y) < (x', y')$ means that $x = x', y \leq y'$. This set is ordered but not simply ordered and not directed.

(4.3) $A$ consists of all the subsets of a given set $E$ and $a \leq b$ when $a \subseteq b$. This system is directed by $>$ and not simply ordered. Its ordering will occur frequently and is sometimes called ordering by inclusion.
A subset \( A' \) of a directed set \( A = \{a; >\} = \{a; <\} \) is said to be cofinal [cofinal] in \( A \) whenever for every \( a \in A \) there is an \( a' \in A' \) such that \( a < a' [a > a'] \). Thus if \( A = \{a_n\} \) is a monotone numerical sequence, then any subsequence \( A' \) is cofinal in \( A \). In point of fact, cofinal systems play in many respects a role analogous to that of subsequences.

In a partially ordered system \( A \) the subset \( A' \) is said to have \( a_0 \) for an upper (lower) bound whenever \( a < a_0 [a > a_0] \) for every \( a \in A' \). The element \( a_0 \) is said to be maximal for \( A \) if \( a > a_0 \Rightarrow a_0 > a \). If the ordering is proper then this definition specializes to the usual one: \( a_0 \) is maximal if no \( a \neq a_0 \) is such that \( a > a_0 \).

(4.4) If \( \{\lambda; >\} \) is a countable directed system, either it contains a maximal element or it contains a cofinal simply ordered sequence.

Let \( \{\lambda\} = \{\lambda_1, \lambda_2, \cdots\} \). Choose \( \lambda'_1 = \lambda_1 \), and choose \( \lambda'_s \) so that \( \lambda'_s > \lambda_s \), \( \lambda'_{s-1} \geq \lambda'_{s} \). If such a choice is impossible at the \( n \)th step, then \( \lambda'_{n-1} \) is a maximal element. If the choice can always be carried out, then \( \{\lambda'_s\} \) is a sequence cofinal in \( \{\lambda\} \).

5. Zorn's theorem. We now introduce a theorem which will be used in a number of proofs. It is logically equivalent to the well-ordering postulate, but in a form which can be used to replace arguments based on well-ordering, particularly transfinite induction, by a simpler procedure.

We give three statements of the theorem, which are easily proved equivalent.

(5.1) Theorem of Zorn. If in a partially ordered system \( A \) each simply ordered subset has an upper bound in the system, then there exists at least one maximal element \( a \in A \), with \( a > a_0 \) for a preassigned \( a_0 \).

(a) A property \( P \) of sets is said to have finite character if whenever it holds for every finite subset of a set \( X \) it also holds for \( X \) itself, and conversely.

(b) If a property \( P \) of some subsets of a set \( X \) has finite character then there exists at least one subset \( Y \) of \( X \) with property \( P \) such that any subset containing \( Y \) which has property \( P \) is equal to \( Y \).

(c) Every partially ordered system contains at least one maximal simply ordered subset; that is, a subset \( B \) which cannot be extended in simple order by an element greater than or less than all elements of \( B \).

This last form of the theorem is perhaps most intuitive; but (b) brings out more clearly the basis for the proofs making use of Zorn's theorem, since the properties involved usually are first defined for finite subsets of some set and then extended.

In the formulation (a) and for the subsets of a given set ordered by inclusion the theorem was given by R. L. Moore [M, 84] but the first general formulation, and particularly its usage as a substitute for transfinite induction are due to Zorn.

§2. TOPOLOGICAL SPACES

6. We shall understand by topological space \( \mathbb{R} \) an aggregate of elements, the points of \( \mathbb{R} \), and an aggregate \( U \) of subsets, the open sets of \( \mathbb{R} \), which satisfy the following axioms:

OS1. The null set and \( \mathbb{R} \) itself are open.

OS2. The union of any number of open sets is open.
OS3. The intersection of two (and hence of any finite number of) open sets is open.

Although a space, as we have defined the concept, is made up of points, the points themselves will not be used in a major way until (§6) is reached.

(6.1) Open base. An open base, or merely a base, for \( R \) is an aggregate \( \{ W_a \} \) of open sets \((\neq \emptyset)\) of \( R \) such that every open set of \( R \) is a union of these basic open sets. The empty set is understood to be a union of an empty aggregate of sets.

If we wish to make a set \( R \) into a space by choosing an aggregate \( \{ W_a \} \) of its subsets as a base, we will need

(6.2) \( \{ W_a \} \) is a base for a topological space if and only if: (a) \( R \) is a union of \( W_a \)'s; (b) the intersection of every two \( W_a \)'s is a union of \( W_a \)'s.

The necessity of these conditions is clear from OS123. If \( R \) is a union of \( W_a \)'s, then OS1 will be satisfied. OS2 is automatically satisfied. The intersection of two unions of \( W_a \)'s is the union of intersections of pairs of \( W_a \)'s; hence if the intersections of pairs of \( W_a \)'s are unions of \( W_a \)'s, then the intersection of two unions of \( W_a \)'s is a union of \( W_a \)'s and OS3 holds.

In the applications it is more convenient to replace (6.2) by the equivalent condition:

(6.3) \( \{ W_a \} \) is a base whenever: (a) every point \( x \) is in some \( W_a \); (b) if \( x \in W_a \cap W_b \) there is a \( W_c \) such that \( x \in W_c \subset W_a \cap W_b \).

Two bases \( \{ W_a \} \), \( \{ W'_b \} \) are said to be equivalent if they are bases for the same topological space. The condition for this is that every \( W_a \) is a union of \( W_b \)'s, and conversely. Or more conveniently in terms of points: if \( x \in W_a \) there is a \( W_b \) such that \( x \in W_b \subset W_a \) and likewise with \( W_a \), \( W_b \) interchanged.

(6.4) Subbase. An aggregate \( \{ W_a \} \) of open sets of \( R \) such that their finite intersections constitute a base is known as a subbase for \( R \). If a space has a subbase it is necessarily topological.

(6.5) Base and subbase at a point. Let \( x \) be a point of \( R \). An aggregate \( \{ W_a \} \) of open sets, all containing \( x \), is a base at \( x \), whenever if \( U \) any open set containing \( x \) there is a set \( W_a \) such that \( x \in W_a \subset U \). An aggregate \( \{ W_a \} \) is a subbase at \( x \) whenever the finite intersections of its sets constitute a base at \( x \).

(6.6) Countable bases. The presence of countable bases is often an important property of a space. In this connection we must mention the two well known axioms due to Hausdorff:

First countability axiom. There is a countable base at each point of \( R \).

Second countability axiom. The space \( R \) has a countable base.

Clearly the second implies the first.

In connection with countable bases we have also the classical:

(6.7) Theorem of Lindelöf. If \( R \) has a countable base and \( V = \bigcup V_a \), where the \( V_a \) are open sets, then there is a countable subcollection \( \{ V_{a_n} \} \) of \( \{ V_a \} \) such that \( V = \bigcup V_{a_n} \).
Let \( \{W_n\} \) be a countable base. Every \( V_n \) is a union of sets \( W_n \). The totality of the sets \( W_n \) which are contained in some \( V_n \) is thus a subcollection \( \{W'_n\} \) of \( \{W_n\} \) and so necessarily countable. We have then \( V = \bigcup W'_n \). Now for each \( W_n \) there is a

\[ V_n \supset W'_n \]

and clearly \( |V_n| \) behaves as required.

From the theorem we deduce

(6.8) If \( \mathfrak{R} \) has a countable base \( \{W_n\} \) then every base \( \{V_n\} \) contains a countable subaggregate \( \{V_{nq}\} \) which is already a base.

By the theorem just proved out of the sets \( V_n \) whose union is \( W_p \) there may be selected a countable subcollection \( \{V'_{nq}\} \) whose union is again \( W_p \). Therefore \( \{V'_{mq}\} \) (all \( p, q \)) is a countable base.

7. Closed sets. The complement \( F = \mathfrak{R} - U \) of an open set is known as a closed set. The properties of closed sets are the duals of those of open sets; explicitly, the duals of OS123 are:

CS1. \( \mathfrak{R} \) and \( \emptyset \) are closed.

CS2. Any intersection of closed sets is closed.

CS3. The union of two (and hence of a finite number of) closed sets is closed.

Conversely, if we had the closed sets satisfying CS123 and defined the open sets as the complements of closed sets, then the open sets would satisfy OS123.

(7.1) Closed base. An aggregate \( \{F_n\} \) of closed sets of \( \mathfrak{R} \) is a closed base whenever every closed set is an intersection of basic closed sets. Clearly:

(7.2) \( \{F_n\} \) is a closed base for \( \mathfrak{R} \) if and only if \( \{\mathfrak{R} - F_n\} \) is an open base for the space.

8. Transformations between spaces. A single- or multi-valued transformation \( T \) is called open[closed] if it takes open sets of \( \mathfrak{R} \) onto open sets of \( T \mathfrak{R} \) [closed sets of \( \mathfrak{R} \) onto closed sets of \( T \mathfrak{R} \)]. Since the image of a union is the union of the images we have

(8.1) If \( \{U_n\} \) is a base for \( \mathfrak{R} \), then a transformation \( T \) of \( \mathfrak{R} \) onto \( \mathfrak{E} \) is open if and only if each \( TU_n \) is open in \( \mathfrak{E} \).

A continuous transformation or mapping is a transformation \( T \), whose inverse \( T^{-1} \) is open. An argument similar to that for (8.1) yields

(8.2) If \( \{V_n\} \) is a subbase for \( \mathfrak{E} \), then a transformation \( T \) of \( \mathfrak{R} \) into \( \mathfrak{E} \) is continuous if and only if each \( T^{-1}V_n \) is open in \( \mathfrak{R} \).

If both \( T \) and \( T^{-1} \) are single-valued and continuous, then \( T \) is called a topological transformation or a homeomorphism. Clearly:

(8.3) A one-one transformation \( T \) is topological if and only if both \( T \) and \( T^{-1} \) are open.

(8.4) A transformation of \( \mathfrak{R} \) into \( \mathfrak{E} \) is continuous if and only if its inverse is closed.

A formal description of topology may now be given:
(8.5) Definitions. A topological property of a topological space $R$ is a property of $R$ which remains invariant under topological transformations. Topology is the study of the topological properties of topological spaces.

Examples. (8.6) A rigid plane motion is a topological transformation. A folding over of the plane is a continuous transformation but it is not topological.

(8.7) Topological equivalence. The relation expressing that one space is the topological transform of another is evidently an equivalence, and is called topological equivalence.

9. Some examples of topological spaces.

(9.1) Euclidean spaces. Consider the set $X^*$ of all the ordered sets of $n$ real numbers $[x_1, \cdots, x_n]$. The subsets of $X$ defined by inequalities

$$a_i < x_i < b_i, \quad i = 1, 2, \cdots, n,$$

are known as $n$-intervals, written $I^n$, or merely intervals when $n = 1$. Since $[I^1]$ is immediately seen to verify (6.2) it may be chosen as a base in a topology for $X^*$. The resulting topological space, or any other topologically equivalent, is known as an Euclidean $n$-space, written $E^n$, also as a real line for $n = 1$. The open sets of $E^n$ are sometimes called regions.

Strictly speaking “Euclidean $n$-space” should be applied only to certain metric spaces described more accurately in (44.1). However, in this and other similar instances it will be generally more convenient to enlarge the meaning of a well known term in the above manner, rather than to have recourse to a more involved terminology.

Let $I$ be the base just defined for $E^n$ and let $B$ be the set of the rational $n$-intervals, i.e., corresponding to the $a_i, b_i$ all rational. If $x \in I^n$, there is an element of $B$ between $x$ and $I^n$; hence every $I^n$ is a union of sets of $B$. Moreover every element of $B$ is an $I^n$. Therefore $B$ may serve as a base for $E^n$. In other words $E^n$ has a countable base namely $B$.

(9.2) Let $R$ be any point set and let the open sets be defined as all the subsets of $R$ so that the points themselves are open. The verification of the axioms OS123 is now trivial. The topology thus affixed to $R$ is known as the discrete topology. Its chief function is to make statements for topological spaces valid for arbitrary point sets, it being always understood when this is done that the discrete topology is assigned to the set.

(9.3) Let $R$ be a set ordered by $\prec$. Define as an open set any subset $U$ such that $x \in U$ and $x < x' \rightarrow x' \in U$. Then the sets $U$ verify OS123. In fact OS3 is fulfilled in the stronger form:

OS3'. Any intersection of open sets is open.

$R$ is known as an ordered space. Let a set $F$ have the property that $x \in F$ and $x' < x \rightarrow x' \in F$. Then $R - F = U$ is open, and so $F$ is closed. Conversely, if $F$ is closed it has the property just considered. From this follows that the closed sets of $R$ satisfy the same axioms OS123 as the open sets of $R$. Noteworthy examples of ordered spaces are the complexes (III, 1). In their theory however the fact that they are topological spaces is not important.

(9.4) An interesting example of ordered space is the real line $L$: $- \infty < x < + \infty$, considered as a set ordered by $\leq$. The open sets are then the “rays” $a \leq x \prec \infty$, and the closed sets the rays $- \infty \leq x \leq a$. This topology is manifestly different from the customary topology of $L$ as an $E^1$.

10. Additional topological concepts. The new concepts to be introduced must of course be expressed directly or indirectly in terms of the primitive elements, the open sets.

(10.1) The interior of a set $A$, written $\text{Int} A$, is the open set which is the union
of all the open sets $\subset A$ (greatest open set contained in $A$). If $x \in \text{Int } A$, $x$ is said to be an interior point of $A$.

(10.2) The closure $\overline{A}$ of $A$ is the closed set which is the intersection of all the closed sets $\supset A$ (least closed set containing $A$).

(10.3) $A$ is dense in $\mathcal{R}$ if $\overline{A} = \mathcal{R}$.

(10.4) The boundary $\partial A$ of $A$ is the intersection of the closures of $A$ and its complement: $\partial A = \overline{A} \cap \mathcal{R} - A$.

(10.5) A neighborhood of $A$ is any open set containing $A$.

Many formal properties may be derived directly from the definitions. Thus:

(10.6) Interiors are open sets, and the interior of an open set $U$ is $U$ itself: $\text{Int } U = U$. Closures and boundaries are closed sets, and the closure of a closed set $F$ is $F$ itself: $\overline{F} = F$.

(10.7) $A$ is dense in $\mathcal{R} \iff A \cap U_n \neq \emptyset$ for every set $U_n$ of a base $\{U_n\}$.

Noteworthy and readily proved properties of the closure are:

\begin{align*}
(10.8a) \quad & A \subset \overline{A}, \\
(10.8b) \quad & \emptyset = \emptyset, \\
(10.8c) \quad & \overline{A} = \overline{A}, \\
(10.8d) \quad & \overline{A \cup B} = \overline{A} \cup \overline{B}.
\end{align*}

It may be shown that if we take (10.8 abc) as axioms for a closure operator and define a set $F$ as closed by: $\overline{F} = F$ (10.6), then we obtain a collection $\{F\}$ satisfying CS123. Thus following F. Riesz and Kuratowski, one may describe topological spaces in terms of a suitably restricted closure operator.

Additional properties of the closure needed later will now be considered.

\begin{align*}
(10.9) \quad & \bigcap_{\mathcal{R}} A_n \subset \bigcap_{\mathcal{R}} A_n \quad \text{Int } U \cap A \supset U \text{ Int } A_n \\
(10.10) \quad & \mathcal{R} - \overline{A} = \text{Int } (\mathcal{R} - A).
\end{align*}

This last property may also be expressed as: the complement of $\overline{A}$ is the union of all the open sets which do not meet $A$. It leads to the following important property (the only one of the present set where "points" are mentioned):

(10.11) The closure $\overline{A}$ is the set of all the points $x$ such that every neighborhood of $x$ meets $A$.

For if $x \in \overline{A}$, no neighborhood $U$ of $x$ can be in $\mathcal{R} - A$ and so every such neighborhood meets $A$. On the other hand if this last property holds then $x \in \mathcal{R} - \overline{A}$, since otherwise $\mathcal{R} - \overline{A}$ would be a neighborhood of $x$ disjoint from $A$. Therefore $x \in \overline{A}$.

(10.12) Let $\mathcal{R}$, $\mathcal{E}$ be topological spaces and $T$ a mapping $\mathcal{R} \to \mathcal{E}$. Then if $A$ is any subset of $\mathcal{R}$ we have $T(\overline{A}) \subset \overline{T(A)}$.

For $T^{-1}(\overline{T(A)})$ is closed and $\supset A$, hence $T^{-1}(\overline{T(A)}) \supset \overline{A}$ which yields one of (10.12).

11. Topologization of subsets. Let $A$ be any subset of the space $\mathcal{R}$ and let $\{U_n\}$, $\{F_n\}$ be the aggregates of open sets and closed sets of $\mathcal{R}$. We see at
once that any of the properties OSi which hold for \( \{U_a\} \) also hold for \( \{A \cap U_a\} \), provided only that we replace \( A \) by \( A \) in their statement. The same is true for \( \{F_a\} \), OSi, and \( \{A \cap F_a\} \). This leads to adopting throughout the present work the rule

(11.1) **Principle of Relativization.** Any subset \( A \) of a topological space \( \mathbb{R} \) is turned into a topological space by choosing as its open sets the intersections with \( A \) of the open sets of \( \mathbb{R} \). In this statement “open sets” may be replaced by “closed sets.”

(11.2) **Example.** Under the principle of relativization the subsets of any Euclidean space are topological spaces.

Observe that \( B \) might well be closed in \( A \) but not closed in \( \mathbb{R} \). For example, let \( L \) be the real line. If \( A \) is the interval \( 0 < x < 1 \), and \( B \) the set \( 0 < x \leq 1/2 \), then \( B \) is closed in \( A \) but not in \( L \).

(11.3) **The closure in \( A \) of a subset of \( A \) is the intersection with \( A \) of its closure in \( \mathbb{R} \).

(11.4) **Application.** Let \( T \) be a mapping \( \mathbb{R} \rightarrow \mathbb{S} \) and let \( A \subset \mathbb{R} \). Then \( T \mid A \) is a mapping of \( A \) onto a subset \( B \) of \( \mathbb{S} \), for \( T \mid A \) is continuous in the relative topologies. In particular \( T \) is a mapping of \( \mathbb{R} \) onto its image \( T\mathbb{R} \).

12. **Topological products.** Let \( \{\mathbb{R}_a\} \) be a collection of topological spaces, and let \( \{U_{a,k}\} \) be the aggregate of open sets of \( \mathbb{R}_a \). We have already defined the set-product \( P\mathbb{R}_a \). We now agree to topologize it by choosing as a base the sets

\[ V = PU_{a,(k)} \]

where \( U_{a,(k)} = \mathbb{R}_a \) except for a finite set of \( a \)’s depending on \( V \). These sets might well be called “basic prisms.” It is easily seen that (6.2) is fulfilled and so the product is a topological space \( \mathbb{R} \).

We notice that \( \{U_a \times P\mathbb{R}_a\} \) is a subbase for \( \mathbb{R} \).

(12.1) **The projection** \( \pi_a : \mathbb{R} \rightarrow \mathbb{R}_a \) **is an open mapping. More generally if**

\[ \{a\} = \{b\} \cup \{c\}, \mathbb{R}' = P\mathbb{R}_b, \mathbb{R}'' = P\mathbb{R}_c, \text{so that } \mathbb{R} = \mathbb{R}' \times \mathbb{R}'', \text{then the projection } \pi : \mathbb{R} \rightarrow \mathbb{R}' \text{ is an open mapping.} \]

It is only necessary to consider the projection \( \pi_a \). If \( U_a \) is open in \( \mathbb{R}_a \) then \( V_a = \{x : x_a \in U_a\} = \pi_a^{-1}U_a \) is open in \( \mathbb{R} \), so \( \pi_a \) is a mapping. Since \( \pi_a V_a = U_a \), \( \pi_a \) is open by (8.1).

The following proposition is expressed in the form in which it usually occurs:

(12.2) **Let** \( \mathbb{S} = \{y\} \) **be a topological space and let** \( f_a(y) \) **be a continuous function on** \( \mathbb{S} \) **to** \( \mathbb{R}_a \). **If we set** \( x = (f_a(y)) = \phi(y) \), **then** \( \phi(y) \) **is likewise continuous on** \( \mathbb{S} \). **We may also say that** \( \phi \) **is a mapping:** \( \mathbb{S} \rightarrow \mathbb{R} \).

Since \( x_a = f_a(y) \) is continuous \( f_a^{-1}U_a = \phi^{-1}V_a \) is open and since \( \{V_a\} \) is a subbase for \( \mathbb{R} \), \( \phi \) is continuous (8.2).

(12.3) **Application to the continuity of functions of several variables.** To simplify matters consider a function of two variables \( f(x, x') \) with ranges \( \mathbb{R}, \mathbb{R}' \) and values in a space \( \mathbb{S} \). By definition \( f \) is merely a function of the point \( (x, x') \) of \( \mathbb{R} \times \mathbb{R}' \) i.e., with range \( \mathbb{R} \times \mathbb{R}' \), with values in \( \mathbb{S}; \) and \( f \) is said to be **continuous in both** \( x, x' \), when it is a continuous mapping \( \mathbb{R} \times \mathbb{R}' \rightarrow \mathbb{S} \). Let \( \{U\}, \{U'\} \) be the open sets of \( \mathbb{R}, \mathbb{R}' \). Since \( \{U \times U'\} \) is a base for \( \mathbb{R} \times \mathbb{R}' \) a n. a. s. c. for the function \( f \) to be continuous is that if \( f(x_0, x'_0) = y_0 \) and \( V \) is any neighborhood
of \( y_0 \) then there exists a neighborhood \( U \times U' \) of \((x_0, x'_0) \in \mathcal{R} \times \mathcal{R}' \) such that the values of \( f \) on \( U \times U' \) are in \( V \). This is the well known condition: There exist neighborhoods \( U, U' \) of \( x_0, x'_0 \) such that \( x \in U, x' \in U' \Rightarrow f(x, x') \in V \).

The extension to any number of variables is obvious.

(12.4) The graph \( G \) of a mapping \( T: \mathcal{R} \to \mathcal{R}' \) is topologically equivalent to \( \mathcal{R} \).

More precisely if \( \pi \) is the projection \( \mathcal{R} \times \mathcal{R}' \to \mathcal{R} \) then \( \pi \mid G \) is a topological mapping \( G \to \mathcal{R} \).

It is already known that \( \pi \mid G \) is one-one, and it is continuous since \( \pi \) is continuous. Therefore we only need to prove \( \pi \mid G \) open. If \( \{U\}, \{U'\} \) are the open sets of \( \mathcal{R}, \mathcal{R}' \) then \( \{U \times \mathcal{R}'\} \) and \( \{\mathcal{R} \times U'\} \) together form a subbase for \( \mathcal{R} \times \mathcal{R}' \). Hence \( \{G \cap \mathcal{R} \times \mathcal{R}'\} \) and \( \{\pi^{-1}\mathcal{R} \cap \mathcal{R} \times \mathcal{R}'\} \) together form a subbase for the subset \( G \). Now \( \pi(G \cap \mathcal{R} \times \mathcal{R}') = U, \pi(G \cap \mathcal{R} \times \mathcal{R}') = \mathcal{T} \) \( \mathcal{U}' \) is an open set of \( \mathcal{R} \), since \( T \) is continuous. Since \( \pi\mid G \) maps the elements of a subbase of \( G \) onto open sets it is open, and (12.4) follows.

(12.5) If \( \mathcal{R} = \mathcal{P}\mathcal{R}_a, A_n \subseteq \mathcal{R}_a, A = \mathcal{P}A_n \), then \( \bar{A} = \mathcal{P}\bar{A}_n \). Or, explicitly: the closure of a product is the product of the closures.

Since \( A \) is a product in order that \( x = \{x_n\} \in \mathcal{A} \) a n. a. s. c. is that every neighborhood of \( x \) meet \( A \), and hence that every set of the subbase \( \{U_n \times \mathcal{P}\lambda_n\mathcal{R}_b\} \) \( (U_n \) are the open sets of \( \mathcal{R}_b \) containing \( x \) meet \( A \). Hence the n.a.s.c. is: for every \( a \) every neighborhood of \( x_a \) must meet \( A_a \), or \( x_a \in \bar{A}_a \) for every \( a \), and this is (12.5).

If the \( A_n \) are closed then \( \bar{A}_n = A_n \), and hence \( \bar{A} = A \) or:

(12.6) A product of closed sets is a closed set.

This may also be proved directly as follows. If the \( A_n \) are closed then \( A = \bigcap A_n \), \( G_a = \mathcal{A} \times \mathcal{P}\mathcal{L}_n\mathcal{R}_b \). Since \( \mathcal{R}_a \) is open, \( \mathcal{L}_a \) is closed, and so is \( A \).

(12.7) An Euclidean \( n \)-space \( \mathbb{E}^n \) is the product of \( n \) real lines: \( L_1 \times \cdots \times L_n \).

If we replace in this product one factor say \( L_i \) by an interval \( \lambda \) of \( L_i \), there is obtained a strip "perpendicular" to \( L_i \), and the totality of these strips forms a subbase. Thus in the Euclidean plane the horizontal and vertical strips form a subbase.

(12.8) Parallelotopes, cells, spheres. We have already defined the interval as a subset \( a < x < b \) of the real line. Its closure \( a \leq x \leq b \) \( (a \neq b) \) is known as a segment. Let \( \lambda_1, \cdots, \lambda_n \) be intervals, and \( l_i = \lambda_i \), the corresponding segments. The product \( \mathcal{P}\lambda_i \) is an \( n \)-interval \( I^n \). The product \( \mathcal{P}\lambda_i \) is known as an \( n \)-parallelotope. The set \( S^{n-1} = \mathcal{P}\lambda_i = \mathbb{S}\mathcal{P}^n \) is called a topological \( (n - 1) \)-sphere. The topological zero-sphere consists of two points, the topological one-sphere of the perimeter of a rectangle.

The terms "parallelotope," "sphere" are also applied to any sets topologically equivalent to \( \mathcal{P}^n, S^{n-1} \). However, a set topologically equivalent to \( I^n \) is generally called an \( n \)-cell.

The number \( n \) for the \( n \)-parallelotope, \( n \)-cell or \( n \)-sphere is called its dimension.

For the present this designation is merely to be understood in a formal way. Later (VIII, 15), we shall identify \( n \) with the topological dimension.
Owing to their importance it is advisable to recognize paralleloptopes, cells and spheres even when they occur in a form unrelated to the products. Most models may be deduced from:

(12.9) A bounded convex region $\Omega$ (region with bounded coordinates) in an Euclidean $n$-space $E^n$ is an $n$-cell; its closure $\bar{\Omega}$ is an $n$-parallelotope and its boundary $\partial \Omega$ is a topological $(n - 1)$-sphere.

Let $x_i$ be running coordinates for the space and choose for $n$-cell the set:

$\Omega_0 : 0 < x_i < 1$. Take a point $a$ on $\Omega$ and a point $a_0$ on $\Omega_0$. Any ray $\lambda$ from $a$ meets the boundary $\partial \Omega$ in a single point $p$. Draw from $a_0$ a ray $\lambda_0$ parallel to $\lambda$ and in the same direction, and let it meet $\partial \Omega_0$ in $p_0$. Let $T$ be the transformation whereby a point $x_0$ dividing $a_0p_0$ in a given ratio between 0 and 1 goes into the point $x$ dividing $ap$ in the same ratio, while $a_0 \to a$, $p_0 \to p$. $T$ is manifestly a topological transformation, and since $T\Omega_0 = \Omega$, $T\partial \Omega_0 = \partial \Omega$, (12.9) follows.

**APPLICATION.** Let $E^n$ be referred to the coordinates $x_i$. Then the Euclidean spherical region

$$\sum x_i < 1$$

is an $n$-cell. Its closure, the set

$$\sum x_i \leq 1,$$

is an $n$-parallelotope. The boundary, the Euclidean $(n - 1)$-sphere, is a topological $(n - 1)$-sphere. This is, of course, the justification for the term "sphere."

(12.10) Let $\{l_n\}$ be a countable collection of segments. The product $P^n = \prod l_n$ is known as the **Hilbert paralleloptope**. If $P^n$ is parametrized by $0 \leq x_i \leq 1$, then the "strips" defined by one condition of the form $a_0 < x_i < b_0$, $0 \leq x_i < b_n$, $a_0 < x_n \leq 1$, make up a subbase for $P^n$.

Here again strictly speaking the Hilbert paralleloptope as we have defined it, is only a topological image of the set commonly designated by that name.

(12.11) Let $X$, $Y$ denote, respectively, the segments $0 \leq x \leq 1, 0 \leq y \leq 1$ and let $\{Y_s\}$ be a system indexed by $X$, where $Y_s = Y$. Then $\prod Y_s = Y^X$ is the set of all functions $f$ on $X$ to $Y$. Let $U_s$ be an interval of $Y_s$. Then the set $V_s = \{f | f(x) \in U_s\}$ is open in $Y^X$ and $\{V_s\}$ is a subbase for this space. The space thus obtained has many important properties of great interest in analysis. For instance, the subset $Q$ of $Y^X$ which represents the continuous functions on $[0 - 1]$ to $[0 - 1]$ is "very thinly spread" in $Y^X$.

The space $Y^X$ is also interesting as a special case of what we shall call later (25.2) a "compact paralleloptope."

13. **Topological identification.** We have seen (2.4) that a relation of equivalence $R$ between the elements of a set $X$ yields a new set $Y$ by identification of the elements in each equivalence class $y$. We also have an associated transformation $T$ of $X$ onto $Y$ whereby $Tx = y$. Suppose that $X$ is a topological space and let $Y$ be topologized by specifying $V \subseteq Y$ to be open whenever $T^{-1}V$ is open in $X$. Then OS123 are readily verified, and so $Y$ is a topological space. This space is said to be obtained from $X$ by topological identification.
EXAMPLES. (13.1) The set described as "projective plane" in (2.6) receives by
topological identification a definite topology and it is the set thus topologized which is referred
to henceforth as projective plane.
(13.2) Let \( X = \{x\} \) be the real line and let the relation \( R \) be defined by the condition:
x and \( x' \) are in the relation \( R \) whenever \( x = x' \mod 1 \) (\( x - x' \) is an integer). This is mani-
festly a relation of equivalence and topological identification yields the space \( Y \) referred
to as the real line \( \mod 1 \), or also the circumference.

(13.3) Topological imbedding. Let \( \mathcal{R} \) contain a set \( S \) such that there is a
topological mapping \( t: \mathcal{S} \to \mathcal{S} \). If we replace every point \( x \in \mathcal{S} \) by \( t^{-1}x \) both
in \( \mathcal{R} \) and in its open sets, we obtain what is known as a topological imbedding or
immersing of \( \mathcal{S} \) into \( \mathcal{R} \). It is an imbedding in the sense of (2.7) since it replaces
\( \mathcal{R} \) by \( (\mathcal{R} - \mathcal{S}) \cup \mathcal{S} \).

§3. AGGREGATES OF SETS. COVERINGS. DIMENSION

14. In view of the fundamental role of aggregates of sets and coverings it is
important to settle the nomenclature as rapidly as possible.

We shall be dealing with aggregates of subsets of a given space \( \mathcal{R} \). Let
\( \mathcal{A} = \{A_a\} \) be such an aggregate. The set \( \{A_a\} \) of the closures of the \( A_a \) is
denoted by \( \mathcal{R} \). Given a second collection \( \mathcal{B} = \{B_\beta\} \) we shall write:
\( \mathcal{A} \cup \mathcal{B} \) or \( \mathcal{A} \vee \mathcal{B} \) = the union of \( \mathcal{A} \) and \( \mathcal{B} \);
\( \mathcal{A} \wedge \mathcal{B} = \{A_a \cap B_\beta\} \);
\( \mathcal{A} \times \mathcal{B} = \{A_a \times B_\beta\} \);
\( \mathcal{A} \bowtie \mathcal{B} = \) every \( A_a \) is in some \( B_\beta \); we say also that \( \mathcal{A} \) is a refinement of \( \mathcal{B} \) or
refines \( \mathcal{B} \).

As a special case of \( \mathcal{A} \wedge \mathcal{B} \) one of the collections, say \( \mathcal{A} \), may consist of a
single set \( A \) so that \( \mathcal{A} \wedge \mathcal{B} \) is now \( \{A \wedge B_\beta\} \).

The order of \( \mathcal{A} \) is the largest number \( p \) if one exists such that some \( p + 1 \) sets
of \( \mathcal{A} \) intersect; if \( p \) does not exist the order is said to be infinite.

The finiteness properties of the aggregates are important. We say that \( \mathcal{A} \) is:
point-finite whenever every point of \( \mathcal{R} \) belongs to at most a finite number
of \( A_a \);
neighborhood-finite whenever every point of \( \mathcal{R} \) has a neighborhood \( N \) which
meets at most a finite number of \( A_a \);
locally finite whenever every \( A_a \) meets at most a finite number of \( A_a \);
finitely covered by \( \mathcal{B} \) whenever \( \mathcal{A} \bowtie \mathcal{B} \) and every \( B_\beta \) contains at most a finite
number of \( A_a \).

Notice that when \( \mathcal{A}, \mathcal{B} \) are point-finite or neighborhood-finite so is \( \mathcal{A} \vee \mathcal{B} \).

Two aggregates \( \mathcal{A} = \{A_a\}, \mathcal{B} = \{B_\beta\} \) are said to be similar whenever there
may be established a one-one transformation \( \tau: \{\alpha\} \to \{\beta\} \) such that
\( A_n \cap \cdots \cap A_{n'} \neq \emptyset \Leftrightarrow B_\tau A_n \cap \cdots \cap B_\tau A_{n'} \neq \emptyset \). The transformation \( \mathcal{A} \to \mathcal{B} \)
defined by \( A_a \mapsto B_\tau A_a \) is known also as a similitude.

By a covering of \( \mathcal{R} \) is meant an aggregate \( \mathcal{K} \) whose sets \( A_a \) have \( \mathcal{R} \) for union:
\( \bigcup A_a = \mathcal{R} \) (every point \( x \) belongs to an \( A_a \)). An open [closed] covering is a
covering by open [closed] sets.
Let \( \mathcal{A} \) be a collection of coverings. We say that the subcollection \( \mathcal{A}_s \) is cofinal in \( \mathcal{A} \) whenever every \( \mathcal{A}_s \) has an \( \mathcal{A} \) refinement.

Notice that a neighborhood-finite covering \( \mathcal{A} \) may be characterized thus: There exists an open covering each of whose sets meet at most a finite number of sets of \( \mathcal{A} \).

15. Dimension. The general theory of dimension, of Menger and Urysohn, fully developed for separable metric spaces, has not reached very far beyond these spaces. An important reason is that several equivalent definitions, all natural and which agree for separable metric spaces, seem to part company for other, less simple, spaces. A full treatment of these questions is wholly outside the scope of the present treatise, and they will be touched upon here and there only in those phases of interest in algebraic topology. Let us say at all events, that while in the early definition of Menger-Urysohn the "local" point of view predominates, we shall adopt the definition, inspired by Lebesgue, in terms of the order of coverings, as it is most closely related to our general purpose.

(15.1) **Definition.** Let \( K = \{ \mathcal{A}_s \}_{s > \alpha} \) be a class of coverings of a topological space \( \mathcal{R} \), which is directed by refinement: \( \mathcal{A}_s > \mathcal{A}_t \iff \mathcal{A}_s \) refines \( \mathcal{A}_t \). For a given \( \mathcal{A}_s \), consider all the \( \mathcal{A}_s > \mathcal{A}_t \), and let \( n_s \) be the least order of all such \( \mathcal{A}_t \). The \( K \)-dimension of \( \mathcal{R} \) is \( \sup n_s \).

Among the noteworthy classes \( K \) are: all the finite open or all the finite closed coverings, all the point-finite, or neighborhood-finite open or closed coverings. If \( \mathcal{R} \) is topological then each of these has the property that \( \mathcal{A}_s \wedge \mathcal{A}_t \) refines both \( \mathcal{A}_s \) and \( \mathcal{A}_t \), and is in the class. Hence each may serve to define a dimension. We have thus the dimensions by finite open or closed coverings, \( \cdots \). The most generally utilized is the first, and it is to this dimension by finite open coverings that the term **dimension**, written \( \dim \mathcal{R} \), is applied in the sequel.

Little is known regarding the mutual relations between the various dimensions and there are few, if any, very general properties. A simple property is:

(15.2) **If** \( F \) **is a closed set in** \( \mathcal{R} \) **then** \( \dim F \leq \dim \mathcal{R} \).

Any finite open covering of \( F \) is of the form \( \{ F \cap U \} \), where \( \mathcal{U} = \{ U \} \) is a finite collection of open sets of \( \mathcal{R} \). Since \( \{ U_i, \mathcal{R} - F \} \) is a finite covering of \( \mathcal{R} \), if \( \dim \mathcal{R} = n \), the covering has a refinement \( \mathcal{S} = \{ V_j \} \) whose order does not exceed \( n \). The sets \( \{ F \cap V_j \} \) are then a refinement of the covering \( \{ F \cap U_i \} \) of \( F \), whose order does not exceed \( n \). Therefore \( \dim F \leq n = \dim \mathcal{R} \).

§4. CONNECTEDNESS

16. There is perhaps no simpler intuitive property of a space than connectedness.

(16.1) **Definition.** A topological space \( \mathcal{R} \) is said to be connected when it is not the union of two non-void disjoint open sets.
If \( R = U \cup V \), where \( U \) and \( V \) are open and disjoint, then \( U \) and \( V \) are also closed, so that \( R \) is the union of two disjoint closed sets, and conversely. Therefore in the definition "open sets" may be replaced by "closed sets." Moreover the property of the definition is seen to be equivalent to the following: The null-set and \( R \) itself are the only subsets of \( R \) which are both open and closed.

Let \( A, B \) be two subsets of \( R \) which satisfy the so-called Hausdorff-Lennes separation condition:

\[
(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset.
\]

Explicitly neither set meets the closure of the other. We prove:

(16.3) Theorem. A n.a.s.c. for the connectedness of a subset \( C \) of \( R \) is that it admits of no decomposition \( C = A \cup B \) wherein \( A, B \) are not empty and satisfy the Hausdorff-Lennes condition.

This characteristic property is frequently taken as the definition of connectedness.

If \( C = A \cup B \) and (16.2) holds, then \( C \cap \bar{A} \subseteq C \setminus B \subseteq A \), and hence \( A \) is closed in \( C \). Similarly \( B \) is closed in \( C \), and hence \( B \) and \( A \) are open in \( C \) and \( C \) is not connected.

If \( C \) is not connected, then \( C = A \cup B \), where \( A \) and \( B \) are disjoint, non-empty, open and closed in \( C \); hence \( (\bar{A} \cap B) \cup (A \cap \bar{B}) = (C \cap \bar{A} \cap B) \cup (A \cap \bar{B} \cap \bar{B}) = (A \cap B) \cup (A \cap B) = A \cap B = \emptyset \). Thus the theorem is proved.

17. Connected aggregates. The definition of connectedness for aggregates of sets rests upon the simple properties of certain finite collections, the chains.

Let us call topological chain or merely chain a finite collection \( A = A_1, A_2, \ldots, A_r = A' \) such that consecutive sets of the collection intersect. The \( A_i \) are the links of the chain. The chain is said to join \( A \) to \( A' \), and if every \( A_i \) is member of a collection \( \mathcal{A} = \{A_i\} \), the chain is said to join \( A \) to \( A' \) in \( \mathcal{A} \), and it is called an \( \mathcal{A} \)-chain. In particular when \( r = 1 \) then \( A = A' \) and the chain consists of one link \( A \).

Let us take a particular set \( A \) in \( \mathcal{A} \) and let \( \mathcal{A}_1 \) be the subaggregate of \( \mathcal{A} \) consisting of all the sets which may be joined to \( A \) by a chain in \( \mathcal{A} \). The aggregate \( \mathcal{A}_1 \) is called a component of \( \mathcal{A} \). The following properties are immediate:

(17.1) The set \( A \) belongs to the component \( \mathcal{A}_1 \) which it serves to determine.
(17.2) The component determined by any set of \( \mathcal{A}_1 \) is \( \mathcal{A}_1 \) itself.

In other words the components are independent of the individual sets which serve to determine them and they depend on \( \mathcal{A} \) alone.

(17.3) Two components of \( \mathcal{A} \) with a common set \( A \) coincide, or equivalently: distinct components are disjoint.
(17.4) Each component of a locally finite aggregate is composed of a countable number of sets.

For we may then obtain, say \( \mathcal{A}_1 \) as follows: Take the sets \( A_i \) (finite in number) which meet \( A_1 \in \mathcal{A}_1 \), then the sets \( A_{i_1} \) (finite in number) which meet the sets \( A_{i_1} \), etc. The totality of the sets thus obtained is \( \mathcal{A}_1 \) and it is countable.
An aggregate $\mathfrak{A}$ consisting of a single component is said to be connected. It is characterized by the property that any two of its sets may be joined by an $\mathfrak{A}$-chain. This corresponds in every way to the intuitive concept of connectedness.

(17.5) Let $\mathfrak{A}, \mathfrak{B}$ be two aggregates such that $\mathfrak{A} > \mathfrak{B}$ and that every set of $\mathfrak{B}$ contains a set of $\mathfrak{A}$. Then if $\mathfrak{A}$ is connected so is $\mathfrak{B}$.

For let $B, B' \in \mathfrak{B}$. By assumption we have in $\mathfrak{A}$ two sets $A \subset B$ and $A' \subset B'$.

Since $\mathfrak{A}$ is connected there is a chain $A, A_1, \cdots, A_r, A'$ joining $A$ to $A'$ in $\mathfrak{A}$. Choose for each $A_i$ a set $B_i \supset A_i$. Then $B, B_1, \cdots, B_r, B'$ is a chain joining $B$ to $B'$ in $\mathfrak{B}$. Therefore $\mathfrak{B}$ is connected.

18. We now link up connectedness in sets and in aggregates by:

(18.1) A n.a.s.c. for connectedness of a space $\mathcal{R}$ is that all the coverings of any one of the following families be connected: (a) all the open coverings; (b) all the locally finite open coverings; (c) all the finite open coverings; (d) a family cofinal in any one of (a)–(c) (refinements as in 17.5).

The proof of (a, b, c) is the same, while combined with (17.5) they yield (d), so we merely consider (a). If the condition holds all the open coverings are connected. Therefore in every decomposition $\mathcal{R} = U \cup V$, $U$ and $V$ open, necessarily $U \cap V \neq \emptyset$, or $\mathcal{R}$ is connected. Thus the condition is sufficient.

To prove necessity let $\mathcal{R}$ be connected and let the open covering $\mathcal{U}$ be disconnected. $\mathcal{U}$ has then at least two components. Let $U$ be the open set which is the union of all the elements of one of the components, and $V$ the open set which is the union of the remaining elements. We then have $\mathcal{R} = U \cup V$, $U$ and $V$ are open and $U \cap V = \emptyset$. But this is ruled out since $\mathcal{R}$ is connected. This proves necessity and hence (18.1).

19. We shall utilize the result just proved to derive a certain number of simple properties of connected sets. Unless otherwise stated they are supposed to be subsets of a given topological space $\mathcal{R}$.

(19.1) A union of connected sets of which every pair intersect is itself connected (18.1, 17.1).

(19.2) Whenever in a sequence of connected sets $A_1, A_2, \cdots$ each meets the next one, their union is connected.

For $A_1, A_1 \cup A_2, A_1 \cup A_2 \cup A_3, \cdots$ are all connected and contain $A_1$. Hence their union which is $\cup A_1$ is connected.

(19.3) If $A$ is connected so is $\bar{A}$.

Let $\mathcal{U}$ be a covering of $\bar{A}$ without sets not meeting $\bar{A}$. Then $A \setminus \mathcal{U}$ is a covering of $A$. Since $A \setminus \mathcal{U} > \mathcal{U}$ these two aggregates are related as in (17.5). It follows that in the sequence of sets and aggregates of sets: $A, A \setminus \mathcal{U}$, $\mathcal{U}, \bar{A}$ the connectedness of each implies that of the following one. This proves (19.3).

(19.4) If $A, B$ are connected so is $A \times B$.

If $(x, y), (x, y') \in A \times B$, both are in the union of the connected sets $A \times y$, $x \times B$ with the common point $(x, y)$ and (19.4) follows.

(19.5) The image of a connected set under a mapping is connected.
For let $T$ map $A$ continuously onto $B$ and let $\mathcal{U} = \{U_a\}$ be an open covering of $B$. Then $\{T^{-1}U_a\}$ is an open covering of $A$ and hence connected. Therefore any two sets $T^{-1}U_a$ and $T^{-1}U_b$ may be joined by a chain of such sets. Since $T$ is single-valued, $TT^{-1}U_a = U_a$, and so the images of the links of this chain make up a $\mathcal{U}$-chain joining $U_a$ to $U_b$, hence $\mathcal{U}$ is connected and so is $B$.

20. From (19.1) follows that the union $C(x)$ of all the connected sets containing a given point $x$ is connected. If $y \neq x$ and $y \in C(x)$ then $C(y) = C(x)$. For otherwise their union would be a connected set containing $x$ and $\emptyset \subset C(x)$ which contradicts the definition of $C(x)$. Thus $C(x)$ is uniquely defined by any one of its points. The set $C(x)$ is called a component of $\mathfrak{N}$. We have similarly of course components of any subset of $\mathfrak{N}$.

Since $C(x)$ contains all the connected sets containing $x$ and is itself connected, it is the maximal connected set containing the point.

(20.1) The $n$-cell, $n$-parallelotope, Hilbert parallelotope, $n$-sphere ($n > 0$), are all connected.

We first show that a segment $l: 0 \leq t \leq 1$ is connected. For if it is not we have $l = A \cup B$, where $A, B$ are closed and disjoint. Let $0 \in A$ and set $a = \sup \{t \mid t \in A; t < B\}$. Since $A$ is closed $a \in A$. Furthermore whatever $\eta > 0$ there is a point of $B$ in the interval $a, a + \eta$. Hence $a \in B = B$, and so $A, B$ are not disjoint, contrary to assumption.

Since any two points of one of the sets in (20.1) can be joined by a "closed arc" (one-parallelotope, topological image of a segment), by the result just proved the sets are connected.

§5. COMPACT SPACES

21. It is not too much to say that all the spaces of chief interest in general topology, and even more so in algebraic topology, are compact spaces or their subsets. This is largely due to the fact that in dealing with compact spaces one may frequently replace infinite collections by finite collections.

We emphasize at the outset the following important departures from hitherto accepted terminology: (a) with Bourbaki we shall replace the term bicompact of Alexandroff-Urysohn by the term compact; (b) what has been known hitherto as compact (following Fréchet who introduced the concept) shall be called countably compact; (c) following Alexandroff-Hopf, a compact metric space shall be called a compactum. It is important that these modifications be kept in mind. The chief justification for adopting them, aside from convenience, are first that “bicompact metric” = “compact metric” and that “compact” (non-metric) spaces in the earlier sense occur but rarely.

22. (22.1) DEFINITION. A collection of sets is said to have the finite intersection property whenever every finite subcollection has a non-empty intersection.

(22.2) The following properties of a topological space are equivalent.

P1. If $\{U_a\}$ is any open covering of $\mathfrak{N}$, then some finite subcollection of $\{U_a\}$ is already a covering.

P2. If $\{F_a\}$ is a collection of closed sets with the finite intersection property, then the intersection of the whole collection is non-empty.
For P2 is equivalent to: if the sets of \( |F_a| \) have a void intersection the same holds for some finite subcollection, and this is the dual of P1, hence equivalent to P1.

On the strength of (22.2) we lay down the

(22.3) Definition. A topological space satisfying any one of P12 (and hence both) is said to be compact.

Notice that the concept of compactness is primitive in the sense that it may be expressed without reference to the other properties of open or closed sets.

In the applications it is convenient to have:

(22.4) If the \( U_a \) in P1 are restricted to a particular open base, or the \( F_a \) in P2 to a particular closed base, then we still have equivalent conditions.

Let in fact \( \mathcal{B} = \{ V_b \} \) be any covering of \( \mathcal{R} \) and suppose P1 to hold in the restricted manner. Let \( \mathcal{U}' = \{ U'_a \} \) be the set of the elements of the base which are contained in any \( V_b \). Since every \( V_b \) is a union of elements of \( \mathcal{U}' \), \( \mathcal{U}' \) is also a covering. By assumption it has a finite subcovering \( \{ U'_{a_1}, \ldots, U'_{a_n} \} \). Each \( U'_{a_i} \) is in some set \( V_i \) of \( \mathcal{B} \). Hence \( \{ V_i \} \) is a finite subcovering of \( \mathcal{B} \) and so P1 holds. The treatment of P2 is wholly similar and is omitted.

(22.5) Compactness of subsets. A subset \( A \) of a topological space \( \mathcal{R} \) is compact when and only when one of the following two equivalent properties holds:
(a) if \( \{ U_a \} \) is any open covering of \( A \) by open sets of \( \mathcal{R} \) then some finite subcollection of \( \{ U_a \} \) is already a covering; (b) if \( \{ F_a \} \) is a collection of closed sets of \( \mathcal{R} \) such that \( \{ A \cap F_a \} \) has the finite intersection property, then the sets \( F_a \) have a non-empty intersection which meets \( A \).

This is an immediate consequence of the principle of relativization (11.1).

23. (23.1) A closed subset \( F \) of a compact space \( \mathcal{R} \) is also compact.

For a collection \( \{ F_a \} \) of closed subsets of \( F \) with the finite intersection property is also a similar collection for \( \mathcal{R} \) itself, and so \( \cap F_a \neq \emptyset \), proving \( F \) compact.

(23.2) If a compact space \( \mathcal{R} \) is mapped onto a subset \( A \) of a topological space \( \mathcal{S} \) then \( A \) is compact.

Let \( \tau \) be the mapping and \( \{ F_a \} \) a collection of closed sets of \( A \) with the finite intersection property. Then \( \{ \tau^{-1} F_a \} \) is a similar collection for \( \mathcal{R} \). Hence

\[
\cap \tau^{-1} F_a \neq \emptyset
\]

and therefore \( \cap F_a \neq \emptyset \), proving \( A \) compact.

(23.3) The union of a finite number of closed subsets of a space is compact if and only if each subset is compact.

Let \( F = \cup F_a \). If \( \mathcal{F} \) is compact, then, since each \( F_a \) is a closed subset of \( F \), each \( F_a \) is compact. If each \( F_a \) is compact, consider any open covering of \( F \); each \( F_a \) is covered by a finite number of its sets (22.5a) and hence \( F \) is covered by a finite number of its sets. Thus P1 holds and \( F \) is compact.

(24.1) **Theorem.** *An arbitrary product of compact spaces is compact* (Tychonoff; proof after Bourbaki).

Let \( \mathcal{R} = \mathcal{P}(\mathcal{R}) \), where the \( \mathcal{R} \) are compact. Given in \( \mathcal{R} \) a family \( \mathcal{G} = \{F_\alpha\} \) of closed sets with the finite intersection property we must show that there is a point common to all the \( F_\alpha \).

By Zorn's theorem \( \mathcal{G} \) is contained in a family \( \mathcal{G} = \{G_\alpha\} \) of sets (not necessarily closed) with the finite intersection property and maximal relatively to this property. As a consequence: (a) any finite intersection of sets of \( \mathcal{G} \) is in \( \mathcal{G} \); (b) a set meeting every set of \( \mathcal{G} \) is in \( \mathcal{G} \).

Let \( \pi_\alpha \) be the projection \( \mathcal{R} \to \mathcal{R}_\alpha \) and set \( \mathcal{G}_\alpha = \{\pi_\alpha G_\beta\} \). Since \( \mathcal{G}_\alpha \) is a family of closed sets in \( \mathcal{R}_\alpha \) with the finite intersection property there is an \( x_\alpha \) common to all its sets. Let \( x = \{x_\alpha\} \) and let \( N \) be any neighborhood of \( x \). If \( \{N_\alpha\} \) are the neighborhoods of \( x_\alpha \) then \( \{\pi_\alpha^{-1} N_\alpha\} \) is a subbase at \( x \), and so for some finite set \( \lambda_1, \ldots, \lambda_4 \) we have: \( x \in \bigcap(\pi_\alpha^{-1} N_\lambda) \subseteq N \). Since \( x_\alpha \in \pi_\alpha G_\beta \), \( N_\lambda \), meets \( \pi_\alpha G_\beta \). Hence \( \pi_\alpha^{-1} N_\lambda \), meets \( G_\beta \), and so it is in \( \mathcal{G} \). It follows that \( \bigcap(\pi_\alpha^{-1} N_\lambda) \in \mathcal{G} \), hence \( N \in \mathcal{G} \). Therefore \( N \) meets \( F_\alpha \) and consequently \( x \in F_\alpha = F \). This proves the theorem.

(24.2) **If** \( \mathcal{R}, \mathcal{S} \) **are compact then every finite open covering \( \mathcal{B} = \{W_i\} \) of \( \mathcal{R} \times \mathcal{S} \) has a refinement \( \mathcal{U} \times \mathcal{B} \) where \( \mathcal{U} \) and \( \mathcal{B} \) are finite open coverings of \( \mathcal{R} \) and \( \mathcal{S} \).

Let \( \{U\}, \{V\} \) be the open sets of \( \mathcal{R}, \mathcal{S} \), so that \( \{U \times V\} \) is a base for \( \mathcal{R} \times \mathcal{S} \). For each \( (x, y) \in \mathcal{R} \times \mathcal{S} \) select a \( W \), \( (x, y) \), then a \( U^* \times V^* \) between \( (x, y) \) and \( W \). Thus \( \{U^* \times V^*\} \), \( y \) fixed, is a covering of the compact set \( \mathcal{R} \times y \), and so there is a finite subcovering \( \{U^{*,*} \times V^{*,*}\} \). If \( V^{**,*} = U^{**,*} \), then \( \{U^{*,*} \times V^{*,*}\} \) is a covering of \( \mathcal{R} \times V^{*,*} \). Since \( \{V^{*,*}\} \) is a covering of the compact set \( \mathcal{S} \) it has a finite subcovering \( \{V^*_i\} \). If \( U_i = \{U_i^*\} \) and \( U = U_i \wedge \cdots \wedge U_r \), then \( U \times \mathcal{B} \) is readily shown to behave as asserted.

25. Applications.

(25.1) Segments are compact.

Let \( l: 0 \leq x \leq 1 \), be a segment. It has for base all the sets \( V \) of the following types: \( 0 \leq x < a, a < x < b, b < x \leq 1 \), where \( a, b \) are rational and \( 0 < a, b < 1 \). Therefore by (22.4) we merely need to show that a covering \( \mathcal{B} \) by such sets has a finite subcovering. Since the set of all the \( V \)’s is countable so is \( \mathcal{B} \). Let then \( \mathcal{B} = \{V_\alpha\} \) and suppose that it has no finite subcovering. The sets \( W_\alpha = l - (V_1 \cup \cdots \cup V_\alpha) \) are never empty. Since \( W_{\alpha+1} \subseteq W_\alpha \), and each \( W_\alpha \) is a finite set of disjoint segments, there may be selected among these segments one, say \( l_\alpha \), such that \( l_{\alpha+1} \subseteq l_\alpha \) throughout. It follows then from elementary properties of the Dedekind cut that \( \cap l_\alpha \neq \emptyset \), and so it contains a point \( x \). Since \( x \in W_\alpha \) for every \( n \), it is contained in no \( V_\alpha \), which contradicts the fact that \( \mathcal{B} \) is a covering and (25.1) follows.

(25.2) **Let** \( \alpha \) **be any cardinal number and l a segment. Then l" is a compact set known as “compact parallelootope” (24.1, 25.1).**
(25.3) The $n$-parallelotope $P^n$ and the Hilbert parallelotope $P^*$ are compact.
(25.4) All spheres $S^n$ are compact.

For $S^n$ is closed in $P^{n+1}$, and so (25.4) follows from (23.1) and (25.2).

(25.5) Any power $C^n$ of a circumference $C$ is compact and is known as a "toroid" and $\alpha$ is called the "dimension" of the toroid (24.1).

(25.6) Every closed and bounded subset of an Euclidean space $\mathbb{E}^n$ is compact.

By "bounded subset of $\mathbb{E}^n$" we mean here a set $A$ such that the coordinates of its points are bounded. Since $A$ is in some $P^n$, and closed in $P^n$, (25.5) follows from (25.3) and (23.1).

(25.7) Real projective spaces are compact.

For the set $A$ of all points of $\mathbb{E}^n$ which satisfy

$$\sum x_i^2 \leq 1$$

is closed and bounded, and hence compact. Since a real projective $n$-space is the image of $A$ under a mapping it is likewise compact (23.2).

26. Compacting. This refers to the operation of imbedding topologically a space in a compact space. The basic result is the

(26.1) Theorem. Every topological space $\mathbb{E}$ may be mapped topologically onto a dense subset of a compact space $\mathbb{S}$ such that $\dim \mathbb{S} \leq \dim \mathbb{E}$ (Wallman [a]).

We will define the points, the closed sets and the open sets of $\mathbb{S}$, and for later purposes develop their properties somewhat beyond the immediate requirements of the theorem.

Notations. The points, open sets and closed sets of $\mathbb{E}$ are denoted by $x$, $U$, $F$, and the same for $\mathbb{S}$ by $X$, $U$, $F$.

The points of $\mathbb{S}$. Let $\varphi$ denote the union of a closed set of $\mathbb{E}$ and of a finite point set of $\mathbb{E}$. By a basic set is meant a collection $\xi = \{\varphi_a\}$ with the finite intersection property. By Zorn's theorem $\xi$ is contained in a similar collection which is maximal with respect to this property. Such a collection will be called a maximal basic set (= m.b.s.). The points $X$ of $\mathbb{S}$ are the m.b.s. and if $X = \{\varphi_a\}$ then the $\varphi_a$ are called the coordinates of $X$. As in (24) the maximality of $X$ implies that: (a) every finite intersection of coordinates of $X$ is a coordinate of $X$; (b) every set $\varphi$, and in particular, every set $\varphi$ meeting every coordinate of $X$ is also a coordinate.

Every $x \in \mathbb{E}$ is a set $\varphi$, and so it is a coordinate of at least one maximal basic set $X = \{\varphi_a\}$. Since $X$ is a basic set $x \cap \varphi_a \neq \emptyset$, hence $x \in \varphi_a$ and $\cap \varphi_a = x$. Suppose that $x$ is a coordinate of $X' = \{\varphi'_a\} \neq X$. Then some $\varphi'_b \notin X$. Since $x \in \varphi'_b$, $X$ may be augmented by $\varphi'_b$ without ceasing to be a basic set, which contradicts the assumption that $X$ is maximal. Therefore $X' = X$. Thus:

(26.2) Every point $x$ of $\mathbb{E}$ is a coordinate of a unique m.b.s. which is written $X(x)$. If $x \neq x'$ then $x \cap x' = \emptyset$, hence $x' \notin X$ and $X(x') \neq X(x)$. Therefore

(26.3) The transformation $T$: $x \rightarrow X(x)$ is univalent.
Let $X$ be a point which is not an $X(x) : X \in \mathcal{S} \rightarrow T\mathcal{R}$. If $\varphi \in X$ then $\varphi = f_a \cup x_1 \cup \cdots \cup x_r$. Since $X \neq T x_i$, none of the $x_i$ may belong to all the coordinates of $X$. Therefore for each $i$ there is a $\varphi_{a_i}$ which does not contain $x_i$. Hence $\varphi = \cap \varphi_{a_i}$ does not contain any $x_i$. Now $(\varphi \cap \varphi \cap \varphi \neq \emptyset) \Rightarrow (f_a \cap \varphi \cap \varphi \neq \emptyset)$ whatever the coordinate $\varphi$ of $X$. Therefore $f_a \cap \varphi \neq \emptyset$ and hence $f_a$ is a coordinate of $X$ also. Thus:

(26.4) If $X \in T\mathcal{R}$ has the coordinate $\varphi = f_a \cup x_1 \cup \cdots \cup x_r$, then the closed set $f_a$ is also a coordinate of $X$.

27. The closed sets of $\mathcal{E}$. Let $f$ be a closed set of $\mathcal{R}$ and $\Phi(f)$ the set of all the points $X$ with the coordinate $f$. We verify at once:

(27.1) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathcal{R}) = \mathcal{E}, \quad f \neq f' \iff \Phi(f) \neq \Phi(f')$;

(27.2) $\Phi(\cap f_a) = \cap \Phi(f_a)$

whenever $\{f_a\}$ is finite;

(27.3) $f \subset f' \Rightarrow \Phi(f) \subset \Phi(f')$.

Less obvious is the relation

(27.4) $\Phi(f \cup f') = \Phi(f) \cup \Phi(f')$.

Let $f_1 = f \cup f'$. Since $f_1 \supset f$, by (27.3): $\Phi(f) \subset \Phi(f_1)$, and similarly $\Phi(f') \subset \Phi(f_1)$. Therefore

(27.5) $\Phi(f) \cup \Phi(f') \subset \Phi(f \cup f')$.

Suppose now $X \in \Phi(f_1) - \Phi(f')$. Since $f_1$ is a coordinate of $X$ and $f'$ is not, there is a finite intersection $\varphi$ of coordinates of $X$ which include $f_1$, and such that $\varphi$ does not meet $f'$. Since $X$ is a maximal basic set, $\varphi$ itself must be a coordinate of $X$ and so $\varphi \subset f_1 - f' \subset f$, hence $X \in \Phi(f)$. This proves (27.5) with the inclusion reversed and (27.4) follows.

Referring to (6.2, 7.2) and by (27.1, 27.4) the collection $\{\Phi(f)\}$ may be chosen as a closed base for $\mathcal{E}$, turning it into a topological space. We shall show that $\mathcal{E}$ is compact. Since $\{\Phi(f)\}$ is a base, we merely need to show that if $\mathcal{F} = \{\Phi(f_a)\}$ has the finite intersection property then $\bigcap \Phi(f_a) \neq \emptyset$. Now by (27.2) when $\mathcal{F}$ has the finite intersection property so has $\{f_a\}$. That is to say, $\{f_a\}$ is a basic set. It follows that there is a m.b.s. $X$ with the $f_a$ as coordinates. Thus $X \in \Phi(f_a)$, $\bigcap \Phi(f_a) \neq \emptyset$, and $\mathcal{E}$ is compact.

The open sets of $\mathcal{E}$. Let $u = \mathcal{R} - f$ and set $\Omega(u) = \mathcal{E} - \Phi(f) = \text{an open set of } \mathcal{E}$. A point $X$ is in $\Omega(u)$ when and only when it does not have $f$ as a coordinate, or when and only when it has a coordinate $\varphi \subset u$. Since $\{\Phi(f)\}$ is a closed base for $\mathcal{E}$, $\{\Omega(u)\}$ is an open base. From (27.1), ..., (27.4) follows then by dualization:

(27.6) $\Omega(\emptyset) = \emptyset, \quad \Omega(\mathcal{R}) = \mathcal{E}, \quad u \neq u' \iff \Omega(u) \neq \Omega(u')$;

(27.7) $\Omega(u_a) = \cup \Omega(u_a)$,

whenever $\{u_a\}$ is finite;
Further properties of the $\Omega(u)$ follow.

(27.10) If $\mathcal{U} = \{u_1, \ldots, u_n\}$ is a finite open covering of $\mathcal{S}$ then $\Omega(\mathcal{U}) = \{\Omega(u)\}$ is one for $\mathcal{S}$ and in addition: (a) $\mathcal{U} < \mathcal{U}' \Rightarrow \Omega(\mathcal{U}) < \Omega(\mathcal{U}')$; (b) order $\Omega(\mathcal{U}) = \text{order } \mathcal{U}$; (c) $u_i \rightarrow \Omega(u_i)$ defines a similitude $\mathcal{U} \rightarrow \Omega(\mathcal{U})$.

The covering property is a consequence of (27.6) and (27.7), while (a), (b), (c) follow from (27.6, \ldots, 27.9).

(27.11) Any finite open covering $\mathcal{B}$ of $\mathcal{S}$ has a finite refinement $\Omega(\mathcal{B})$.

Since $|\Omega(u)|$ is a base for $\mathcal{S}$ there is a refinement $\{\Omega(u_i)\} = \Omega(\mathcal{U}_i)$ of $\mathcal{B}$.

28. All the elements for the proof of the compacting theorem are now at hand. We have already shown that $\mathcal{S}$ is compact and we have a univalent transformation $T: \mathcal{R} \rightarrow \mathcal{S}$. Let $T \mathcal{R} = \mathcal{R}$. If $x \in u$ then $X(x) = Tx$ has the coordinate $x$ in $u$, and so $X(x) \in \Omega(u)$. Conversely, $X(x) \in \Omega(u)$ implies that $X(x)$ has a coordinate $\varphi$ in $u$, and since $x \in \varphi$, likewise $x \in u$. Therefore $x \in u \iff X(x) \in \Omega(u) \cap \mathcal{R}$. It follows that $T$ induces a one-one transformation of the elements of the base $\{u\}$ for $\mathcal{R}$ into those of the base $\{\Omega(u) \cap \mathcal{R}\}$ for $\mathcal{R}$. Consequently $T$ imbeds $\mathcal{R}$ topologically as a subset $\mathcal{R}$ of $\mathcal{S}$.

Every $u \neq \emptyset$ contains at least one point $x$ of $\mathcal{R}$ and so $\Omega(u)$ contains $X(x) = Tx \in \mathcal{R}$. Therefore $\Omega(u)$ meets $\mathcal{R}$ and $\mathcal{S} - \mathcal{R}$ contains no $\Omega(u)$, hence no open set since $|\Omega(u)|$ is a base. It follows that $\mathcal{S} = \mathcal{R}$. Thus the imbedding is dense.

Let finally $\dim \mathcal{R} = n$ and let $\mathcal{B}$ be a finite open covering of $\mathcal{S}$. By (27.11) it has a finite refinement $\Omega(\mathcal{B})$, where $\mathcal{B}$ is a finite open covering of $\mathcal{S}$. Since $\dim \mathcal{R} = n$, $\mathcal{B}$ has a refinement $\mathcal{U}'$ of order not exceeding $n$, and $\Omega(\mathcal{U}')$ is a refinement of $\mathcal{B}$ of order likewise not exceeding $n$. Therefore $\dim \mathcal{S} \leq n$. This completes the proof of the theorem.

29. Locally compact spaces. The compacting process just given, while very general, usually provides a far more involved space than one would wish to have. Consider, for example, the interval $\lambda$: $0 < x < 1$, and let $f, f'$ be two infinite convergent sequences tending towards 0 or 1, but having no common terms. Each is the coordinate of a m.b.s., and the two m.b.s., say $X, X'$ thus obtained must be distinct since $f \cap f' = \emptyset$. Thus in the case under consideration the space $\mathcal{S}$ of (26.1) is such that $\mathcal{S} - \mathcal{R}$ contains at least as many points as there are disjoint sequences $\rightarrow 0, 1$. On the other hand if $C$ is a circumference and $y \in C, \lambda$ is topologically equivalent to $\mathcal{R} = C - y$, and so $C$ is a compacting space such that $C - \mathcal{R}$ is a point. This is a special case of a theorem which we shall now prove. First a

(29.1) Definition. A topological space $\mathcal{R}$ is said to be locally compact whenever every point $x$ of $\mathcal{R}$ has a neighborhood $\mathcal{N}$ whose closure $\overline{\mathcal{N}}$ is compact. Thus the interval, the real line, indeed any Euclidean space, are locally compact but not compact. They show that the locally compact class is very extensive.
(29.2) A n.a.s.c. for $\mathfrak{R}$ to be locally compact is the existence of an open base whose elements have compact closures.

Sufficiency is obvious. Suppose $\mathfrak{R}$ locally compact and let $\{U\}$ be a base and $\{V\}$ the open sets with compact closures. Since $U \cap V \subset V$, we have $\overline{U} \cap V \subset V$, and hence $\overline{U} \cap V$ is compact. If $W$ is any open set and $x \in W$, there is a $U$ between $x$ and $W$ and a $V \ni x$. Hence $U \cap V$ is an open set between $x$ and $W$ whose closure is compact, and so $\{U \cap V\}$ is a base whose sets have compact closures. This proves necessity and hence also (29.2).

(29.3) Theorem. A locally compact space $\mathfrak{R}$ may be compacted by the addition of a single point $y$.

Let $F$ denote the closed sets of $\mathfrak{R}$. Define the closed sets of $\mathfrak{R}' = \mathfrak{R} \cup y$ as all the sets $F \cup y$ and all the sets $F$ which are compact. The verification for $\mathfrak{R}'$ of the conditions CSi of (7) is a consequence of the same for $\mathfrak{R}$, and so $\mathfrak{R}'$ is a topological space. Since the closed sets of $\mathfrak{R}$ are the intersections with $\mathfrak{R}$ of those of $\mathfrak{R}'$, $\mathfrak{R}$ is topologically imbedded in $\mathfrak{R}'$. Let $\{f_n\}$ be a collection of closed sets of $\mathfrak{R}'$ possessing the finite intersection property. Separate the $f_n$ into two groups. The first made up of sets $f_n$ which are compact, and also closed sets of $\mathfrak{R}$ itself. The second group consists of sets $f_n$ such that $f_n = F_n \cup y$, where $F_n$ is closed in $\mathfrak{R}$. Suppose that there exist sets $f_n$, and let $f_{n_0}$ be one of them. We have $\cap f_n = \cap (f_n \cap f_{n_0})$. The sets $f_n \cap f_{n_0}$ are closed in the compact set $f_{n_0}$ and their collection has the finite intersection property. Hence their intersection is non-empty and the same holds for $\{f_n\}$.

Suppose now that there are no sets $f_n$. We have then

$$\cap f_n = \cap (F_n \cup y) \ni y \neq \emptyset.$$ 

Since $\cap f_n \neq \emptyset$, $\mathfrak{R}'$ is compact. The theorem is therefore proved.

(29.4) Remark. Since local compactness has not been utilized in the proof, the theorem is valid for any topological space $\mathfrak{R}$. However, if $x \in \mathfrak{R}$ has the neighborhood $N$ and $\mathfrak{R}$ is not compact then $N \ni y$. Hence when $\mathfrak{R}$ is not locally compact the open sets of $\mathfrak{R}'$ do not behave very well and so the theorem is of value only in the locally compact case.

§6. SEPARATION AXIOMS

30. The theory developed so far rests exclusively upon the axioms OSi in which the points are nowhere mentioned. Thus the points have merely been the primitive elements of which the sets considered are composed. To express it in another way the properties with which open or closed sets have been implemented do not as yet enable us to distinguish between the individual points by means of these sets: It may well happen that there exist pairs of distinct points $x, y$ such that every open or closed set containing one of the two also contains the other. This is certainly remote from the situation in the familiar spaces, where usually the points are closed, and where in fact no two are on the same total aggregate of open sets.
We require then a suitable “separation” axiom for the points. The most frequently utilized are the following which we describe in the “$T_r$-nomenclature” of Alexandroff-Hopf [A-H, 58]:

**Axiom T$_1$.** Of each pair of distinct points at least one has a neighborhood which does not contain the other.

**Axiom T$_2$.** Each point of every pair of distinct points has a neighborhood which does not contain the other.

**Axiom T$_3$.** (Hausdorff’s separation axiom.) Every pair of distinct points have disjoint neighborhoods.

A topological space which verifies Axiom $T_1$ is known as a $T_1$-**space**, although $T_3$-spaces are commonly called Hausdorff spaces. They include all the spaces of classical geometry and analysis.

The following proposition is an immediate consequence of the $T_r$-axioms together with the principle of relativization:

(30.1) The subsets of a $T_r$-space are $T_r$-spaces.

Again from the $T_r$-axioms together with the definition of topological products we deduce:

(30.2) A product of $T_r$-spaces is a $T_r$-space.

In our ascending scale of axioms the following property shows that with the $T_1$-class points begin to assume their customary properties:

(30.3) A n.a.s.c. in order that the points of a topological space $\mathfrak{R}$ be closed sets is that $\mathfrak{R}$ be a $T_1$-space.

For let $x$, $y$ be any two distinct points of $\mathfrak{R}$. A n.a.s.c. for $x$ to be closed is that $\mathfrak{R} - x$ be open. Since $y$ is merely any point of $\mathfrak{R} - x$ a n.a.s.c. is that given any $y \neq x$ there exist an open set $U$ between $y$ and $\mathfrak{R} - x$, i.e., such that $y \in U$, $x \notin \mathfrak{R} - U$. In other words the required condition is that $\mathfrak{R}$ satisfies Axiom $T_1$.

We prove also for an ulterior purpose:

(30.4) If $\mathbb{S}$ is a Hausdorff space and $T$ is a mapping $\mathfrak{R} \to \mathbb{S}$, then the graph $G$ of $T$ in $\mathfrak{R} \times \mathbb{S}$ is closed.

Let $(x_0, y_0) \in G$, or $y_0 = T x_0$ and let $(x_0, y_1) \in G$, or $y_1 \neq y_0$. Since $\mathbb{S}$ is a Hausdorff space $y_0, y_1$ have disjoint neighborhoods $V_0, V_1$. Since $T$ is continuous $x_0$ has a neighborhood $U$ such that $x_0 \in U \Rightarrow T x \in V_0$ hence $T x \in V_1$. Hence $(U \times V_1) \cap G = \emptyset$ or $(x_0, y_1)$ has the neighborhood $U \times V_1$ free from points of $G$. Therefore $\mathfrak{R} \times \mathbb{S} - G$ is open and $G$ is closed.

**Examples.** (30.5) All the examples considered hitherto in the chapter except ordered spaces (9.3) are $T_r$-spaces. We state explicitly that cells, spheres, paralleloipeds, Euclidean and projective spaces as well as all their subsets, finally discrete spaces, are all $T_r$-spaces.

(30.6) The following example essentially due to Alexandroff-Urysohn [a], describes a space which is $T_1$ but not $T_2$. The space is the real line $L: -\infty < x < +\infty$. For $x_0 \leq 0$ a base at $x_0$ consists of the intervals with the center $x_0$. For $x_0 > 0$ a base $[U]$ at $x_0$ is
made up as follows: \( U \) is an interval \( 0 < a < z < b \) of center \( x \) together with the interval 
\(-b < x < -a \) with the center \(-x\) removed. It is readily seen that as between \( x \) and 
\(-x \) Axiom \( T_1 \) holds but \( T_2 \) fails to hold.

(30.7) Any ordered space which contains at least one ordered pair: \( z < x' \), is a \( T_3 \)-space 
but not a \( T_1 \)-space. Thus the real line \( L \) with the points ordered as in (9.4) by \( \preceq \) is a \( T_3 \)-space 
but not even a \( T_1 \)-space. Under its customary topology however \( L \) is a \( T_1 \)-space.

31. Limits. A few words about this important concept will not be out of 
place. Let \( \mathbb{R} \) be a topological space. A sequence \( \{x_n\} \) of points of \( \mathbb{R} \) is said 
to have for limit the point \( x \) of \( \mathbb{R} \) or to tend or converge to \( x \), written \( \{x_n\} \rightarrow x \), 
or \( x_n \rightarrow x \), whenever corresponding to any neighborhood \( U \) of \( x \) there is an 
integer \( p \) such that \( n > p \rightarrow x_n \in U \). When \( \{x_n\} \) has a limit it is said to be 
convergent.

(31.1) Theorem. Let \( \mathbb{R} \) have countable bases at each point and let \( T \) be a 
transformation \( \mathbb{R} \rightarrow \mathbb{R}' \). Then a n.a.s.c. for \( T \) to be continuous, i.e., that it be 
a mapping, is that if \( \{x_n\} \rightarrow x \) then \( \{Tx_n\} \rightarrow Tx \). This is the situation in 
particular when \( \mathbb{R} \) has a countable open base.

The proof of necessity is elementary and requires no restriction on the bases. 
Conversely, suppose the condition fulfilled and yet \( T \) fail to be continuous. 
There exist then \( x \) and \( x' = Tx \), with a neighborhood \( U' \) of \( x' \) such that \( z \) 
is not an interior point of \( T^{-1} U' \). We may construct a countable base \( \{U_n\} \) at 
\( x \) such that \( U_{n+1} \subset U_n \). Then \( U_n \) contains a point \( x_n \in T^{-1} U' \). Therefore 
\( x_n \rightarrow x \) and yet \( Tx_n \not\rightarrow x' \), contrary to assumption. This proves (31.1).

(31.2) Theorem. In a Hausdorff space limits are unique.

Suppose that a sequence \( \{x_n\} \) converges to two distinct limits \( x, x' \). There 
exist disjoint neighborhoods \( U, U' \) of \( x, x' \) such that for \( n \) sufficiently high 
\( x_n \in U \cap U' = \emptyset \) which is absurd. This proves (31.2).

32. Compact subsets of Hausdorff spaces. Many of the important and 
better known characteristic properties of compact sets appear first in the Hausdorff 
class.

(32.1) A compact subset \( A \) of a Hausdorff space \( \mathbb{R} \) is closed.

Let \( x \in \mathbb{R} - A \) and \( y \in A \). There exist disjoint open neighborhoods \( U_x(z) \) of \( z \) 
and \( U_y(y) \) of \( y \). Since \( A \) is compact and has the open covering \( \{A \cap U_y(y_1)\} \) 
there is a finite subcovering. Hence there is a finite set \( \{y_i\} \) such that \( A \subset V = \bigcup U_y(y_i) \). If \( W = \bigcap U_y(z) \), we have then \( A \subset V, V \cap W = \emptyset \), and 
since \( W \) is open, so is \( \mathbb{R} - A \) which implies (32.1).

(32.2) A continuous image of a compact space into a Hausdorff space is closed 
(32.2, 32.1).

(32.3) A continuous transformation of a compact space into a Hausdorff space is 
a closed transformation (32.2).

(32.4) A one-one mapping of a compact space onto a Hausdorff space is topological; hence both spaces must be compact Hausdorff spaces.
For \( T \), the mapping, is continuous and closed hence both \( T \) and \( T^{-1} \) are continuous.

(32.5) If \( A \) and \( B \) are disjoint compact subsets of a Hausdorff space, then they have disjoint neighborhoods.

Let \( y \in A \), \( x \in B \). As shown in the proof of (32.1) there exist disjoint neighborhoods \( V_\alpha \) of \( A \), and \( W(x) \) of \( x \) (the \( V \), \( W \) there considered). Since \( \{ B \cap W(x) \} \) covers the compact set \( B \), it has a finite subcovering \( \{ B \cap W(x_i) \} \). Hence this time \( V = \cap V_\alpha \) and \( W = \cup W(x_i) \) are disjoint neighborhoods of \( A, B \).

33. Normality. The separation axioms alone are in general not powerful enough to reach down to the usual spaces. For example, they do not suffice to characterize metric spaces. For this reason further restrictions are required and one of the most important, given presently, is a separation axiom for closed sets analogous to Hausdorff's axiom. The basic definition is:

(33.1) Definition. A topological space \( \mathcal{R} \) is said to be normal whenever every two disjoint closed sets \( F, F' \) have disjoint neighborhoods: \( F \subseteq U, F' \subseteq U' \), \( U \cap U' = \emptyset \).

In point of fact normality, like compactness, is a primitive concept, in the sense that it may likewise be expressed without reference to the other properties of open or closed sets. One must also bear in mind that normality does not imply, nor is implied by any one of the separation axioms \( T_\alpha \). Of course mutual relations do exist. Thus if \( \mathcal{R} \) is \( T_1 \) and normal it is necessarily a Hausdorff space.

The dual form of (33.1) is

(33.2) If \( \{ U, U' \} \) is an open covering of \( \mathcal{R} \) then there exists a closed covering \( \{ F, F' \} \) such that \( F \subseteq U \) and \( F' \subseteq U' \).

(33.3) Definition. Given an open covering \( \mathcal{U} = \{ U_\alpha \} \) of \( \mathcal{R} \), if there exists for each \( \alpha \) an open set \( V_\alpha \) such that \( V_\alpha \subseteq U_\alpha \), and that \( \mathcal{B} = \{ V_\alpha \} \) is a covering, we shall say that \( \mathcal{U} \) has been shrunk to \( \mathcal{B} \), also that \( \mathcal{U} \) is shrinkable.

A stronger result than (33.2) is:

(33.4) Every point-finite (in particular every finite or locally finite) open covering \( \mathcal{U} \) of a normal space \( \mathcal{R} \) is shrinkable.

(a) \( \mathcal{U} \) is finite. Although the proof for this case is covered by the general proof, it is so simple, that we give it first. Let \( \mathcal{U} = \{ U_1, \ldots, U_n \} \). Since the closed sets \( F = \mathcal{R} - U_1, F' = \mathcal{R} - ( \cup \{ U_i \mid i \neq 1 \} ) \) are disjoint, they have disjoint open neighborhoods \( U, V \). From \( V_1 \subseteq \mathcal{R} - U \) follows \( V \subseteq \mathcal{R} - U \subseteq U_1 \). Since \( \mathcal{R} - ( \cup \{ U_i \mid i \neq 1 \} ) \subseteq V \) we know that \( \{ V_1, U_2, \ldots, U_n \} \) is an open covering. We proceed to shrink the \( U_i, i \neq 1 \), in the same way, proving the theorem.

(b) General case. Let \( \mathcal{U} = \{ U_\alpha \}, A = \{ \alpha \} \) and let \( \phi(\alpha) \) be a function on \( A \) such that: (a) \( \phi(\alpha) = U_\alpha \) or else \( \phi(\alpha) = V_\alpha \), \( V_\alpha \subseteq U_\alpha \); (b) \( \{ \phi(\alpha) \} \) is a covering.
Order $\Phi = \{\varphi\}$ by the relation: $\varphi < \varphi'$ whenever $\varphi'(\alpha) = \varphi(\alpha)$ if $\varphi(\alpha) = V_\alpha$. It is readily shown that if $\Phi' = \{\varphi\} \subset \Phi$ is simply ordered, and $\varphi'(\alpha) = \bigcap \varphi' = \sup \varphi'$. Therefore by Zorn's theorem $\Phi$ contains an element $\varphi$, such that $\varphi > \varphi \rightarrow \varphi = \varphi_1$. It remains to be shown that $\varphi(\alpha) = V_\alpha$ for every $\alpha$. Suppose indeed that $\varphi(\beta) = U_\beta$ and set $F = \mathbb{R} - \bigcup \varphi(\alpha) \setminus [\alpha \neq \beta]$. We show as above that there is an open set $V_\beta$ such that $F \subset V_\beta$, $V_\beta \subset U_\beta$. Hence $\varphi$ such that $\varphi(\alpha) = \varphi_1(\alpha)$, $\alpha \neq \beta$, and $\varphi_2(\beta) = V_\beta \neq \varphi_1$ and $\varphi_2 > \varphi_1$. This contradiction proves that $\varphi(\beta) = V_\beta$ and (33.4) follows.

Notice that point-finiteness is required to prove that $\varphi''$ is a covering: Let $U_{a_1}, \ldots, U_{a_n}$ be the sets $U_a$ containing $x$. From some $\varphi_0$ on, none of them will be modifed so that if $\varphi' > \varphi_0$ then $\varphi'(\alpha) = \varphi_0'(\alpha) = \varphi''(\alpha) = U_{a_1}$ or $V_{a_1}$. Since $\varphi'$ is a covering, for some $\alpha_i: x \in \varphi_0(\alpha_i) = \varphi''(\alpha_i)$.

A direct generalization of the property of (33.1) is:

(33.5) If $\{F_i\}$ is a finite collection of closed sets in the normal space $\mathbb{R}$, there can be found for each $F_i$, an open set $U_i \supset F_i$ such that $F_i \cap \cdots \cap F_j = \emptyset \Leftrightarrow U_i \cap \cdots \cap U_j = \emptyset$.

Suppose first that $\{G_i\}$ is a finite collection of nonintersecting closed sets. Then $\{\mathbb{R} - G_i\}$ is a finite open covering and so by (33.4) there exist open sets $W_i$ such that $W_i \subset \mathbb{R} - G_i$, and that $\{W_i\}$ is a covering. Therefore $U_i = \mathbb{R} - W_i$ is an open set such that $G_i \subset U_i$ and that $\cap U_i = \emptyset$.

In the general case consider all the combinations $\alpha = \{\alpha_1, \ldots, \alpha_j\}$ of indices such that $\cap F_{\alpha_i} = \emptyset$. By the result just obtained there exist corresponding neighborhoods $U^*_{\alpha_i}$ of the $F_{\alpha_i}$ such that $\cap U^*_{\alpha_i} = \emptyset$. For a given $i$ let $U_i = \bigcap \{U^*_{\alpha_i} \mid x_i \in \alpha_i\}$. Clearly $\{U_i\}$ is such that $F_i \subset U_i$, and that $F_i \cap \cdots \cap F_j = \emptyset \rightarrow U_i \cap \cdots \cap U_j = \emptyset$. The implication in the other direction is obvious.

(33.6) A compact Hausdorff space is normal (23.1, 32.5).

34. Urysohn's characteristic function. Normality is intimately related to the existence of nonconstant real continuous functions. Let $A, B$ be disjoint sets and $f$ a real continuous function on $\mathbb{R}$ whose values on $A, B$ are constant and distinct. Then for suitable constants $a, b$ the function $a + \frac{1}{2}b$ has the same properties and takes its values in the segment $[0, 1]$. Urysohn has considered more particularly continuous functions $f$ on $\mathbb{R}$ to $[0, 1]$ such that $f(A) = 0$, $f(B) = 1$. If such a function exists it is called a characteristic function of the pair $(A, B)$. Of particular importance is:

(34.1) Urysohn's Lemma. Normality is equivalent to the existence of a characteristic function for every pair of disjoint closed sets $(F, F')$.

Suppose that $F, F'$ have the characteristic function $f$ and set $U = \{x \mid f(x) < 1/4\}$, $U' = \{x \mid f(x) > 3/4\}$. Since $f$ is continuous $U, U'$ are open and as they are thus disjoint neighborhoods of $F, F'$, $\mathbb{R}$ is normal.

Conversely, let $\mathbb{R}$ be normal and $F, F'$ disjoint closed sets. There exists an open set $U(1/2)$ such that $F \subset U(1/2), U(1/2) \subset \mathbb{R} - F'$. We treat similarly the pairs of disjoint closed sets $(F, \mathbb{R} - U(1/2)), (U(1/2), F')$, and thus obtain
$U(1/4), U(3/4),$ etc. This produces an open set $U(t)$ for every dyadic proper fraction $t = m/2^n$ with the property: $t < t' \Rightarrow U(t) \subseteq U(t')$. Given any $x \in \mathbb{R}$ we define $f(x) = y$, where $y = \sup \{ t \mid x \notin U(t) \}$. The function $f$ is single-valued and has the proper range. Furthermore $f = 0$ at $F$, $f = 1$ at $F'$. If $\lambda = \alpha \beta$ is a subinterval of $0 - 1$, or else closed at an end point $a, b$ which is then 0 or 1, we have $f^{-1}(\lambda) = \bigcup \{ U(t) \mid t < b \} - \bigcap \{ U(t) \mid t > a \}$ is an open set. Therefore $f$ is continuous, and the lemma is proved.

A noteworthy application of Urysohn’s lemma is the proof (after Alexandroff-Hopf [A–H, 75]) of:

(34.2) Tietze’s Extension Theorem. Any mapping $f$ of a closed subset $F$ of a normal space $\mathbb{R}$ into an $n$-parallelotope $P^n$ or into the Hilbert parallelotope $P^n$ has an extension $\varphi: \mathbb{R} \to P^n$ or $P^n$ as the case may be.

If $P^n$ or $P^n$ are referred to coordinates $x_1, x_2, \ldots$, each of these will be a real continuous single-valued function on $F$ and (34.2) will follow from:

(34.3) Any real continuous single-valued function $f(x)$ on $F$ has an extension $\varphi(x)$ to $\mathbb{R}$: $\varphi | F = f$, which is also real continuous and single-valued.

Evidently $f$ may be replaced by $1/2 + (1/\pi) \arctan f$ whose values are in the segment $[0 - 1]$. Therefore we may assume that this is already the case for $f$. Given any two disjoint closed sets $A, B$ in $\mathbb{R}$, we denote their characteristic function in $\mathbb{R}$ by $\Phi(A, B; x)$. Functions $\{ f_n, \varphi_n \}$, where the range of the $f_n$ is $F$, and the range of the $\varphi_n$ is $\mathbb{R}$, are now introduced as follows:

$$f_0 = f, f_{n+1} = f_n - \varphi_n;$$

setting now $F_n = \{ x \mid f_n(x) \leq (1/3)(2/3)^n \}$, $F'_n = \{ x \mid f_n(x) \geq (2/3)(2/3)^n \}$, we have two disjoint closed sets, and take

$$\varphi_n = \frac{1}{2}\Phi(F_n, F'_n; x).$$

These relations yield a determination of the functions in the order $f_0, \varphi_0, f_1, \varphi_1, \ldots$, and an elementary recurrence leads to the inequalities:

$$0 \leq \varphi_n \leq \frac{1}{2}\Phi, \quad 0 \leq f_n \leq \frac{1}{2}\Phi.$$ 

Introduce now

$$s_n(x) = \varphi_0 + \cdots + \varphi_n.$$ 

From (34.4) follows that the series $\sum \varphi_n(x)$ is uniformly convergent on $\mathbb{R}$, and since the $\varphi_n$ are continuous and single-valued lim $s_n(x)$ exists and is a continuous and single-valued function $\varphi(x)$ on $\mathbb{R}$. From the relation

$$s_n(x) = f(x) - f_{n+1}(x), \quad x \in F,$$

follows then that $\varphi | F = f$. This proves (34.3) and hence (34.2).

35. Tychonoff spaces. These spaces originally called completely regular by their discoverer Tychonoff are given by the
(35.1) **Definition.** A Tychonoff space $\mathcal{R}$ is a Hausdorff space such that for every point $x$ and neighborhood $U$ of $x$ there is a characteristic function of the pair $x, \mathcal{R} - U$.

We verify immediately:

(35.2) Every subset of a Tychonoff space is a Tychonoff space.

(35.3) Every normal Hausdorff space is a Tychonoff space.

The following two definitions are designed to introduce two important concepts needed immediately:

(35.4) **Definition.** Let $\mathcal{R}$ be a topological space, $\{\mathcal{R}_\alpha\}$ an indexed system of topological spaces and for each $\alpha$ let $f_\alpha$ be a function on $\mathcal{R}$ to $\mathcal{R}_\alpha$. Then $\kappa = \{f_\alpha\}$ is said to be a separating class for $\mathcal{R}$ whenever for any two distinct points $x, y$ of $\mathcal{R}$ there is an $\alpha$ such that $f_\alpha(x) \neq f_\alpha(y)$.

(35.5) **Definition.** Under the same conditions let $\{V_\alpha\}$ be the open sets of $\mathcal{R}_\alpha$. Then $\kappa$ is said to be a basic class for $\mathcal{R}$ whenever $\{f_\alpha^{-1}V_\alpha\}$ is a subbase for $\mathcal{R}$.

(35.6) When $\mathcal{R}$ is a $T_0$-space, a basic class $\kappa$ is necessarily separating.

Since $\mathcal{R}$ is a $T_0$-space if $x \neq y$ there is an open set $U$ such that say $x \in U$, $y \notin U$, and hence an $f_\alpha^{-1}V_\alpha$ such that $x \in f_\alpha^{-1}V_\alpha$, $y \notin f_\alpha^{-1}V_\alpha$ and so clearly $f_\alpha(x) \neq f_\alpha(y)$.

Returning to Tychonoff spaces we prove:

(35.7) **Lemma.** Let $\mathcal{R}$ be a Tychonoff space. Then the class $\kappa = \{f_\alpha\}$ of all continuous functions on $\mathcal{R}$ to the segment $[0 - 1]$ is a basic class for $\mathcal{R}$.

Consider the $f_\alpha$ as mapping $\mathcal{R}$ on the segment $0 \leq y \leq 1$. Since $f_\alpha$ is continuous the set $V_\alpha = f_\alpha^{-1}[y \mid 0 \leq y < 1]$ is open. Take now any open set $U$ of $\mathcal{R}$, and $x \in U$ and let $f_\alpha$ be the characteristic function of $x, \mathcal{R} - U$. Evidently $x \in V_\alpha \subset U$. Therefore $\{V_\alpha\}$ is a base and $\kappa$ is basic.

The fundamental theorem for the spaces under consideration is:

(35.8) **Theorem.** Every Tychonoff space $\mathcal{R}$ may be mapped topologically into a compact parallelepiped and every subset of such a parallelepiped and indeed of any compact Hausdorff space is a Tychonoff space (Tychonoff).

Let $\kappa = \{f_\alpha\}$ be as in (35.7) and set $A = \{a\}$. For each $a$ introduce a segment $l_a: 0 \leq y_a \leq 1$ and set $P^a = \mathcal{P}_{l_a}$. If $y = \{y_a\}, y_a = f_a(x)$, then $x \rightarrow y$ defines a transformation $T: \mathcal{R} \rightarrow P^a$.

(a) $T$ is univalent (35.6, 35.7).

(b) $T$ is open. For $\{V_\alpha\}$ being as before, $TV_\alpha = \{y \mid y_a < 1\} \cap T\mathcal{R}$ is open in $T\mathcal{R}$ and since $\{V_\alpha\}$ is a base, $T$ is open.

(c) $T$ is continuous (12.2).

Properties (a), (b), (c) prove that $T$ imbeds $\mathcal{R}$ topologically in $P^a$. 

Since every compact Hausdorff space \( \mathfrak{R} \) is normal (33.6), by (35.3) and (35.2) \( \mathfrak{R} \) and its subsets are Tychonoff spaces.

(35.9) **Remark.** The proof of (35.8) goes through step by step if \( k \) is replaced by any subclass \( k_0 = \{ f_k \} \) such that \( |V_k| \) is a base.

36. **Separation properties and compacting.** The influence of separation on the two compacting processes that have been given is described in the two propositions to follow.

(36.1) **Theorem.** The space \( \mathfrak{R} \) and the compact space \( \widehat{\mathfrak{R}} \) of the general compacting theorem (26.1) are related in their separation properties as follows: When \( \mathfrak{R} \) is \( T_b \), \( T_1 \), or Hausdorff normal, so is \( \widehat{\mathfrak{R}} \).

Let the notations be those of (26, 27, 28). Take two distinct points \( X, X' \) of \( \widehat{\mathfrak{R}} \) both in \( \mathfrak{T} \mathfrak{R} = R \). That is to say, \( X = X(x), X' = X(x'), x \neq x' \). Suppose first \( \mathfrak{R} \) to be \( T_\phi \). There exists then an open set \( u \) of \( \mathfrak{R} \) containing say \( x \) but not \( x' \). Since \( x, x' \) are coordinates of \( X, X' \) we have \( X \mathfrak{O}(u), X' \mathfrak{O}(u) \), and so the \( T_{\phi} \)-axiom holds for the pair \( (X, X') \). Suppose now \( \mathfrak{R} \) is \( T_1 \). There exist disjoint coordinates \( \varphi, \varphi' \) of \( X, X' \). If \( \varphi = f_a \cup x_1 \cup u \cdots \cup x_r \), by (26.4) \( f_a \) is also a coordinate of \( X \) and \( f_a \cup \varphi' = \emptyset \). Hence if \( u = \mathfrak{R} - f_a \) then \( \mathfrak{R} \mathfrak{O}(u), X' \mathfrak{O}(u) \) and the situation is as before. Therefore \( \widehat{\mathfrak{R}} \) is \( T_\phi \).

Suppose now \( \mathfrak{R} \) to be \( T_1 \). Since the points of \( \mathfrak{R} \) are closed sets all the coordinates \( \varphi_a \) are closed sets. If \( X \neq X' \) they have disjoint coordinates \( f_a, f_b \) and so \( \mathfrak{O}(\mathfrak{R} - f_a), \mathfrak{O}(\mathfrak{R} - f_b) \) are neighborhoods of \( X, X' \) which are, respectively, free from \( X', X \). Therefore \( \widehat{\mathfrak{R}} \) is \( T_1 \).

Suppose finally \( \mathfrak{R} \) to be normal Hausdorff. We have again disjoint coordinates \( f_a, f_b \) of \( X, X' \). Since \( \mathfrak{R} \) is normal there exist disjoint open neighborhoods \( u, u' \) of \( f_a, f_b \). Therefore \( \mathfrak{O}(u), \mathfrak{O}(u') \) are disjoint neighborhoods of \( X, X' \) in \( \widehat{\mathfrak{R}} \) and so \( \widehat{\mathfrak{R}} \) is Hausdorff. Since it is compact it is also normal, and the proof of (36.1) is completed.

37. For locally compact spaces we have the stronger

(37.1) **Theorem.** When the locally compact space \( \mathfrak{R} \) of (29.3) is a \( T_{\phi} \)-space so is the associated compact space

\[
\mathfrak{R}' = \mathfrak{R} \cup y.
\]

We may as well exclude at the outset the trivial case of \( \mathfrak{R} \) compact. We denote by \( U = \{ U \} \) the collection of all the open sets \( U \) of \( \mathfrak{R} \) such that \( U \) is compact.

(37.2) A set \( U \) is also open in \( \mathfrak{R}' \).

For \( F = \mathfrak{R} - U \) is not compact, else \( \mathfrak{R} \) would be the union of the two compact sets \( F, \overline{U} \), and hence compact (23.3). It follows that \( F' = F \cup y \) is closed in \( \mathfrak{R}' \) and so \( U = \mathfrak{R}' - (F' \cup y) \) is open in \( \mathfrak{R}' \).

(37.3) \( U \) is a base for \( \mathfrak{R} \).
Let \( V \) be an open set of \( \mathcal{R} \) and \( x \in V \). Then some \( U \) contains \( x \) and \( V \cap U \subset U \). Hence \( V \cap U \in \mathcal{U} \). Since \( x \in V \cap U \subset V \), \( \mathcal{U} \) is a base.

The proof of our theorem is now a simple matter. Any point \( x \in \mathcal{R} \) has a neighborhood \( U \). The set \( U \) is thus closed in \( \mathcal{R'} \) and so \( V = \mathcal{R'} - U \) is a neighborhood of \( y \). By (37.2) \( U \) is also a neighborhood of \( x \) in \( \mathcal{R'} \). Since \( U \cap V = \emptyset \), Axiom \( T_3 \) holds for the pair \( (x, y) \).

Suppose now \( \mathcal{R} \) to be \( T_i \), and let \( x, x' \) be distinct points of \( \mathcal{R} \). The \( T_i \) condition may be fulfilled with neighborhoods out of any base for \( \mathcal{R} \), and in particular out of the base \( \mathcal{U} \) of (37.3). Since the elements of \( \mathcal{U} \) are open in \( \mathcal{R'} \) also, the \( T_i \) condition is fulfilled for the pair \( (x, x') \). Since \( T_3 \) is fulfilled for the pairs \( (x, y) \) so is \( T_i \). Therefore \( \mathcal{R} \) is a \( T_i \)-space.

\section{7. INVERSE MAPPING SYSTEMS}

38. The spaces which are to be introduced here are especially important in the applications to homology. For our purposes it will be quite sufficient to restrict the treatment to Hausdorff spaces.

Let then \( \{\mathcal{R}_k\} \) be a system of Hausdorff spaces indexed by a directed set \( \Lambda = \{\lambda; >\} \) and suppose that whenever \( \lambda > \mu \) there is given a mapping, also known as a projection, \( \pi_\lambda: \mathcal{R}_\lambda \to \mathcal{R}_\mu \) such that \( \lambda > \mu > \nu \Rightarrow \pi_\lambda \circ \pi_\nu = \pi_\lambda = \pi_\nu \). The system \( \Sigma = \{\mathcal{R}_\lambda; \pi_\lambda\} \) of the \( \mathcal{R}_\lambda \) and the \( \pi_\lambda \) is called an inverse mapping system.

Let \( \mathcal{R}^* = \mathcal{P}\mathcal{R}_\Lambda \) and in \( \mathcal{R}^* \) let \( \mathcal{R} \) be the set of all the points \( x = \{x_\lambda\} \) such that \( \lambda > \mu \Rightarrow x_\lambda = x_\mu \). We call \( \mathcal{R} \) the limit-space of the inverse mapping system \( \Sigma \).

Notice incidentally that for \( \lambda < \lambda \) we have \( \pi_\lambda \circ \pi_\lambda = x_\lambda \) or \( \pi_\lambda = 1 \).

Example. \( \{\mathcal{R}_k\} \) is a sequence \( \{C_n\} \) of circumferences, where \( C_n \) is the image of a real variable \( z_n \) reduced mod 1. Choose \( \pi_\lambda = k_n z_n \), \( k_n \) an integer, and define \( \pi_\lambda = \pi_\lambda \circ \pi_\lambda \) \( \cdots \). Then \( \Sigma = \{C_n; \pi_\lambda\} \) is an inverse mapping system. Its limit-space \( \mathcal{R} \), introduced by Vietoris, is known as a solenoid.

As a subset of \( \mathcal{R}^* \) the limit-space \( \mathcal{R} \) receives the relative topology and by (30.1) and (30.2)

\((38.1)\) The limit-space \( \mathcal{R} \) is a Hausdorff space.

\((38.2)\) The topology of the limit-space \( \mathcal{R} \) is frequently described as follows: For each open set \( U_{x_\lambda} \) of \( \mathcal{R}_\lambda \) introduce the set \( V_{x_\lambda} = \{x \mid x \in \mathcal{R}; x_\lambda \in U_{x_\lambda}\} \) and choose \( \{V_{x_\lambda}\} \) as a base for the topology. This topology is readily identified with the "relative topology." Define in fact \( V^*_x = \{x \mid x \in \mathcal{R}^*; x_\lambda \in U_{x_\lambda}\} \). Then \( V_{x_\lambda} = V^*_x \cap \mathcal{R}_\lambda \), and since \( \{V_{x_\lambda}\} \) is a subbase for \( \mathcal{R}^* \), \( \{V_{x_\lambda}\} \) is one for \( \mathcal{R} \) in the relative topology. Furthermore given \( V_{x_\lambda}, V_{x_\mu} \) let \( \lambda > \mu \). Owing to the continuity of the projections \( \pi_\lambda, \pi_\mu \) \( U_{x_\lambda}, (\pi_\lambda)^{-1} U_{x_\mu} \) are open, and so is their intersection \( U_{x_\lambda} \). From this follows that \( V_{x_\lambda} \cap \mathcal{R}^* = V_{x_\lambda} \), so that \( V_{x_\lambda} \) is actually a base for the relative topology, making the identity of the two topologies obvious.

As a subset of \( \mathcal{R}^* \) may be viewed as the graph of the set of relations \( \pi_\lambda x_\lambda = x_\mu \). We find therefore the natural analogue of the property (30.4) for graphs:

\((38.3)\) The limit-space \( \mathcal{R} \) is closed in the product space \( \mathcal{R}^* \).
For \( \lambda > \mu \) introduce \( S_\lambda^\mu = \{ x \mid \pi_\lambda^\mu x_\lambda = x_\mu \} \), and let \( G_\lambda^\mu \) be the graph of \( \pi_\lambda^\mu \). We have: \( S_\lambda^\mu = G_\lambda^\mu \times P_{\pi_\lambda^\mu} \mathcal{R}_\mu \). Since \( G_\lambda^\mu \) is closed in \( \mathcal{R}_\lambda \times \mathcal{R}_\mu \) (30.4), \( S_\lambda^\mu \) is closed in \( \mathcal{R}^* \) (12.6). Therefore \( \mathcal{R} = \cap S_\lambda^\mu \) is also closed in \( \mathcal{R}^* \).

From (23.1), (38.3) and (24.1) we deduce:

(38.4) When the \( \mathcal{R}_\lambda \) are compact so is the limit-space \( \mathcal{R}^* \).

39. We now come to the important:

(39.1) Theorem. If the \( \mathcal{R}_\lambda \) are compact and not empty then the limit-space is likewise not empty (Steenrod [a]).

Let the notations be those of (38.3). Since \( \mathcal{R}^* \) is compact we only need to prove

(39.2) \( S_\lambda^\mu \) has the finite intersection property.

Given a finite set \( \{ S_\lambda^\mu \} \), \( i = 1, 2, \ldots, r \), choose \( \lambda_0 > \lambda_1, \ldots, \lambda_r \). Since \( \lambda_i > \mu \), we also have \( \lambda_0 > \mu_i \). Take any \( x_\lambda \in \mathcal{R}_\lambda \), define \( x_\lambda = \pi_\lambda^\mu x_\lambda = x_\mu = \pi_\mu^\lambda x_\mu \), and let \( x \) be any point of \( \mathcal{R}^* \) with the coordinates \( x_\lambda, x_\mu \). From \( \pi_\mu^\lambda = \pi_\lambda^\mu \pi_\mu^\lambda \) there follows \( \pi_\mu^\lambda x_\lambda = x_\mu \), and so \( x \in S_\lambda^\mu \). This proves (39.2) and hence also (39.1).

A complementary property is:

(39.3) If the \( \mathcal{R}_\lambda \) are compact and \( x_\lambda \) is such that for every \( \lambda > \mu \) the set \( (\pi_\lambda^\mu)^{-1} x_\lambda \neq \emptyset \) then \( \mathcal{R} \) contains a point \( x \) with the coordinate \( x_\lambda \).

If \( \pi_\mu \) is the natural projection \( \mathcal{R}^* \rightarrow \mathcal{R}_\mu \) then \( F = \pi_\mu^{-1} x_\mu \) is closed in \( \mathcal{R}^* \) and (39.3) reduces to: \( F \cap \mathcal{R} \neq \emptyset \), and hence to:

(39.4) \( F, S_\lambda^\mu \) has the finite intersection property.

Choose this time \( \lambda_0 > \lambda_1, \ldots, \lambda_r, \mu \). By hypothesis \( (\pi_\mu^\lambda)^{-1} x_\mu \neq \emptyset \), and so we may take \( x_\lambda \) in that set. We now take \( x \) as before save that in addition its \( \mu \) coordinate is to be \( x_\mu \), a condition which may manifestly be fulfilled. Thus \( x \in F \) and still \( x \in S_\lambda^\mu \). Hence (39.4) holds and (39.3) follows.

40. Let \( \mathcal{M} = \{ \mu; > \} \) be a directed subsystem of \( \Lambda = \{ \lambda; > \} \). Then the spaces \( \mathcal{R}_\lambda \) and projections \( \pi_\mu^\lambda \) give rise to a new inverse mapping system \( \Sigma_1 = \{ \mathcal{R}_\mu; \pi_\mu^\lambda \} \) known as a partial system of \( \Sigma \). If \( \mathcal{M} \) is cofinal in \( \Lambda \) we say that \( \Sigma_1 \) is cofinal in \( \Sigma \).

Let \( \Sigma_1 \) be a partial system of \( \Sigma \) and \( \Theta \) its limit-space. If \( x = \{ x_\lambda \} \in \mathcal{R} \) the coordinates \( x_\lambda \) of \( x \) determine a unique point \( x' \) of \( \Theta \). The transformation \( \tau: \mathcal{R} \rightarrow \Theta \) whereby \( \tau x = x' \) is known as the projection of \( \mathcal{R} \) into \( \Theta \).

(40.1) Let \( \Sigma_1 \) with limit-space \( \Theta \) be a partial system of \( \Sigma \). Then: (a) the projection \( \tau: \mathcal{R} \rightarrow \Theta \) is a mapping; (b) when \( \Sigma_1 \) is cofinal in \( \Sigma \) then \( \tau \) is topological, so that \( \mathcal{R} = \Theta \) are then topologically equivalent.

Let \( \Theta^* = P_{\pi_\mu^\lambda} \mathcal{R}_\mu \). We have then \( \mathcal{R}^* = \Theta^* \times P_{\pi_\mu^\lambda} \mathcal{R}_\lambda \) and the projection \( \pi^*: \mathcal{R}^* \rightarrow \Theta^* \) is continuous. Now if \( x = \{ x_\lambda \} \in \mathcal{R} \) the point \( \pi^* x \) is the point of \( \Theta^* \) whose coordinates are the \( \mu \) coordinates of \( x \), i.e., \( \tau^* x = \tau x \). Hence \( \tau = \pi^* \mid \mathcal{R} \), and so \( \tau \) is also continuous.

Suppose now that \( \Sigma_1 \) is cofinal in \( \Sigma \). We shall show that \( \tau \) is a one-one mapping of \( \mathcal{R} \) onto \( \Theta \). Given \( x' = \{ x_\lambda \} \in \Theta \) if \( x = \{ x_\lambda \} \in \mathcal{R} \) is to be such that
\(\tau x' = z\) then we must have \(x_\lambda = \pi_\lambda^\star x_\mu, \mu > \lambda\). Choose some \(\mu > \lambda\) and let \(x_\lambda\) be defined by this relation. If \(\mu_1 > \mu > \lambda\) then \(\pi_\mu^\star x_{\mu_1} = \pi_\mu^\star(\pi_\mu^\star x_\mu) = \pi_\lambda^\star x_\mu\) so that \(\mu_1\) yields the same value of \(x_\lambda\) as \(\mu\). Take any two indices \(\mu_1, \mu_2\). There is a \(\mu > \mu_1, \mu_2\) and since \(\mu, \mu_1\) and \(\mu, \mu_2\) yield the same value for \(x_\lambda\), so do \(\mu_1, \mu_2\). Therefore \(x_\lambda\) is unique. If \(\lambda > \lambda'\) choose \(\mu > \lambda\). From \(x_{\lambda'} = \pi_\lambda^\star x_\mu = \pi_\lambda^\star (\pi_{\lambda'}^\star x_{\lambda'}) = \pi_\lambda^\star x_{\lambda'}\) follows that \(\{x_\lambda\}\) are in fact the coordinates of a point \(x \in \mathcal{R}\). Since this point has the coordinates \(x_\mu\) of \(x'\) we have \(\tau x = x'\). Thus every point \(x'\) is the image under \(\tau\) of one and only one \(x\) and so \(\tau\) is a one-one mapping.

To prove \(\tau\) topological there remains to show that it is open. Let \(U_\lambda, V_\lambda\) be as in (38.2). Then \(x \in V_\lambda \implies x_\lambda \in U_\lambda\). Choose any \(\mu > \lambda\). Since \(\pi_\mu^\star\) is continuous \((\pi_\mu^\star)^{-1}U_\lambda = U_{\lambda\mu}\) is open in \(\mathcal{R}_\mu\) and \(V_{\lambda\mu} = \{x' : x_\mu \in U_{\lambda\mu}\}\) is open in \(\mathcal{R}\). Since \(\tau V_\lambda = V_{\lambda\mu}\), \(\tau\) is open and hence topological. This proves (40.1).

Two inverse mapping systems are said to be equivalent if there exists a third in which both are cofinal. From (40.1) we deduce:

(40.2) Equivalent inverse mapping systems have topologically equivalent limit-spaces.

41. When \(\{|\lambda|\}\) is a sequence it is more convenient to use \(1, 2, \cdots\) as the indices. We write therefore \(\mathcal{R}_\lambda\) for \(\mathcal{R}_{\lambda_n}\) and \(\pi_n^\lambda\) for \(\pi_{\lambda_n}^\lambda\), and require that for \(n < p < q\) we have \(\pi_n^\lambda \pi_n^\lambda = \pi_n^\lambda\). The system \(\{\mathcal{R}_\lambda, \pi_\lambda^\star\}\) is then called an inverse mapping sequence. From (4.4) and (40.2) we see that

(41.1) When \(\{|\lambda|\}\) is countable, \(\mathcal{R} = \lim \{|\mathcal{R}_\lambda, \pi_\lambda^\star\}\) is topologically equivalent either to some \(\mathcal{R}_\lambda\) or to the limit-space of an inverse mapping sequence cofinal in \(\{\mathcal{R}_\lambda, \pi_\lambda^\star\}\).

It is clear that if \(\{|U_\lambda|\}\) is a base for each \(\mathcal{R}_\lambda\), then \(\{|V_\lambda|\}\) (in the notation of 38) is a base for \(\mathcal{R}\). Thus we have

(41.2) When \(\{|\lambda|\}\) is countable and the \(\mathcal{R}_\lambda\) all have countable bases, \(\mathcal{R} = \lim \{|\mathcal{R}_\lambda, \pi_\lambda^\star\}\) has likewise a countable base.

§ 8. METRIZATION

42. Guided by the Euclidean situation, given a point set \(R\) we call distance-function or merely distance a real function \(d(x, y)\) defined for all \(x, y \in R\) and possessing the following properties:

D1. \(d(x, y) = 0\) when and only when \(x = y\).

D2. (Triangle axiom): \(d(y, z) \leq d(x, y) + d(x, z)\).

From D12 we derive, with Lindenbaum, the other two noted properties of the distance:

D3. \(d(x, y) \geq 0\).

D4. \(d(x, y) = d(y, x)\).

We prove D3 by making \(z = y\) in D2. Regarding D4 making \(x = z\) in D2 and taking account of D1 we have \(d(y, x) \leq d(x, y)\). Since the inequality may be proved also in reverse order D4 follows.

The set \(R\) with an associated distance-function \(d(x, y)\) is called a metric space. We also say that \(d(x, y)\) defines a metric for \(R\).

We shall now discuss the first properties of metric spaces.
(42.1) **Subsets of a metric space.** Let $\mathcal{R}$ be a metric space with the distance $d(x, y)$. If $A \subset \mathcal{R}$ and we take $x, y \in A$, $d(x, y)$ becomes a suitable distance-function for $A$, so that $A$ is metric also.

(42.2) **Distance between two sets.** Diameter of a set. Spheres. The distance $d(A, B)$ between two subsets $A$ and $B$ of $\mathcal{R}$ is $\inf \{d(x, y) \mid x \in A, y \in B\}$. The diameter of $A$ (diam $A$) is $\sup \{d(x, y) \mid x, y \in A\}$. The set of all points $x$ such that $d(x, A) < \varepsilon$ is called an $\varepsilon$ neighborhood of $A$ and denoted by $\mathcal{E}(A, \varepsilon)$. When $A = x_0$, a single point, $\mathcal{E}(x_0, \varepsilon)$ is commonly called a spheroid or sphere; the point $x_0$ is the center of the spheroid and $\varepsilon$ its radius. The analogy with Euclidean spherical regions is obvious.

(42.3) **$\varepsilon$ aggregates, $\varepsilon$ coverings, $\varepsilon$ transformations.** This type of designation with appropriate variations is frequently convenient. The mesh of an aggregate is the supremum of the diameters of its sets. An $\varepsilon$ aggregate or covering is one whose mesh is less than $\varepsilon$. If a space has a finite $\varepsilon$ covering for every $\varepsilon > 0$ it is said to be **totally bounded**. An $\varepsilon$ transformation of a metric space $\mathcal{R}$ into a set $Q = \{y\}$ is a transformation such that mesh $\{T^{-1}y\} < \varepsilon$.

(42.4) In terms of the spheres we may define **regions** as in Euclidean geometry: $U$ is a region whenever $x \in U$ implies $\mathcal{E}(x, \varepsilon) \subset U$ for some $\varepsilon > 0$.

43. The chief justification for considering metric spaces at this juncture lies in the

(43.1) **Theorem.** If a metric space is topologized by choosing regions as open sets it becomes a normal Hausdorff space with a countable base at each point.

The verification of OS123 (6) is immediate. If $x \neq y$, then $d(x, y) = \varepsilon > 0$. Hence $\mathcal{E}(x, \varepsilon/3)$ and $\mathcal{E}(y, \varepsilon/3)$ are disjoint neighborhoods of $x$ and $y$. If $A$ and $B$ are disjoint closed sets, then $\{x \mid 3d(x, A) < d(x, B)\}, \{x \mid 3d(x, B) < d(x, A)\}$, are disjoint open sets containing $A$ and $B$. Thus the space is a normal Hausdorff space. Clearly $|\mathcal{E}(x_0, 1/n)|$ is a countable base at the point $x_0$.

(43.2) The spheroids form a base, and those of center $x$ form a base at $x$.

On the strength of (43.2) we shall say that two distinct metrics are **equivalent** whenever if $\mathcal{E}(x, \varepsilon), \mathcal{E}'(x, \varepsilon)$ are the corresponding spheroids then given $\varepsilon$ and $x$ there is an $\eta$ such that $\mathcal{E}'(x, \eta) \subset \mathcal{E}(x, \varepsilon)$, and vice versa. That is to say the two metrics are equivalent if they induce the same topology in $\mathcal{R}$.

It is easy to see that

(43.3) **The distance-function in $\mathcal{R}$ defines a topology in the subsets which is in accordance with the principle of relativization.**

A topological space $\mathcal{R}$ is said to be **metrizable** whenever it is possible to assign it a **metric**, i.e., a distance function $d(x, y)$ inducing the topology of the space. **Metrization** is the process of assigning a metric to a metrizable space. Frequently for shortness a space is described as metric when it is merely metrizable. In each case the context shows clearly what is meant.

(43.3a) **The closure $\bar{A}$ is the set of all points $x$ such that $d(x, A) = 0$.**

(43.4) **Limits, continuity.** Since metric spaces are Hausdorff spaces with a
countable base in each point, limits in such spaces are unique (31.2) and continuity of mappings of metric spaces into one another may be expressed in terms of limits as in (31.1). Moreover in so far as such spaces alone are involved all questions of continuity and convergence may be dealt with by the "ε, δ" method of classical analysis. Many, if not most of the well known concepts of the latter may be introduced here also. We merely recall the useful concept of uniform continuity: \( R, R' \) being metric the mapping \( T: R \to R' \) is said to be uniformly continuous whenever if \( x \in R \) and \( x' = Tx \) then given any \( \epsilon > 0 \), there is an \( n > 0 \) independent of \( x \) such that \( T \mathcal{E}(x, \eta) \subset \mathcal{E}(x', \epsilon) \).

(43.5) Completeness. Since \( \{ \mathcal{E}(x, \epsilon) \} \) is a base at \( x \), \( \{ x_n \} \to x \) whenever \( \{ d(x, x_n) \} \to 0 \). As is well known a necessary condition in order that \( \{ x_n \} \) converge (to some point) is that Cauchy's condition hold: given any \( \epsilon > 0 \) there is an \( n \) such that \( p, q > n \to d(x_p, x_q) < \epsilon \). Whenever Cauchy's condition implies the convergence of any sequence for which it holds the space \( R \) is said to be complete.

(43.6) Separability. A space is said to be separable whenever it has a countable dense subset. This property is only of interest in connection with metrization, and largely owing to:

(43.7) For a metrizable space \( R \) separability is equivalent to the existence of a countable base.

For this reason we shall call such spaces "separable metric."

At all events whether \( R \) is metric or otherwise, when it has a countable base \( \{ U_n \} \) we may choose a point \( x_n \) on \( U_n \), and since \( \{ x_n \} \) is a countable dense set, \( R \) is separable. Conversely, let \( R \) be metric with the countable dense set \( \{ x_n \} \). To show that \( R \) has a countable base it will be sufficient to show that \( R = \{ \mathcal{E}(x, \rho) \} \), \( \rho \) rational, which is composed of a countable number of spheroids, is a base. Let \( U \) be any neighborhood of \( x \). There is a spheroid \( \mathcal{E}(x, \rho) \subset U \). Since the \( x_n \) are dense in \( R \) we may find an \( x_n \) such that \( d(x, x_n) < \rho/4 \), then choose \( \rho_p \) between \( \rho/4 \) and \( \rho/2 \). As a consequence \( x \in \mathcal{E}(x, \rho) \subset \mathcal{E}(x, \rho) \subset U \). Therefore \( R \) is a base.

(43.8) Metric product. Let \( \{ R_n \} \) be a countable collection of metrizable spaces. Choose a distance \( d_n(x, y) \) for \( R_n \) and metrize the product \( R = \mathbb{P}R_n \) (for the present merely set-product) as follows. If \( x = (x_1, \ldots) \), \( y = (y_1, \ldots) \) are two points of \( R \) we choose a distance-function

\[
d(x, y) = \sum k_n \left( d_n(x_n, y_n) + 1 \right)
\]

where \( \sum k_n \) is any convergent series of strictly positive terms. For instance we may take \( k_n = 2^{-n} \) but any other choice will do. In particular if the number of factors \( R_n \) is finite we may choose

\[
d(x, y) = \sum d_n(x_n, y_n).
\]

The verification of D12 is elementary, so that \( d(x, y) \) defines a metric for \( R \).
and \( \mathfrak{R} \) is metrizable. We call \( \mathfrak{R} \) with the metric (43.9) a metric product of the \( \mathfrak{R}_n \). We shall now prove

(43.10) **Theorem.** The metric (43.9) determines the same topology as previously assigned to the topological product. In particular it is independent of the special choice of \([k_n] \) and of the metrics \( d_n \).

Since the spheroids form a base on each \( \mathfrak{R}_n \), we may choose as a base for \( \mathfrak{R} \) the sets \( U = P U_n \), where \( U_n = \mathfrak{R}_n \) for \( n > m \), and \( U_n \), \( n \leq m \), is a spheroid of \( \mathfrak{R}_n \). Given \( x \in U \) there is a similar \( U'_n \supset x \), \( U' \subset U \), with spheroidal factors \( U'_n = \mathfrak{S}(x, r_n), n \leq m \), where the \( x_n \) are the projections of \( x \). Let \( r = \inf r_n, k = \inf k_n \) for \( n \leq m \) and let \( R = kr/(1 + r) \). Then if \( y \in \mathfrak{S}(x, R) \), we have from (43.9)

\[
\delta_n = k_n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < \frac{kr}{1 + r}, \quad n \leq m,
\]

and hence \( d(x_n, y_n) < r \). Therefore \( y_n \in U_n \), \( y \in U \), and finally \( \mathfrak{S}(x, R) \subset U' \subset U \). Thus there is a spheroid between \( x \) and \( U \).

Conversely, given \( \mathfrak{S}(x, R) \) we may choose \( m \) so large that the remainder of \( \sum k_n \) after \( m \) terms is less than \( R/2 \), then take \( U_n = \mathfrak{S}(x_n, r_n), r_n < R/(2mk_n) \) when \( n \leq m \). As a consequence \( y \in U \Rightarrow d(x, y) < R \) and hence \( x \in U \subset \mathfrak{S}(x, R) \). Therefore \( \{U\} \) and \( \{\mathfrak{S}(x, R)\} \) are equivalent bases for \( \mathfrak{R} \).

44. **Examples.**

(44.1) The \( n \)-dimensional number space \( \mathbb{G}^n \) referred to the coordinates \( x_1, \ldots, x_n \) has the distance-function

(44.2)

\[
d(x, y) = \left( \sum (x_i - y_i)^2 \right)^{1/2}.
\]

The space with this metric is an Euclidean \( n \)-space. It is an elementary matter to identify the resulting topology with that of (9.1).

If we consider \( \mathbb{G}^n \) as the product of the \( n \) lines of the variables \( x_i \), metrized by \( d_i = |x_i - y_i| \), the method of (43.10) leads for \( \mathbb{G}^n \) to the distance-function

(44.3)

\[
d'(x, y) = \sum |x_n - y_n|
\]

which yields again the same topology as (9.1), and hence as (44.2). Therefore for topological purposes the two metrics are interchangeable.

(44.4) Since \( \mathbb{G} \) is metrizable so are all its subsets.

(44.5) Consider the Hilbert parallelepiped \( P = \mathbb{P}t_\lambda \), and let \( l_\lambda \) be parametrized as:

\[
0 \leq x_\lambda \leq 1/n_\lambda. \quad \text{Thus } P = \text{the set of all sets: } (x_1, x_2, \ldots), 0 \leq x_\lambda \leq 1/n. \quad \text{By (43.10) } P \text{ is metrizable and admits the distance-function}
\]

(44.6)

\[
d(x, y) = \sum \frac{|x_n - y_n|}{n^2}.
\]

Consider on the other hand the Euclidean metric for \( P \) defined by

(44.7)

\[
d'(x, y) = \left( \sum (x_n - y_n)^2 \right)^{1/2}.
\]

If \( \mathfrak{S}(x, \epsilon) \) and \( \mathfrak{S}'(x, \epsilon) \) are the spheroids corresponding to the two metrics, it is readily shown
that for fixed $x$ each $\mathcal{S}(x, \varepsilon)$ contains an $\mathcal{S}'(x, \eta)$, and conversely. Hence $\{\mathcal{S}'(x, \varepsilon)\}$ is a base for $\mathcal{P}$, and so $d'(x, y)$ defines an admissible metric for the Hilbert parallelopotope. It is in fact the metric customarily assigned to it.

45. Compacta. A compactum is a compact metric space. The category of compacta partakes therefore of the combined advantages of compactness and metrizability. Its importance is sufficiently indicated if we observe that closed and bounded subsets of Euclidean spaces are compacta.

(45.1) Every compactum $\mathcal{R}$ is separable and hence possesses a countable base.

Define a set $A$ as $\varepsilon$ dense in $\mathcal{R}$ whenever every point of $\mathcal{R}$ is nearer than $\varepsilon$ to $A$. I say that we merely have to prove that there exists in $\mathcal{R}$ a countable set $A$ which is $\varepsilon$ dense whatever $\varepsilon$. For in that case every sphere $\mathcal{S}(x, \rho)$ about a given point $x$ will contain points of $A$. As these spheres form a base for $x$, we will have $x \in \hat{A}$, hence $\hat{A} = \mathcal{R}$.

Now $\{\mathcal{S}(x, \varepsilon)\}, \varepsilon \in \mathcal{R}$, is an open covering of the compact space $\mathcal{R}$. Hence there is a finite subcovering $\{\mathcal{S}(x_i, \varepsilon)\}$. Clearly the set $A(\varepsilon) = \{x_i\}$ is $\varepsilon$ dense. The set $A = \bigcup A(1/n)$ is countable and $\varepsilon$ dense for each $\varepsilon > 0$.

(45.2) If $F$ is closed in $\mathcal{R}$ and $x \in \mathcal{R} - F$ then $d(x, F) > 0$ (43.3a).

(45.3) If $\mathcal{F} = \{F_1, \cdots, F_s\}$ is a finite aggregate of nonintersecting closed sets in the compactum $\mathcal{R}$ there is a constant $c(\mathcal{F}) > 0$ such that every $x \in \mathcal{R}$ is at a distance not less than $c(\mathcal{F})$ from at least one $F_i$.

For otherwise there are points $x$ whose distance from every $F_i$ is not more than $1/n$. The set $G_0$ of all such points is closed and $G_0 \supseteq G_{n+1}$. Since $\mathcal{R}$ is compact and $\bigcap G_n \neq \emptyset$ it contains a point $x$. Clearly $d(F_i, x) = 0$ and since $F_i$ is closed $x \in F_i$. Therefore $x \in \bigcap F_i \neq \emptyset$, contrary to assumption. This proves (45.3).

(45.4) If $F, F'$ are closed in the compactum $\mathcal{R}$ and $d(F, F') = 0$ then $F$ and $F'$ intersect.

For otherwise if $x \in F$ we have $d(x, F') \geq c(F, F') > 0$, and hence $d(F, F') \geq c(F, F') > 0$.

(45.5) For every finite aggregate of closed sets $\mathcal{F}$ in the compactum $\mathcal{R}$ there exists a positive constant $d(\mathcal{F})$ called the Lebesgue number of $\mathcal{F}$, such that if $A \subset \mathcal{R}$, $\text{diam} A < d(\mathcal{F})$, and $A$ meets a collection of sets of $\mathcal{F}$, then these sets have a nonvacuous intersection.

Let $\mathcal{F}_i, i = 1, 2, \cdots, s$, be the subaggregates of $\mathcal{F}$ whose sets do not meet, and let $d(\mathcal{F}_i) = \inf d(\mathcal{F}_i)$. If $A$ behaves as stated it cannot meet the sets of $\mathcal{F}_i$, since otherwise $x \in A$ would imply that $x$ is nearer than $c(\mathcal{F}_i)$ to every set of $\mathcal{F}_i$, which contradicts the definition of $c(\mathcal{F})$. This proves (45.5).

(45.6) For every finite open covering $\mathcal{U} = \{U_i\}$ of the compactum $\mathcal{R}$ there exists a positive constant $d_1(\mathcal{U})$ called the Lebesgue number of $\mathcal{U}$, such that: (a) every point $x$ of $\mathcal{R}$ is on some set $U_i$; and at a distance at least $d_1(\mathcal{U})$ from $\mathcal{R} - U_i$; (b) if $A \subset \mathcal{R}$ and $\text{diam} A < d_1(\mathcal{U})$ then $A$ is in some set $U_i$.

Since $\mathcal{U} = \mathcal{R}$ we have $\bigcap (\mathcal{R} - U_i) = \emptyset$. Therefore $d_1(\mathcal{U}) = d(\mathcal{R} - U_1, \cdots)$ has property (a). If $A$ is chosen in accordance with (b), and $x \in A$ then for some $i$ property (a) holds and hence $A \subset U_i$. 


(45.7) A continuous mapping \( f \) of a compactum \( \mathbb{R} \) into a metric space \( \mathbb{E} \) is uniformly continuous.

By (23.2) the values of \( f \) make up a compactum \( R \) in \( \mathbb{E} \). Given then any \( \epsilon > 0 \) there is a finite open \( \epsilon \) covering \( \mathbb{U} = \{ U_\alpha \} \) of \( R \). It follows that \( \mathbb{B} = \{ f^{-1}U_\alpha \} \) is a finite open covering of \( \mathbb{R} \). Let \( \eta = d_i(\mathbb{B}) \). If \( x' = f(x) \), \( y' = f(y) \) and \( d(x, y) < \eta \), some \( f^{-1}U_\alpha \) contains both \( x \) and \( y \) and hence some \( U_\alpha \) contains both \( x' \) and \( y' \), which implies \( d(x', y') < \epsilon \). Therefore \( f \) is uniformly continuous.

The following two properties of compacta are obvious but often useful:

(45.8) A compactum is totally bounded (42.3).

(45.9) A decreasing sequence of closed sets \( \{ F_n \} : F_{n+1} \subset F_n \), has a non-void intersection and if \( \text{diam} F_n \to 0 \) the intersection is a point.

Sequential compactness. A familiar and very important fact in analysis is the close connection between compactness and convergence (see notably J. Tukey [TJ]). The specialization to separable metric spaces brings to the fore the

(45.10) DEFINITION. The space \( \mathbb{R} \) is said to be sequentially compact whenever every sequence \( \{ x_n \} \) has a subsequence \( \{ x_{n_k} \} \) which converges to a point of \( \mathbb{R} \).

(45.11) A compactum is sequentially compact.

By (45.8) \( \mathbb{R} \) possesses a finite open \( \epsilon \) covering. The closures of its sets make up a finite \( \epsilon \) closed covering \( \mathbb{R} = \{ F_1, \ldots, F_l \} \). Let \( \{ x_n \} \subset \mathbb{R} \). One of the \( F_{1i} \), say \( F_{1i} \), contains an infinite subsequence \( \{ x_{1m} \} \). Since \( F_{1i} \) is a compactum it has an \( \epsilon/2 \) finite closed covering \( \{ F_{1i} \} \), one of whose sets \( F_{1m} \) contains an infinite subsequence \( \{ x_{2m} \} \) of \( \{ x_{1m} \} \), etc. By (45.9): \( \cap F_{1m} = x_0 \) is a point and clearly

\[
\{ x_{nm} \} \to x_0 \quad \text{(diagonal process)}.
\]

(45.12) A sequentially compact metric space \( \mathbb{R} \) is a compactum.

We first prove \( \mathbb{R} \) separable. For any \( \epsilon \) the space has a finite \( \epsilon \) dense set \( A(\epsilon) \). For if this were false we could find a sequence \( \{ x_n \} \) such that \( d(x_m, x_n) \geq \epsilon \) whatever \( m, n, m \neq n \), and no subsequence could converge. It follows that \( \cup A(1/n) \) is a countable dense set, and so \( \mathbb{R} \) is separable.

Since \( \mathbb{R} \) is separable it has a countable base (43.7). Hence (6.7) an open covering \( \{ U_\alpha \} \) of \( \mathbb{R} \) has a countable subcovering \( \{ U'_\alpha \} \). Suppose that the latter has no finite subcovering. Then we may choose an \( x_0 \in \mathbb{R} \) \( - \) \( \cup U'_1 \cup \cdots \cup U'_n \) by hypothesis a subsequence \( \{ x_{n'} \} \) of \( \{ x_n \} \) has a limit \( x_0 \). Since \( \{ U'_{n'} \} \) is a covering we have \( x_0 \in U'_{n'} \) for some \( n \), hence \( x_{n'} \in U'_{n'} \) for \( n' \) above a certain value. Since this is ruled out (45.12) is proved.

(45.13) For metric spaces compactness and sequential compactness are equivalent (45.11, 45.12).

An interesting consequence of (45.11) is:

(45.14) A compactum is complete.

46. Urysohn's metrization theorems. We have now all the elements necessary for dealing with these classical theorems.
(46.1) **Theorem.** Every Tychonoff space with a countable base can be imbedded topologically in the Hilbert parallelopotope \( P^n \) and hence it is metrizable, and for that matter also normal.

(46.2) **Theorem.** A n. a. s. c. for a Hausdorff space with a countable base to be metrizable is normality.

(46.3) **Theorem.** Separable metric spaces are those and only those which may be imbedded topologically in \( P^n \).

(46.4) **Theorem.** A n. a. s. c. for a compact Hausdorff space to be a compactum is that it possess a countable base.

**Proof of (46.1).** Referring to (35.8) and (35.9), the mapping considered in (35.8) exists when the base \( \{ V_s \} \) there considered is replaced by any subcollection forming a base. Now under the hypothesis of (46.1), and by (6.8), there is a countable subcollection \( \{ V_s \} \) which is a base and the mapping of (35.8) is then into \( P^n \). This proves (46.1).

**Proof of (46.2).** Since normal Hausdorff spaces are also Tychonoff spaces (35.3) sufficiency is a consequence of (46.1); and necessity follows from (43.1).

**Proof of (46.3).** Since \( P^n \) is a compactum it has a countable base. Hence the subsets of \( P^n \) are metric with a countable base, and therefore also separable. Conversely, if \( R \) is separable metric it is normal with a countable base and hence by (46.1) it may be imbedded topologically in \( P^n \).

**Proof of (46.4).** Necessity is a consequence of (45.1). Since a compact Hausdorff space is normal (35.0), sufficiency follows from (46.2).

§9. HOMOTOPY. DEFORMATION. RETRACTION

47. These concepts are important not only in their strict form, but also in view of certain noteworthy algebraic analogues which occur in the theory of complexes.

**Homotopy, deformation.** The intuitive concept of a deformation or displacement is clear enough. Duly generalized and made fully rigorous it gives rise to the:

(47.1) **Definitions.** Let \( A, B \) be topological spaces, and \( l \) the segment \( 0 \leq u \leq 1 \). Two mappings \( t_1, t_2 : A \to B \) are said to be homotopic whenever there is a mapping \( T \) of the product \( l \times A \to B \) such that \( T(0 \times x) = t_1 x, T(1 \times x) = t_2 x, x \in A \). If \( t_1 = 1 \), which implies \( A \subseteq B \), then \( t_2 \) is a deformation. The set \( T(l \times x) \) is the path of \( x \). Whenever the space is metric and the paths are all of diameter less than \( \epsilon \) we have an \( \epsilon \) homotopy, or \( \epsilon \) deformation as the case may be.

In a more geometric form the images of \( t_1 A \) and \( t_2 A \) are homotopic whenever the "cylinder" \( l \times A \) may be so mapped in \( B \) that its bases agree with the images \( t_1 A, t_2 A \).
(47.2) Homotopy is an equivalence relation.

Homotopy is:

- symmetric, for if \( T \) is as above then \( T_1 \) such that \( T_1(u \times x) = T((1 - u) \times x) \) bears the same relation to \( t_1, t_2 \) as \( T \) but in reverse order;
- reflexive, for \( T(u \times x) = t_x \) is a mapping \( I \times A \to A \) making \( t_1 \) homotopic to itself;
- transitive, for let \( (t_1, t_2) \) and \( (t_2, t_3) \) be homotopic pairs of mappings \( A \to B \) with \( T', T'' \) as the analogues of \( T \). Define

\[
T(u \times x) = T'((2u) \times x), \quad 0 \leq u \leq 1/2;
\]
\[
T(u \times x) = T''((2u - 1) \times x), \quad 1/2 \leq u \leq 1.
\]

It is clear that \( T(u \times x) \) is continuous in \( u \times x \). We have at once \( T(0 \times x) = t_x, T(1 \times x) = t_x \), and so \( t_1, t_2 \) are homotopic which proves transitivity, hence also (47.2).

Since for fixed \( A, B \) homotopy is an equivalence there are corresponding classes, which are known as homotopy-classes.

(47.3) Let \( t_1, t_2 \) be homotopic mappings \( A \to B \) and let \( t \) be a mapping \( B \to C \). Then \( t t_1, t t_2 \) are homotopic mappings \( A \to C \).

The notations being as before \( tT \) is a mapping \( I \times A \to C \) such that \( tT(0 \times x) = t_1 x, tT(1 \times x) = t_2 x \), proving our assertion.

For mappings into subsets of an Euclidean space \( \mathbb{R}^n \) or parallelepode \( P \) a convenient and intuitive sufficiency condition for homotopy is:

(47.4) The notations being the same suppose that \( B \) is a subset of \( \mathbb{R} = \mathbb{R}^n \) or \( P \). If for every \( x \) the points \( t_1 x \) and \( t_2 x \) coincide or else may be joined by a segment of \( \mathbb{R} \) which is in \( B \) then \( t_1 \) and \( t_2 \) are homotopic.

For let \( \lambda(x) = t_x \) when \( t_x = t_2 x \), and \( \lambda(x) \) = the segment joining \( t_1 x, t_2 x \) when they are distinct. Then (in vector notation)

\[
T(u \times x) = (1 - u)(t_1 x) + u(t_2 x)
\]
defines a mapping \( I \times A \to \mathbb{R} \) making \( t_1, t_2 \) homotopic mappings \( A \to \mathbb{R} \). Since \( T(u \times x) \in \lambda(x) \) we have \( T(I \times A) \subset B \), so that \( t_1, t_2 \) are in fact homotopic as mappings \( A \to B \) also.

(47.5) Retraction. This convenient concept, formulated by Borsuk, is closely related to homotopy.

(47.6) Definitions. Let \( A, B \) be topological spaces, with \( A \subset B \). A retraction of \( B \) onto \( A \) is a mapping \( t: B \to A \) such that \( t \mid A = 1 \). When \( t \) exists \( A \) is called a retract of \( B \). If \( t \) is a deformation keeping every point \( x \) of \( A \) fixed (i.e., \( x \) is its own path) then \( t \) is also called a deformation retraction and \( A \) is then said to be a deformation retract of \( B \).

The notations being the same \( A \) is called a neighborhood retract of \( B \) when it has a neighborhood in \( B \) for which it is a retract.