CHAPTER III

COMPLEXES

A complex is a particular type of partially ordered set with complementary properties designed to carry an algebraic superstructure, its homology theory. Complexes thus appear as the tool par excellence for the application of algebraic methods to topology.

For the present we shall deal chiefly with finite complexes and give a complete treatment of their homology and cohomology groups and duality theory. Polyhedral and Euclidean complexes are discussed as special examples. Infinite complexes are likewise considered as well as a special class, the simple complexes, introduced by A. W. Tucker, and may be said to have all the main algebraic attributes of the polyhedral type. It is for simple complexes that an intersection theory is developed in (V), and the combinatorial manifolds of (V) are also simple complexes.

Summation notation. It is the same as in tensor calculus: non-dimensional indices (usually clear from the context) repeated up and down are to be summed unless an explicit statement is made to the contrary. Thus $g'x_i$ stands for $\sum_i g'x_i$.

Kronecker deltas. They are the well known numbers defined by $\delta^i_j = 0$ for $i \neq j$, $\delta^i_i = 1$ for $i = j$.

Designations for some special groups. We will write as in (II): $\mathbb{Z}$ = the group of the integers, $\mathbb{Z}_m$ = the group of the residues mod $m$, $\mathbb{B}$ = the group of the reals mod 1, $\mathbb{R}$ = the additive group of the rational numbers (rational group).

If $G = \{g\}$ is any group then $\langle g \alpha \rangle$ is a group under the composition law $g\alpha - g'\alpha = (g - g')\alpha$, and this group is written $Ga$. The designations $G(m)$, $G^*(m)$, $G[m]$ are as in (II, 20.9).

The function $\beta(p)$. Convenient in many calculations it is defined by

$$\beta(p) = (-1)^{\frac{p(p-1)}{2}},$$

and we notice the useful relations:

$$\beta(-p) = (-1)^{\frac{p(p+1)}{2}},$$

$$\beta(p)\beta(q) = (-1)^{pq}\beta(p + q).$$

General references: Alexander [b, c], Alexandroff [f], Alexandroff-Hopf [A-H, Part 2], Hopf [a], Lefschetz [L, I, VII; L4], Mayer [a, c], Poincaré [b], Seifert-Threlfall [S-T], Steenrod [a], Tucker [a], Veblen [V], Whitney [d].

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§1. COMPLEXES. DEFINITIONS AND EXAMPLES

1. (1.1) Definitions. A complex $X$ is a set $\{x\}$ of elements ordered by a proper reflexive ordering relation $<$ (I, 4) together with two associated functions of the elements and element pairs whose values are integers: one, $\text{dim } x$, the dimension of $x$, also denoted by means of an index as $x^p$, the element then being called $p$-dimensional or a $p$-element, and the other, $[x:x']$, the incidence number of $x$ and $x'$, subject to the following conditions:

K1. $x' \prec x \iff \text{dim } x' \leq \text{dim } x$;
K2. $[x:x'] = [x':x]$;
K3. $[x:x'] \neq 0 \Rightarrow x < x'$ or $x' < x$, and $|\text{dim } x - \text{dim } x'| = 1$.
K4. For every pair of elements $x, x''$ whose dimensions differ by two there is at most a finite number of $x'$ such that $[x:x'][x''':x''] \neq 0$ and then

\[ \sum_{x'} [x:x'][x':x''] = 0. \]

When the complex is finite K4 may be replaced by the simpler condition:
K4'. For every pair of elements $x, x''$ whose dimensions differ by two, the relation (1.2) holds.

The dimension of $X$, written $\text{dim } X$ is $\text{sup dim } x$. When its value $n$ is finite $X$ is sometimes called an $n$-complex.

(1.3) Let $\alpha(x)$ be a function of $x$ whose values are $\pm 1$. If $[x:x']$ is replaced by $\alpha(x)\alpha(x')[x:x']$ conditions K1234 are fulfilled and so we still have a complex, say $X'$. In conformity with the usual conventions we agree to consider $X'$ as identical with $X$. Thus the function $[ : ]$ for a given $X$ is to be considered as not unique but only given to within a factor $\alpha(x)\alpha(x')$. The different sets of incidence numbers thus arising are said to be admissible, the passage from one to the other is described as reorienting $X$. The function $\alpha(x)$ is known as an orientation function and we say that $x$ has been reoriented if $\alpha(x) = -1$, and that it has preserved its orientation otherwise.

(1.4) Remark. The definition of complexes adopted here is essentially Tucker’s [a] and differs from his only in that: (a) the dimensions are not restricted to being greater than or equal to 0 which will be of importance in dealing with duality; (b) the complexes need not be finite. Indeed persistent attention to infinite complexes will characterize our treatment.

The complexes which we have just introduced are often called “abstract complexes.” Other general types have been considered in the literature notably by M. H. A. Newman [a] and W. Mayer [a]. Newman’s type is designed chiefly to preserve as many as possible of the properties of polyhedra and for many purposes it is decidedly too “geometric.” In Mayer’s type on the other hand only the properties which flow from the incidence numbers are preserved and the type is thus too “algebraic.” Tucker’s type may be said to occupy a reasonable intermediate position.

2. There are three important sets associated with any element $x \in X$: the star of $x$, written $\text{St } x$, the closure of $x$, written $\text{Cl } x$ and the boundary of $x$, written $\partial x$. Their defining relations are
St \( x = \{ x' \mid x < x' \} \), \quad Cl \( x = \{ x' \mid x' < x \} \),
\[ \emptyset x = Cl x - x = \{ x' \mid x' < x, x' \neq x \}. \]

An analogue \( St x - x \) of \( \emptyset x \) may be formally introduced but will not be needed in the sequel.

We say that \( x' \) is a face of \( x \) when \( x' < x \) (a proper face when \( x' \neq x \)), also that \( x \) and \( x' \) are incident when \( x' > \) or \( < x \). By the incidence relations in \( x \) we shall mean the incidences \( < \) together with all the incidence numbers.

The notions of star, closure and boundary of a single element may be generalized as follows: if \( Y \) is any subaggregate of \( X \) the star of \( Y \), written \( St Y \), is the union of the stars of all the elements of \( Y \). Similarly the closure of \( Y \), written \( Cl Y \), is the union of the closures of all the elements of \( Y \). \( St Y \) is the union of all the elements \( > \) some element of \( Y \), while \( Cl Y \) is the union of all the elements \( < \) some element of \( Y \). The boundary of an open subcomplex \( Y \) is \( \emptyset Y = Cl Y - Y \).

A subcomplex of \( X \) is a subaggregate \( Y = \{ x' \} \) of \( X \) such that with the same dimensions and incidence relations as in \( X \), conditions K1234 hold in \( Y \) alone. It is clear that the verification of the complex conditions for \( Y \) merely requires the verification of K4 alone.

We say that the subcomplex \( Y \) of \( X \) is
- open whenever \( St Y = Y \), or \( x \in Y \leftrightarrow St x \subseteq Y \);
- closed whenever \( Cl Y = Y \), or \( x \in Y \leftrightarrow Cl x \subseteq Y \).

Immediate consequences are:

(2.1) If one of the sets \( Y, X - Y \) is an open subcomplex the other is a closed subcomplex, and conversely.

(2.2) Any union or intersection of open or closed subcomplexes is, respectively, an open or a closed subcomplex.

The aggregates \( St Y, Cl Y, \emptyset x \), are subcomplexes of \( X \). The proof merely requires that we verify K4. Let us do so for the first. If \( x, x'' \in St Y \), the only significant contribution to \( \sum_{x' \in X, x' < x} [x:x'][x':x''] \) occurs say when \( x < x' \) and from elements \( x' \) between both. But in that case \( x' \in St Y \) also, so that the relation in question holds in \( St Y \) alone. Evidently \( St x \) is an open subcomplex, and \( Cl x \) a closed subcomplex; since \( Cl x \) is closed so is \( \emptyset x \).

The \( p \)-section \( X^p \) of \( X \) is the set of all the elements of \( X \) whose dimension does not exceed \( p \). \( X^p \) is likewise a closed subcomplex of \( X \). For the union of all \( St x \), \( \dim x > p \), is an open subcomplex and \( X^p \) is its complement.

When the dimensions of the elements of \( X \) are greater than or equal to 0, the zero-dimensional elements are frequently called the vertices of \( X \).

(2.3) Connectedness and components. The component of any element \( x \) is the set of all \( x' \) such that there exists a finite collection \( x = x_1, \ldots, x_r = x' \) in which any two consecutive elements are incident. We will then say briefly that \( x, x' \) are in the relation \( R \). It is not difficult to see that:

(2.4) Properties (I, 17.1, \ldots, 17.4) hold for \( X \) and the present definition of components.
Furthermore we also prove readily:

(2.5) The relation $R$ is equivalent to each of the following:

(a) $\text{St} \ x, \text{St} \ x'$ are in the same component of $\{\text{St} \ x\}$ in the sense of (I, 17);

(b) $\text{Cl} \ x, \text{Cl} \ x'$ are in the same component of $\{\text{Cl} \ x\}$ in the same sense.

It is also a consequence of the definition that

(2.6) The component $X'$ of $x$ contains both $\text{Cl} \ x$ and $\text{St} \ x$. Hence $X'$ is both an open and a closed subcomplex of $X$.

The complex $X$ is said to be connected whenever it consists of a single component, i.e., when any two elements are in the relation $R$.

3. The complex $X$ is said to be star-finite, or closure-finite whenever every $\text{St} \ x$ or $\text{Cl} \ x$ is finite, and to be locally finite when it has both properties. Notice these properties:

(3.1) Finiteness $\rightarrow$ local finiteness.

(3.2) When $X$ is star- or closure-finite there is at most a finite number of elements between $x$ and $x''$ and hence $K4$ may then be replaced by the simpler condition $K4'$.

(3.3) Every component of a locally finite complex $X$ is countable.

Let $Y$ be a component of $X$ and $x$ any element of $Y$. Consider the sequence $Y_1 = x, Y_2, \ldots$, where $Y_{n+1} = \text{St} \text{Cl} \ Y_n$. Every $Y_n$ is finite and since $\cup Y_n = Y$, $Y$ is countable.

4. Let $X_1 = \{x_1\}, X_2 = \{x_2\}$ be two complexes and suppose that there exists a one-one, order-preserving transformation $T: \{x_1\} \rightarrow \{x_2\}$ such that: (a) $\dimTx_2 = \dim x_1 + k$ where $k$ is a fixed integer; (b) the numbers $[x_1 : x_i]$ are appropriate incidence numbers for $Tx_1, Tx_i$. Whenever $k = 0$ we call $T$ an isomorphic transformation or isomorphism $X_1 \leftrightarrow X_2$ and say that $X_1$ and $X_2$ are isomorphic. When $k \neq 0$ we say that we have a weak isomorphism and refer to $X_1, X_2$ as weakly isomorphic.

It is clear that these two types of isomorphisms give rise to equivalence classes but we will not refer to them particularly in the sequel as they are not sufficiently broad for the applications.

Dual complex. Given the complex $X = \{x\}$, let us introduce a new set of elements $X^* = \{x^*\}$ such that $x \leftrightarrow x^*$ is a one-one correspondence with the following properties: (a) $x < x' \iff x^* < x^*$; (b) $\dim x^* = - \dim x$; (c) $[x^* : x^*'] = [x : x']$. We verify immediately that conditions K1234 continue to be fulfilled, so that $X^*$ is also a complex. It is known as the dual of $X$. Clearly $X^{**} = (X^*)^* \cong X$. If we agree to choose as the elements $x^{**}$ the elements $x$ themselves: $x^{**} = x$, then we will have $X^{**} = X$. Thus $X, X^*$ will be dual to one another.

By way of notation if $x^*_i$ is any element of $X$ its image $x^*_i$ in $X^*$ is conveniently denoted by $x^*_i$. Thus when the dimensional index of an element is a subscript it denotes the negative of the true dimension of the element.

The reader will not have missed the fact that our definitions have been so couched as to continue to give free play to the dualism which permeates the general theory of ordered sets. This is the chief justification for imposing symmetry in the incidence numbers and for introducing negative dimensions. The so-called dual complexes hitherto considered in
5. Simplicial complexes. As we shall see later (VII, VIII) this is the dominant type wherever complexes occur in topology.

(5.1) We must first define the simplex. A \( p \)-simplex \( \sigma^p \) is merely any set of \( p + 1 \) objects \( \{A_0, \ldots, A_p\} \), known as the vertices of \( \sigma^p \). It will generally be assumed that they are assigned a definite order modulo an even permutation and we will write accordingly \( \sigma^p = A_0 \cdots A_p \), the specified order being the one in which the \( A_i \) are named. The simplex \( \sigma \) with its vertices ordered as just stated is said to be oriented. The number \( p \) is the dimension of \( \sigma^p \). The simplices whose vertices are among those of \( \sigma^q \) are known as the faces of \( \sigma^p \), more precisely its \( q \)-faces for those of dimension \( q \). In particular \( \sigma^0 \) has \( p + 1 \) zero-faces, the vertices \( A_i \), and a single \( p \)-face, namely itself.

If \( \sigma_1 = A_0 \cdots A_q \), \( \sigma_2 = A_{q+1} \cdots A_p \), where \( A_0, \ldots, A_p \) are all distinct, we write \( \sigma^p = \sigma_1 \sigma_2 \), call \( \sigma^p \) the join of \( \sigma_1, \sigma_2 \) and also \( \sigma_1, \sigma_2 \) opposite faces of \( \sigma^p \). The symbol \( \sigma_1 \cdots \sigma_p \) is defined by recurrence.

(5.2) We come now to the simplicial complex. The elements of a simplicial complex \( K \) are simplices. They make up a set \( \{\sigma\} \) such that if \( \sigma \) is in the set then every face of \( \sigma \) is likewise in the set. The dimension of \( \sigma \) is as defined in (5.1). The relation \( \sigma' < \sigma \) means that \( \sigma' \) is a face of \( \sigma \) in the sense of (5.1).

The incidence numbers are defined as follows:

\( (d) \) If \( \sigma_1 \) is the face opposite the vertex \( A \) in \( \sigma \) set \( \epsilon = \pm 1 \) according as \( \sigma \) is or is not ordered like \( A\sigma_1 \); set \( \epsilon = 0 \) in all the other cases (\( \sigma_1 \) not a face opposite a vertex of \( \sigma \) nor the other way around). Then \( [\sigma_1 : \sigma] = [\sigma : \sigma_1] = \epsilon \).

Example. \( \sigma = A_1A_2A_3, \sigma_1 = A_1A_3 \). Since \( A_1A_3A_4 \) is not \( A_1A_2A_4 \) modulo an even permutation we have \( [\sigma_1 : \sigma_1] = -1 \). On the other hand for instance \( [\sigma : A_1] = 0 \).

It remains to verify that \( K \) is a complex. Since \( K^{123} \) are manifestly satisfied it is only necessary to verify \( K^4 \). It reduces here to:

\[ \sum_{\sigma_1} [\sigma : \sigma_1][\sigma_1 : \sigma_2] = 0. \]

The verification is trivial unless the situation may be so arranged that \( \sigma_2 \) is the opposite face of a one-simplex \( AB \) of \( \sigma \), and \( \sigma_1 \) is then, except for ordering \( \sigma_1 \), one of the simplices \( A\sigma_2, B\sigma_2 \). Furthermore if the order is changed in any simplex occurring in (5.3) the left-hand side will at most change sign. Therefore the order of the vertices may be chosen as specified by the above symbols. And now (5.3) reduces to

\[ [AB\sigma_2:A\sigma_2][A\sigma_2:B\sigma_2] + [AB\sigma_2:B\sigma_2][B\sigma_2:A\sigma_2] = 0. \]

Therefore \( K \) is a complex.

(5.4) If the ordering of the vertices is changed the effect upon the incidence numbers is the same as applying to \( K \) an orientation function \( \alpha(\sigma) = (-1)^d \),
where $v$ is 0 or 1 according as the permutation of the vertices of $\sigma$ is even or odd. Under our conventions this does not modify $K$.

To orient a simplicial complex $K = \{\sigma\}$ is to orient every $\sigma \in K$. The order assigned to the vertices of each $\sigma$ is the one which is to serve in calculating the incidence numbers in (5.2). A convenient mode often utilized in orienting $K$ is to range its vertices $\{A_i\}$ in a definite order, then to orient every $\sigma$ as $\sigma = A_i \cdots A_j$, $i < \cdots < j$.

(5.5) **Special terms.** A one-dimensional complex is sometimes called a linear graph, or merely a graph. If $L$ is a closed subcomplex of the simplicial complex $K$ then $L$ is also a simplicial complex. The complement $K - L$ is known as an open simplicial complex. By contrast $K$ itself is sometimes called a closed simplicial complex.

The boundary $S^n = \partial \sigma^{n+1}$ of a $\sigma^{n+1}$ is a closed simplicial $n$-complex, sometimes called an $n$-sphere. The zero-sphere $S^0$ consists of two vertices.

(5.6) While $K$ may be infinite, it is clearly closure-finite but need not be star-finite.

(5.7) An alternate scheme for the incidence numbers. It is convenient on occasion to define the incidence numbers in the following way: If $\sigma = \sigma'A$ then the new incidence numbers, denoted temporarily by $[\ : \ ]'$ are $[\sigma: \sigma']' = [\sigma': \sigma] = 1$ and all the other incidence numbers are zero. Clearly $[\sigma': \sigma']' = (-1)^n[\sigma': \sigma']$. That is to say, the new incidence numbers correspond to reorientation by means of $\alpha(\sigma') = \beta(-p)$, where $\beta(p)$ is as in the Introduction. Thus they are admissible incidence numbers for $K$.

(5.8) **Remark.** Unless otherwise stated the incidence numbers will always be selected in accordance with (5.2).

(5.9) **Duals.** The dual $K^*$ of the simplicial complex $K$ is defined as for any complex. Its elements are denoted by $\sigma_p^*$, and in particular the dual of $A_i$ is written $A^i$. If $\sigma_p^* = A^j \cdots A^k$ we write $\sigma_p^* = A^j \cdots A^k$, and call $A^j$, $\cdots$, $A^k$ the vertices of $\sigma_p^*$. If $\sigma = A_0 \sigma^{n-1}$ or $\sigma' = \sigma^0 \sigma'$ then we also write $\sigma_\sigma = A^k \sigma_{p-1}$ or $\sigma_p = \sigma_{p} \sigma_{q}$, as the case may be. The incidence numbers may be defined directly in $K^*$ as in $K$ by the rule $[A_\sigma : \varepsilon] = [\sigma : A_\varepsilon] = 1$ in the case of (5.2), $[\sigma : A_\varepsilon] = [\sigma : A_\varepsilon] = 1$ in the case of (5.7), and all the other $[\ : \ ]$ zero. We also have $\dim A^k = 0$, $\dim \sigma_\sigma = -p$ and $\sigma_\sigma < \sigma_q$ signifies that the set of vertices of $\sigma_\sigma$ contains the set of vertices of $\sigma_q$. In other words, the passage from $K$ to $K^*$ consists essentially: (a) in ordering the subsets of $\{A^i\}$ by the inclusions of their complements; (b) in replacing the dimensions by their negatives.

6. **Polyhedral complexes.** We will consider polyhedral complexes in an Euclidean space $\mathbb{E}^n$ and indicate the extension to those in the Hilbert parallelootope (6.14).

(6.1) **Definitions.** A polyhedral complex in $\mathbb{E}^n$ is a countable locally finite complex $\Pi = \{E\}$ with the following properties:

(a) a $p$-dimensional element $E^p$ is a $p$-cell which is a bounded convex region of some $\mathbb{E}^p$ of $\mathbb{E}^n$;
(b) the cells are disjoint;
(c) the union of the cells of $\text{Cl} \ E^p$ is $\bar{E}^p$;
(d) If $\varphi(E)$ is the union of the cells $E'$ of $E$, then $\varphi(E) \cap E = \emptyset$.

The set $UE$ is known as a polyhedron, written $|U|$.

Notice that (c) implies that $E' < E$ when and only when it is $E$ or else a cell $\subset \bar{E} - E$. In other words, $E' < E$ is equivalent to: $E' = E$ or else $E'$ is a face of $E$ in the commonly accepted sense.

Evidently (d) holds automatically when the polyhedral complex is finite. Its purpose is to eliminate certain topological complications which are foreign to the structure of complexes (see 6.2).

The incidence numbers are described below but they necessitate an extensive discussion of Euclidean coordinates.

(6.2) Examples. The regular solids are well known finite polyhedral complexes. The subdivision of the plane by the lines $x, y = 0, \pm 1, \pm 2, \cdots$, is a good example of an infinite polyhedral complex. On the other hand the set of segments:

$$l_n : 0 \leq x \leq 1, y = 0; \quad l_n : 0 \leq x \leq 1, y = \frac{1}{n}, \quad n = 1, 2, \cdots,$$

is not a polyhedral complex since $\varphi(l_0) = \bigcup l_n$ is such that $\varphi(l_0) \supset l_n$.

It may be noticed that under our definition a given point set $A$ may admit of a decomposition in disjoint cells in two distinct ways, one of which gives rise to a polyhedral complex, and the other fails to do so. In particular (c) may cease to hold. Thus let $\lambda$ be the set $0 < u \leq 1$. The interval together with $u = 1$ is not a polyhedral complex. However, the intervals $1/(n+1) < u < 1/n$ together with their end points decompose $\lambda$ into the cells of a polyhedral complex.

(6.3) The incidence numbers in $\Pi$ will be described in terms of auxiliary coordinate systems in the spaces of the cells. As is well known, an Euclidean space $\mathbb{E}^n$ may be viewed as a linear variety $L$ in a real vector space $\mathbb{B}$, i.e., a vector space over the field of reals. An $\mathbb{E}^p$ in $\mathbb{E}^n$ is a linear $p$-dimensional variety $L'$ contained in $L$. The points of $\mathbb{E}^p$ may be represented as vectors $a + z$, where $|z|$ spans a $p$-subspace $\mathbb{B}_1$ of $\mathbb{B}$. If $\{b^1, \cdots, b^p\}$ is a base for $\mathbb{B}_1$ then we have $x = x_i b^i$, and $\{x_1, \cdots, x_p\}$ is a coordinate system for $\mathbb{E}^p$. The point $a$ is known as the origin of the system.

We will suppose once for all that every $\mathbb{E}^p$ has been assigned a definite coordinate system called its basic coordinate system. Let $\{x_1, \cdots, x_p\}$ be the one of $\mathbb{E}^p$. Then if $\{y_1, \cdots, y_p\}$ is any coordinate system for $\mathbb{E}^p$ we will have:

$$y_i = a^i_j x_j + a_i, \quad \alpha = |a_i^j| \neq 0.$$

The number $\varepsilon^p = \alpha / |\alpha| = \pm 1$ is known as the characteristic number of the coordinate system $\{y_i\}$.

An $\mathbb{E}^{p-1} \subset \mathbb{E}^p$ partitions $\mathbb{E}^p$ into two convex regions $\mathbb{E}^p, \mathbb{E}^{p'}$. We may choose
a coordinate system \( \{ x_i \} \) for \( \mathcal{G}^p \) such that \( x_p = 0 \) represents \( \mathcal{G}^{p-1} \). The two regions \( \mathcal{G}^p, \mathcal{G}^{p-1} \) are then the two sets \( x_p > 0, x_p < 0 \). Since we may choose a new coordinate system in which \( x_p \) is replaced by \(-x_p\), and the other coordinates are unchanged, we may assume \( \{ x_i \} \) such that say \( \mathcal{G}^p \) is the region \( x_p > 0 \). The coordinates \( \{ x_1, \ldots, x_{p-1} \} \) of any point of \( \mathcal{G}^{p-1} \) define a coordinate system for \( \mathcal{G}^{p-1} \), with characteristic number say \( \varepsilon^{p-1} \). We will introduce incidence numbers

\[
(6.3b) \quad [\mathcal{G}^p : \mathcal{G}^{p-1}] = -[\mathcal{G}^{p-1} : \mathcal{G}^{p-2}] = \varepsilon^{p-1}.
\]

Let the basic coordinate system \( \{ x_i \} \) of \( \mathcal{G}^p \) serving to determine the characteristic numbers be replaced by another \( \{ \tilde{x}_i \} \). We have then relations

\[
(6.3c) \quad \tilde{x}_i = m_i^i x_i + n_i, \quad \mu = |m_i^i| \neq 0.
\]

If \( \mu > 0 \) the characteristic numbers are unchanged, if \( \mu < 0 \) they are all changed in sign. To orient \( \mathcal{G}^p \) is to assign to it a coordinate system \( \{ x_i \} \) modulo a transformation (6.3c) with \( \mu > 0 \). It is said to have its orientation reversed if the basic coordinate system undergoes a transformation (6.3c) with \( \mu < 0 \).

(6.4) We are now ready for the incidence numbers of \( \Pi \). Let \( \mathcal{G}^p \) denote the subspace of the space \( \mathcal{G}^n \) of \( \Pi \) containing \( E^p_i \) and suppose \( E^p_{\mu-1} < E^p_i \). The subspace \( \mathcal{G}^{p-1}_i \) divides \( \mathcal{G}^p_i \) into two regions one of which, say \( \mathcal{G}^{p-1}_i \) contains \( E^p_i \), and we define \([E^p_i : E^{p-1}_i] = [E^p_i : E^{p-1}_i] = (\mathcal{G}^p_i : \mathcal{G}^{p-1}_i] = \pm 1\). All the other incidence numbers which are not determined by this rule are set equal to zero. Thus K123 hold and we merely have to verify K4.

(6.5) Suppose \( E^{p-2}_i < E^p_i \). In \( \mathcal{G}^p_i \) let \( \mathcal{G}^p_k \) be a plane meeting \( \mathcal{G}^{p-2}_k \) at a single point \( A \in E^p_k \). The intersection of \( \mathcal{G}^p_i \) with \( E^p_k \) is a closed convex plane polygonal region with the vertex \( A \). In such a region each vertex is incident with exactly two edges. Hence \( E^{p-2}_i \) is the common face of exactly two \((p-1)\)-faces \( E^{p-2}_i, E^{p-1}_i \) of \( E^p_i \). There are now two possibilities:

(a) \( \mathcal{G}^{p-2}_i = \mathcal{G}^{p-1}_i \). Let \( \mathcal{G}^{p-1}_i, \mathcal{G}^{p-1}_j \) be the two regions of the partition of \( \mathcal{G}^{p-1}_i \) by \( \mathcal{G}^{p-2}_i \). Since \( E^{p-1}_i \neq E^{p-2}_i \) and they have the common face \( E^{p-2}_i \), there must be one in each of the two regions, say \( E^{p-1}_i \subset \mathcal{G}^{p-1}_i, E^{p-1}_j \subset \mathcal{G}^{p-1}_j \). Hence under our definition of the incidence numbers:

\[
\]

On the other hand:

\[
[E^p_i : E^{p-1}_i] = [E^p_i : E^{p-1}_j] = (\mathcal{G}^p_i : \mathcal{G}^{p-1}_n] = \pm 1.
\]

Hence

\[
(6.6) \quad \sum_m [E^p_i : E^{p-1}_m] [E^{p-1}_m : E^{p-2}_k] = 0
\]

and K4 holds.

(b) \( \mathcal{G}^{p-1}_i \neq \mathcal{G}^{p-1}_j \). Choose a coordinate system \( \{ x_1, \ldots, x_p \} \) for \( \mathcal{G}^p_i \) such that \( \mathcal{G}^{p-1}_i, \mathcal{G}^{p-1}_j \) are, respectively, represented by \( x_{p-1} = 0 \) and \( x_p = 0 \) and that on \( E^p_i \) : \( x_{p-1} > 0, x_p > 0 \). As a consequence on \( E^{p-1}_i : x_p > 0 \) and on \( E^{p-1}_j : x_{p-1} > 0 \).
$x_{p-1} > 0$. For we must have, respectively, $x_p \geq 0$, $x_{p-1} \geq 0$ and equality is excluded. To proceed further let us introduce the characteristic numbers $\epsilon_1^p, \epsilon_2^p, \epsilon_3^p, \epsilon_4^p, \epsilon_5^p$ of $\mathcal{C}_1^p, \mathcal{C}_2^p, \mathcal{C}_3^p$ with respect to the coordinate systems $\{x_1, \ldots, x_p\}, \{x_1, \ldots, x_{p-2}, x_p\}, \{x_1, \ldots, x_{p-1}\}, \{x_1, \ldots, x_{p-2}\}$. We have then

$$[E_1^p : E_1^{p-1}] = -\epsilon_1^p \epsilon_2^{p-1}; \quad [E_1^p : E_2^{p-1}] = \epsilon_1^p \epsilon_3^{p-1};$$

$$[E_1^p : E_3^{p-2}] = \epsilon_2^p \epsilon_4^{p-2}; \quad [E_1^p : E_4^{p-2}] = \epsilon_3^p \epsilon_5^{p-2}. $$

That the last three incidence numbers have the correct value is clear. Regarding the first, to determine it we may utilize for $\mathcal{C}_1^p$ the coordinate system $\{x_1, \ldots, x_{p-2}, x_{p-1}, x_p\}$ with $x_p = x_{p-1}, x_{p-1} = x_p$. This system has the characteristic $\epsilon_1^p = -\epsilon_3^p$. The space $\mathcal{C}_1^{p-1}$ is now represented by $x_p = 0$, and has the coordinate system $\{x_1, \ldots, x_{p-2}, x_{p-1}\}$ with the same characteristic $\epsilon_1^{p-1}$ as before. Hence

$$[E_3^p : E_1^{p-1}] = \epsilon_1^p \epsilon_3^{p-1} = -\epsilon_1^p \epsilon_3^{p-1}.$$ 

Substituting the incidence numbers in (6.6) we find that this relation holds here also. Thus $K_4$ is fulfilled in all cases and so $\Pi$ is a complex.

(6.7) The characteristic numbers $\epsilon_3^p$ have been defined throughout relative to fixed orientations of the spaces $\mathcal{C}_1^p$. If these are modified one will obtain new characteristic numbers $\epsilon_3'^p$. Setting $\alpha(E^p) = \epsilon_1^p \epsilon_2^p$ we find then that the effect upon the incidence numbers $[E_3^p : E_1^{p-1}]$ is equivalent to applying the orientation function $\alpha(E)$. Thus the polyhedron as a complex is independent of the basic coordinate systems which serve to determine the $\epsilon_3^p$ and hence the incidence numbers.

(6.8) A polyhedral complex is countable and locally finite.

This is an immediate consequence of the definition (6.1).

(6.9) Euclidean complexes. We must first define Euclidean simplices, the constituent parts of Euclidean complexes.

We will consider again a fixed $\mathcal{C}^n$ and as in (6.3) take a representation of the space as a subset of a real vector space $\mathbb{R}$. Let then $\sigma'$ be a simplex whose vertices $\{a_i\}$ are independent points of $\mathcal{C}^n$, that is to say, contained in no $\mathcal{C}_i^{n-1}$ of $\mathcal{C}^n$ or equivalently contained in a unique $\mathcal{C}_i$ of $\mathcal{C}^n$. We associate with $\sigma'$ a set $\sigma''$, known as an Euclidean $p$-simplex, and composed of the points of $\mathcal{C}^n$ given by:

(6.10) $x = y^i a_i, \quad 0 < y^i < 1, \quad \sum y^i = 1, \quad p > 0; \quad x = a_i, \quad p = 0.$

The $y^i$ are the barycentric coordinates of $x$. To the face $\sigma' = a_i \cdots a_j$ of $\sigma''$ there corresponds the set of points obtained by replacing above $0 < y^i$ by $0 = y^i$ for $k \neq i, \ldots, j$. It is the $a_i$ associated with $\sigma'$ and is known as a face of $\sigma''$. We transfer to $\sigma''$ and all its faces the terminology and concepts introduced for $\sigma'$ and thus we have notably the incidences, and incidence numbers of (5).

(6.11) $\sigma''$ is a $p$-cell; its boundary $\mathcal{B}\sigma''$ is a topological $(p - 1)$-sphere and $\sigma''$ is a closed $p$-cell.
Let \( \mathbb{G}^p \) be the space of \( \sigma^p \) (i.e., the \( \mathbb{G}^p \supset \sigma^p \)). It is an immediate consequence of (6.10) that: (a) \( \sigma^p \) is convex; (b) if \( A \in \sigma^p \), any ray \( AL \) issued from \( A \) in \( \mathbb{G}^p \) intersects \( \sigma^p \) in a segment \( AB, B \neq A, B \in \partial \sigma^p \). Now \( \sigma^p \) is contained in some parallelootope \( P \) of \( \mathbb{G}^n \). Since \( \partial \sigma^p \) is thus a closed subset of the compactum \( P \) it is likewise a compactum. Hence \( A \in \partial \sigma^p \rightarrow d(A, \partial \sigma^p) > \rho > 0 \) (I, 45.2) and therefore \( \mathbb{G}(A, \rho) \cap \mathbb{G}^p \subset \sigma^p \). Thus \( A \) is an interior point of \( \sigma^p \) in \( \mathbb{G}^p \) and so \( \sigma^p \) is a convex region of \( \mathbb{G}^p \). Hence (6.11) is a consequence of (I, 12.9).

(6.12) Let now \( K = \{ \sigma \} \) be a countable locally finite simplicial complex whose vertices \( \{ a_i \} \) are points of an \( \mathbb{G}^n \) and such that:

(a) the vertices of any \( \sigma^p \in K \) are independent and therefore determine an Euclidean simplex \( \sigma^p \) of \( \mathbb{G}^n \);

(b) \( \sigma \neq \sigma' \rightarrow \sigma \cap \sigma' = \emptyset \);

(c) if \( \varphi(\sigma_i) \) is the union of all the \( \sigma_i' \), then \( \varphi(\sigma_i) \cap \sigma = \emptyset \).

If we transfer to \( |\sigma_i| \) the dimensions, incidences "is a face of," and incidence numbers prevailing in \( K \), it becomes a complex \( K^* \cong K \), known as an Euclidean complex. We also speak of \( K^* \) as an Euclidean realization of \( K \), of \( K \) as an antecedent of \( K^* \).

It follows from the definition of \( K^* \) that every \( x \in \sigma^p \in K^* \) satisfies a relation

\[ x = y^{i_1}a_{i_1} + \cdots + y^{i_p}a_{i_p} \]

where if \( x \in \sigma^p \), and \( \sigma \) has the vertices \( a_{i_1}, \ldots, a_{i_p} \), then \( y^{i_1}, \ldots, y^{i_p} \neq 0 \), and all the other \( y^j \) are zero. The \( y^j \) are the barycentric coordinates of \( x \) and are uniquely determined by the point.

(6.13) \( K^* \) is a polyhedral complex.

Let \( K^* \) be the \( q \)-section of \( K \). Since (6.13) is trivial for \( K^* \), we may assume it for \( K^{p-1} \) and prove it for \( K^p \). At all events all the requisite conditions except those referring to the incidence numbers are fulfilled. Thus we merely have to show that admissible incidence numbers of \( K \) are suitable for \( K^* \) as a polyhedral complex.

Let the vertices \( \{ a_i \} \) be ranged in some fixed order and let \( \sigma^p, \sigma_r^p \) have the common vertices \( a_{i_0}, \ldots, a_{i_q}, i_0 < \cdots < i_q \). If \( \mathbb{G}^p \) is the space of \( \sigma^p \), we agree to choose as its basic coordinate system \( \{ x_1, \ldots, x_p \} \) a system with origin at \( a_{i_0} \) and such that \( a_{i_0}, h > 0 \), has the coordinates \( h^k \) (Kronecker deltas). Let \( \sigma^{p-1}, \sigma_r^{p-1} \) be the faces of \( \sigma, \sigma^p \) with the vertices \( a_{i_0}, \ldots, a_{i_{q-1}}, a_{i_{q-1}}, \ldots, a_{i_q} \) and let the incidence numbers in \( K \) be defined by taking the vertices in the increasing order of the subscripts and in accordance with (5.7). Then

\[ [\sigma^p: \sigma_q^p] = (-1)^{p-q} = [\sigma^p_r: \sigma_r^{p-1}] \]

Therefore (6.13) is proved.

(6.14) Polyhedral complexes in the Hilbert parallelootope. Let \( P^n \) be referred to the coordinates \( \{ x_1, x_2, \ldots \} \), \( 0 \leq x_n \leq 1/n \). Consider a real vector space \( \mathbb{B} \) defined as follows: its elements are all the ordered countable sets of real numbers \( y = \{ y_1, y_2, \ldots \} \); if \( a \) is real then \( ay = \{ ay \} \); if \( y' = \{ y'_1 \} \) then \( y + y' = \{ y_1 + y'_1 \} \). We may identify \( P^n \) with a subset of a real variety \( L \) in \( \mathbb{B} \), where
L is so chosen that it contains no proper linear subvariety \( \supseteq P' \). If we interpret everywhere an \( \mathcal{G}^p \) as being a set \( L' \subseteq P' \), where \( L' \) is a linear \( p \)-dimensional variety contained in \( L \), then all the preceding considerations are applicable.

We will thus obtain polyhedral complexes, Euclidean simplexes and complexes in \( P' \) which have exactly the same properties as before.

(6.15) If \( K_\sigma = \{ \sigma \} \) then in accordance with (6.1) the set \( \bigcup_{\sigma_r} \) is designated by \( | K_\sigma \rangle \). Similarly if \( K_\sigma \) is a closed subcomplex of \( K_\sigma \) then the union of the \( \sigma_r \in K_\sigma - L_\sigma \) is written \( | K_\sigma - L_\sigma \rangle \).

§2. HOMOLOGY THEORY OF FINITE COMPLEXES

(a) GENERALITIES

7. The group-theoretic role of the algebraic structure imposed upon a complex receives its full significance through the medium of the chain-groups and certain associated subgroups and factor-groups. The groups related to finite complexes are to be investigated first and to the full. They will serve as a fundamental pattern for all later developments.

7.1 Let then \( X = | x \rangle \) be a finite complex, and \( G \) an additive group. In the terminology of (II, 8) we use \( \{ x_i \} \) as a base to form the \( p \)-chains over \( G \), or chains

\[ C^p = g^i x_i^p, \quad g^i \in G. \]

These are all finite since \( X \) is finite, and their group \( \mathcal{P} (Gx^p) \) is denoted by \( \mathcal{G}^p (X, G) \). Instead of "\( C^p \) is a chain of \( X \)," we will also say more simply "\( C^p \) is contained in \( X \)," written \( C^p \subseteq X \).

7.2 If there are no \( p \)-elements it is convenient to introduce formally a group \( \mathcal{G}^p (X, G) \) consisting solely of zero.

7.3 The set of all the \( x_i^p \) appearing in \( C^p \) with a coefficient \( g^i \neq 0 \) together with all their faces is a closed subcomplex of \( X \) denoted by \( | C^p \rangle \).

7.4 Instead of "chain over \( \mathcal{G}^p \), \( \mathcal{S}^{m}, \mathcal{B} \), over the rational group," we will say "integral, mod \( m \), mod 1, rational chain" and similarly later for related entities (cycles, homology groups, etc.), the meaning being clear from the context.

7.5 Notice that in dealing with finite complexes the groups of chains arising out of \( \mathcal{P}^p \) may be considered as merely those arising out of \( \mathcal{P} \) over a discrete \( G \), and the distinction between the two possible types of chains of (II, 8) disappears.

7.6 When \( G \) is a discrete field the chain-groups over \( G \) and the related groups introduced below will all be vector spaces over \( G \) and will conform with the basic convention (II, 22.2) for such spaces. That the homomorphisms and multiplications which will arise are always linear will generally be obvious.

8. The chain-boundary, or merely boundary of \( C^p \) is the \((p-1)\)-chain

\[ FC^p = \sum_i g^i [x_i^p : x_i^{p-1}] x_i^{p-1}. \]
The boundary operator F is thus defined simultaneously for all groups G whatever. When FC^p = 0, C^p is said to be a p-cycle of X over G. Notice that Fx^p is a chain of Ci x^p, and hence FC^p is a chain of Ci C^p. Identifying for convenience x^p_1 with the integral p-chain 1.x^p_1, F defines the integral boundary of x^p_1 as

\[ Fx^p_1 = \sum_i [x^p_i : x^p_{i-1}] x^p_{i-1}. \]  

Since the \([x^p_i : x^p_{i-1}]\) are integers, the finite sum \(g^i[x^p_i : x^p_{i-1}]\) is an element of G so FC^p is an element of \(C^{p-1}\). We notice that when \(x^p_i\) has no \((p-1)\)-faces, \(Fx^p_i = 0\) for every G. In particular if \(q\) is the lowest dimension of all the elements of X we have \(Fx^q_i = 0\) and hence \(FC^q = 0\) whatever \(C^q\): all the chains of the lowest dimension are cycles. This is not an exception as it corresponds merely to the fact that \(C^{p-1} = 0\).

According to (II, 8.4) the operation F determines a homomorphism \(C^p \rightarrow C^{p-1}\). The transformed group \( \tilde{C}^p = F(C^{p+1}) \) is known as the group of the bounding p-chains over G. The homomorphism \(F: C^p \rightarrow C^{p-1}\) has a kernel \(\tilde{B}^p\) in \(C^p\) whose elements are the p-cycles over G, and \(\tilde{B}^p\) is known as the group of the p-cycles over G. If there are no p-elements then as in (7.2) we define formally \(\tilde{B}^p = \tilde{B}^p = 0\).

(8.3) As a consequence of K4 (K4' of 1 in fact since X is finite) we have \(FFC^p = 0\) whatever \(C^p\) and whatever G. This relation is generally expressed in the operator form

\[ FF = 0. \]

In fact condition K4' is strictly equivalent to (8.3a) for \(G = \tilde{B}\) and it is at least as frequently expressed in this form. Our formulation offered the formal advantage of being wholly divested of any connection with the chain-groups.

(8.4) It is a consequence of (8.3) that \(\tilde{B}^p\) is a subgroup not merely of \(C^p\) but actually of \(C^p\). Or in words: every boundary is a cycle. For \(G = \tilde{B}\) this is merely another formulation of K4'.

9. Since \(\tilde{B}^p = F^{-1}\tilde{w}_{p-1}\), where \(\tilde{w}_{p-1}\) is the zero of \(C^p\), \(\tilde{B}^p\) is a closed subgroup of \(C^p\). Since \(\tilde{B}^p\) is closed \(\tilde{B}^p \subset \tilde{B}^p\). Thus \(\tilde{B}^p\), which is also a subgroup (II, 3.2), is actually a closed subgroup of \(\tilde{B}^p\). In accordance with (II, 5) we may therefore introduce the factor-group:

\[ \tilde{B}^p(X, G) = \tilde{B}^p(X, G) / \tilde{B}^p(X, G). \]

It is known as the pth homology group of X over G, and its elements are the pth homology classes of X over G. If \(C^p = D^p \subset \tilde{B}^p\), we then write with Poincaré the homology:

\[ C^p \sim D^p. \]

It is hardly necessary to point out that the homologies (9.2) combine like linear equations with integral coefficients, i.e., like arithmetical congruences.
The homology groups have various important characteristic numbers, notably their rank. The rank \( R^p \) of the \( p \)th integral homology group is known as the \( p \)th Betti number of the complex. The calculation of these numbers will be illustrated in some of the examples.

(9.3) The reader will have no difficulty in proving also

\[
\mathcal{C}^p(X, G) / \mathcal{B}^p(X, G) \cong \mathfrak{g}^{p-1}(X, G).
\]

However, while of some interest, (9.4) rarely occurs in the applications.


(10.1) Reorientation convention for chains. We shall agree that if \( X \) is reoriented by the orientation function \( \alpha(x) \), the element \( g\alpha^p \) of \( \mathcal{C}^p \) is to be replaced by \( \alpha(x^p)g\alpha^p \). In other words the chain-groups are to undergo a simultaneous automorphism \( \alpha \) that may be described as:

\[
\alpha : x \rightarrow \alpha(x) x.
\]

Referring to (8.1) we find that \( \alpha F = F \alpha \) (\( \alpha \) commutes with \( F \)). It follows that \( \mathcal{B}^p, \mathfrak{g}^p \) are unchanged by \( \alpha \), and hence the same holds regarding \( \mathfrak{g}^p \). In other words our convention merely introduces isomorphisms of all the groups \( \mathcal{C}, \mathfrak{g}, \mathcal{B} \).

(10.2) Influence of isomorphisms upon the different groups. An isomorphism \( X \rightarrow X' \) induces likewise isomorphisms of the groups \( \mathcal{C}^p(X, G), \ldots, \mathfrak{g}^p(X, G) \), with the corresponding groups of \( X' \). A weak isomorphism \( X \rightarrow X' \) raising dimensions \( n \) units induces isomorphisms of the groups \( \mathcal{C}^p(X, G), \ldots \), with the groups \( \mathcal{C}^{p+n}(X, G), \ldots \). A similar remark applies to all the groups of complexes introduced later and will not be repeated.

(10.3) Separation of dimensions. While we are separating dimensions throughout, this is not absolutely necessary. We could have defined a chain over \( G \) as any expressions \( C = g_i z_i, g_i \in G \), with resulting groups \( \mathcal{C}, \ldots, \mathfrak{g} \) related as before. Evidently \( \mathcal{C} = \mathfrak{g} \mathcal{B} \), \ldots \). However, the more interesting parts of the theory of complexes arise precisely from the comparison of certain dimensions, and so the scheme which we are following is preferable.

(10.4) Simplicial complexes and their duals. Let the notations be as in (5), notably as in (5.9) regarding the dual. If \( C^p = \sum g_i \sigma_i^p \) is a chain of \( K \), then \( AC^p \) denotes the chain \( \sum g_i(A\sigma_i^p) \), where if either \( A\sigma_i^p \) is not a simplex of \( K \), or \( A \) is a vertex of \( \sigma_i^p \) then the term \( g_i(A\sigma_i^p) \) is to be set equal to zero. A similar convention is adopted for \( K^* \) and its chains.

It is convenient to have the explicit expression of \( F\sigma^p, F\sigma^{p*} \) under both schemes (5.2, 5.7):

(a) under the scheme (5.2)

\[
FA_0 \cdots A_p = \sum (-1)^q A_0 \cdots A_{q-1} A_{q+1} \cdots A_p, \quad p > 0;
\]

(b) under the scheme (5.7) \( ([\sigma A : \sigma] = 1) \):

\[
FA_0 \cdots A_p = \sum (-1)^{p-q} A_0 \cdots A_{q-1} A_{q+1} \cdots A_p, \quad p > 0;
\]
(c) under both schemes \( F\gamma^0 = 0 \).

In \( K^* \), we have if \( \gamma_0 = \sum A^i \), then:

(d) under the scheme (5.2):

\[
F\sigma_p = \gamma_0 \sigma_p; \quad FC_p = \gamma_0 C_p;
\]

(e) under the scheme (5.7):

\[
F\sigma_p = \sigma_p \gamma_0; \quad FC_p = C_p \gamma_0.
\]

We verify directly \( F\gamma_0 = 0 \), so \( \gamma_0 \) is a zero-cycle of \( K^* \). This zero-cycle plays the role of a boundary operator for \( K^* \): multiplied to the left under (d), and to the right under (e).

The usual scheme is (d), and so \( \gamma_0 \) will generally act as a left multiplier.

(10.5) **Boundary relations in polyhedral complexes.** Let \( \Pi = \{E\} \) be a polyhedral complex and let \( E^n \in \Pi \) have as its \((n - 1)\)-faces \([E^{n-1}_i]\). Then \([E^n: E^{n-1}_i]\) = \( \epsilon^i = \pm 1 \) (6.4), and so:

\[
FE^n = \epsilon^i E^{n-1}_i.
\]

Therefore:

(10.6) **Every** \((n - 1)\)-face of \( E^n \) **appears in** \( FE^n \) **with a coefficient** \( \pm 1 \).

Since \( E^n \) is a bounded region in an \( \mathcal{G}^n \), \( \mathcal{B}E^n \), \( n > 0 \), cannot be contained in a finite set of \( \mathcal{G}^p \)'s, \( p < n - 1 \). Therefore \( E^n \) has at least one \((n-1)\)-face and hence:

(10.7) \( E^n, n > 0 \), **is not a cycle.**

In \( \Pi \) we have in more general form:

(10.8) \[
FE^n_i = \eta_i(n - 1)E^{n-1}_i,
\]

where \( \eta_i(n - 1) = \pm 1 \) if \( E^{n-1}_i \) is a face of \( E^n_i \), and \( \eta_i(n - 1) = 0 \) otherwise.

\section*{§3. HOMOLOGY THEORY OF FINITE COMPLEXES}

(b) **Integral groups**

11. Further progress is contingent upon a full investigation of the integral groups. They are assumed taken with discrete topology and it is to these that the symbols \( \mathcal{C}, \mathcal{B}, \mathfrak{g}, \mathfrak{g} \) shall refer in the present section.

Since \( \mathcal{C}^p \) is a free group on a finite number of generators, we shall naturally utilize properties of such groups as given in (II, §2).

Set for convenience \( [x^{\gamma^p}_{i+1}: x^p_i] = \eta_i(p) \), and denote by \( \gamma(p) \) the matrix of these numbers, or \( p \)th incidence matrix of \( X \). The group \( \mathfrak{g}^p \) of the \( p \)-cycles is the subgroup of the elements \( \gamma^p = g'x^p_i \) of \( \mathcal{C}^p \) which satisfy the relations \( F\gamma^p = 0 \) or

\[
g'\eta_i(p - 1) = 0.
\]

Since \( \mathfrak{g}^p \) is a subgroup of a free group of finite rank it is itself a free group of finite rank (II, 10.1). Since the topology is discrete \( \mathfrak{g}^p = \mathfrak{g}^p \). The latter is
the group generated by the elements \( Fx_i^{p+1} = \eta_i(p)x_i^p \). The group \( S^p \) is the factor-group of \( B^p \) by \( B^p \) and so it is isomorphic with the group of the elements \( g^i x_i^p \) such that (11.1) holds and with the relations
\[
(11.2) \quad \eta_i(p)x_i^p = 0.
\]
Thus \( S^p \) is a group on a finite number of generators and so from the basic reduction theorem for such groups (II, 12.8) we conclude:

(11.3) Theorem. The \( p \)th integral homology group of a finite complex \( X \) satisfies a relation:
\[
(11.4) \quad S^p \cong B^p \times P_i^p
\]
where: (a) \( B^p \) is a free group on a number of generators equal to the \( p \)th Betti number \( B^p(X) \); (b) the \( P_i^p \) are cyclic groups in finite number whose orders \( t_i^p \) are finite and such that \( t_i^p \) divides \( t_{i+1}^p \).

The \( t_i^p \) are the \( p \)th torsion coefficients of \( X \) and \( B^p \) its \( p \)th Betti group. From the reduction theorem we have also the complementary result:

(11.5) The torsion coefficients \( t_i^p \) are the invariant factors greater than 1 of the incidence matrix \( \eta(p) \).

The group \( S^p = P_i^p \) is also known as the \( p \)th torsion group of \( X \) and we have:

(11.6) \( S^p \) is isomorphic with the product of the \( p \)th Betti group and the \( p \)th torsion group.

12. Pursuing our investigation we shall obtain a simultaneous reduction of the chain-groups to a form clearly exhibiting their mutual relations.

The group \( S^p \) of the integral \( p \)-chains is a free group on the \( x_i^p \) with the subgroups \( B^p \), \( \gamma^p \), where \( \gamma^p \subseteq S^p \). By (II, 12.6) a base may be chosen for \( S^p \) consisting of elements \( A_i^p \), \( B_i^p \) such that certain multiples \( s_i A_i^p \) make up a base for \( S^p \).

Now \( t_i^p = t_i^p x_i^p \) is a cycle when and only when the \( t_i^p \) satisfy (11.1). But in that case the \( g_i \) themselves satisfy also these relations and so \( \gamma^p \) is likewise a cycle. In other words, \( t_i^p \in S^p \rightarrow \gamma^p \in S^p \). Applying this to the \( A_i^p \) we deduce that \( A_i^p = S^p \), which can only be if the \( s_i \) are unity. Thus:

(12.1) A base \{\( A_i^p \), \( B_i^p \)\} may be chosen for \( S^p \) such that \( \{A_i^p\} \) is a base for the group \( S^p \) of the integral \( p \)-cycles.

Regarding the \( B_i^p \) no element based on them is a cycle, i.e., no \( g^i B_i^p \in S^p \) unless every \( g_i = 0 \).

The same reduction may be carried out for all dimensions. Since the \( A_i^{p+1} \) are cycles, \( F A_i^{p+1} = 0 \) and so \( B^p \) has for generators the chains \( F B_i^{p+1} \) which are in fact \( p \)-cycles (8). Again by (II, 12.6) a base may be chosen for \( S^p \) consisting of cycles \( A_i^p \), \( c_i^p \) such that certain multiples \( s_i^p A_i^p \) form a base for \( S^p \). Furthermore \( t_i^p \) divides \( t_{i+1}^p \) throughout.

Notice that the replacement of the \( A_i^p \) by the \( A_i^p \), \( c_i^p \) as free generators of \( S^p \), is a change of base in \( S^p \) which leaves the \( B_i^p \) undisturbed.
Since $\mathfrak{S}^p = \mathfrak{B}^p / \mathfrak{B}^r$ it is isomorphic with the group on the generators $A^p_r$, $c^p_r$ with the relations:

$$r_i^p A^p_r = 0.$$  

(12.2)

We now divide the $A^p_r$ into two sets. The first will consist of the elements, denoted by $a^p_r$, such that the corresponding $r_i^p = 1$, the second of those, denoted by $b^p_r$, such that the corresponding $r_i^p$, henceforth written $t_i^p$ are greater than 1. The notations are so chosen that $t_i^p$ divides $t_i^{r+1}$. The group $\mathfrak{S}^p$ is now isomorphic with the group on the generators $a^p_r$, $b^p_r$, $c^p_r$ with the relations:

(12.3a) \hspace{1cm} a^p_r = 0, \\
(12.3b) \hspace{1cm} t_i^p b^p_r = 0.

Evidently the $a^p_r$ may be suppressed among the generators so that $\mathfrak{S}^p$ is in fact isomorphic with the group on the $b^p_r$, $c^p_r$ with the relations (12.3b). We have then

$$\mathfrak{S}^p \cong \mathbb{P}(3c^p_r) \times \mathbb{P}(3^* (t_i^p) b^p_r).$$  

By comparing with (11.4) we have then:

$$\mathfrak{B}^p \cong \mathbb{P}(3c^p_r), \mathfrak{X}^p = \mathbb{P}(2^p_r) \cong \mathbb{P}(3^* (t_i^p) b^p_r).$$  

Therefore: (a) the rank $R^p$ of $\mathfrak{B}^p$, or of $\mathfrak{S}^p$, is the number of $c^p_r$; (b) the $t_i^p$ are the orders of the $2^p_r$, i.e., they are the $p$th torsion coefficients of $X$.

13. Since $\{a^p_r, t_i^p b^p_r\}$ are free generators for $\mathfrak{S}^p$, by (II, 12.3) they must be reducible to the set of free generators $\{FB^{p+1}_i\}$ by a unimodular transformation. That is to say there exist relations:

$$t_i^p b^p_r = \lambda_i^i FB^{p+1}_i = F(\lambda_i^i B^{p+1}_i)$$  

$$a^p_r = \mu_i^i FB^{p+1}_i = F(\mu_i^i B^{p+1}_i)$$

with a unimodular matrix

$$\begin{vmatrix} \lambda_i^i \\ \mu_i^i \end{vmatrix}.$$  

It follows that if we set

$$d_i^{p+1} = \lambda_i^i B^{p+1}_i, \quad e_i^{p+1} = \mu_i^i B^{p+1}_i,$$

the $d_i^{p+1}, e_i^{p+1}$ may replace the free generators $B_i^{p+1}$ in the set of free generators $\{A_i^{p+1}, B_i^{p+1}\}$ for $\mathfrak{S}^{p+1}$. Notice that this substitution does not affect $\{A_i^{p+1}\} = \{a_i^{p+1}, b_i^{p+1}, c_i^{p+1}\}$.

14. Let us suppose now that the dimensions in $X$ run from $q$ to $r$. We reduce the bases for $\mathfrak{G}, \mathfrak{G}^{p+1}, \ldots$ in succession as follows:

Group $\mathfrak{G}$. Here $\mathfrak{G}^{-1} = 0$ so $\mathfrak{G}^i = \mathfrak{G}^i$. The first reduction yields the base $\{A_i\}$ and the second the base $\{a_i^r, b_i^r, c_i^r\}$. There are no $d_i^r, e_i^r$. 
Group $\mathfrak{C}^{q+1}$. The first reduction yields the base $\{A_1^{q+1}, B_1^{q+1}\}$, and the second $\{a_1^{q+1}, b_1^{q+1}, c_1^{q+1}, B_1^{q+1}\}$. The $a_1, b_1$ determine the $\epsilon_1^{q+1}, d_1^{q+1}$ and as already observed the choice of these in place of the $B_1^{q+1}$ as generators is tantamount to a change of base for $\mathfrak{C}^{q+1}$. We thus have a third reduction to a final base $\{a_1^{q+1}, \ldots, \epsilon_1^{q+1}\}$ for $\mathfrak{C}^{q+1}$.

Group $\mathfrak{C}^q$. The same reduction may be applied step by step for $p = q + 1 \ldots, r$. We have thus proved the

(14.1) Theorem. The bases for the integral chain-groups of a finite complex may be chosen to consist for each dimension $p$ of five sets of elements $a_p^q, \ldots, \epsilon_p^q$ with the boundary relations:

$$
\begin{align*}
Fe_{q+1}^p &= a_{q+1}^p, & Fa_{q}^p &= 0,
Fd_{q}^{q+1} &= \epsilon_{q+1}^p b_{q}^p, & Fb_{q}^p &= 0,
Fc_{q}^p &= 0,
Fd_{q}^{q} &= \epsilon_{q-1}^p b_{q-1}^p, \\
Fe_{q}^p &= a_{q-1}^p.
\end{align*}
$$

(14.2)

The number of $c_q^q$ is the $q$th Betti number $R^q$, and the $t$'s are the torsion coefficients.

The bases in the reduced form described in the theorem are said to be canonical.

15. There remains the explicit calculation of the $R^q$. Let $\alpha^q$ denote as before the number of elements $x_q^p$, and let $\rho^q$ be the rank of the incidence matrix $\eta(q)$.

Since $\mathfrak{B}^q$ is the subgroup of the elements $g^q x_q^p$ for which (11.1) holds, its rank is $\alpha^q - \rho^q$. Since $\mathfrak{B}^q \cong$ the factor-group of $\mathfrak{B}^q$ by the subgroup of the elements which satisfy the relations (11.2), its rank is $\alpha^q - \rho^q - \rho^q - \rho^q$, or

(15.1)

$$R^q = \alpha^q - \rho^q - \rho^q - \rho^q.$$ 

Since $\eta(q)$ exists only for $q \leq p \leq r - 1$, to make (15.1) hold formally for all dimensions we define $\rho^q = 0$, for $s < q$ or $s > r - 1$. Thus (15.1) provides an explicit expression for the Betti numbers in terms of the incidence matrices.

If we multiply both sides in (15.1) by $(-1)^p$ and add there comes the classical Euler-Poincaré relation for finite complexes:

(15.2)

$$\sum (-1)^p \alpha^p = \sum (-1)^p R^p.$$ 

The common value $\chi(X)$ of these two sums is known as the Euler characteristic of $X$. The relation (15.2) is often convenient for computing Betti numbers, notably when the range of the dimensions in $X$ is small.

(15.3) The Poincaré polynomial. We understand thereby the polynomial

(15.4)

$$P(t, X) = \sum R^p t^p$$

whose coefficients are the Betti numbers of $X$. Under certain circumstances (formation of the product, IV) this polynomial obeys very convenient formal
rules. Moreover it may be used to advantage in describing the Betti numbers of certain simple complexes. Thus for the boundary of the \((n + 1)\)-simplex we shall find later (22.4) that in substance \(P(t, X) = 1 + t^n\).

§4. HOMOLOGY THEORY OF FINITE COMPLEXES

(c) ARBITRARY GROUPS OF COEFFICIENTS

16. Let again \(G\) be an arbitrary topological group and let this time \(\mathfrak{C}, \mathfrak{B}, \mathfrak{F}, \mathfrak{D}\) refer to the groups over \(G\). Following Steenrod \([a]\) we shall give a complete analysis of these groups.

By definition (II, 8)

\[
\mathfrak{C} = \mathbf{P}(Gx^*_p).
\]

Let us designate temporarily by \(z^*_p\) the elements \(a^*_p, \ldots, e^*_p\) of (14.1), where we have a relation

\[
(16.1) \quad z^*_p = \lambda^*_p(p)z^*_p, \quad \lambda(p) = || \lambda^*_p(p) || \text{ unimodular}.
\]

Consider now the group

\[
\mathfrak{C}^p = \mathbf{P}(Gx^*_p).
\]

Referring to (II, 8.4):

\[
g^i z^*_p \rightarrow g^i \lambda^*_p(p)z^*_p
\]

defines a homomorphism \(\tau: \mathfrak{C}^p \rightarrow \mathfrak{C}^p\). Since \(\lambda(p)\) is unimodular, it has an inverse

\[
(16.2) \quad \mu(p) = \lambda^{-1}(p) = || \mu^*_p(p) ||,
\]

and so

\[
g^i z^*_p \rightarrow g^i \mu^*_p(p)z^*_p
\]

defines a homomorphism \(\theta: \mathfrak{C}^p \rightarrow \mathfrak{C}^p\). It is an elementary matter to verify \(\tau \theta = 1, \theta \tau = 1\); hence \(\tau, \theta\) are isomorphisms (II, 4.5), and consequently \(\mathfrak{C}^p \cong \mathfrak{C}^p\).

From the result just obtained we infer that we may represent every element \(C^p \in \mathfrak{C}^p\) in the form:

(16.3) \[
C^p = g^h a^*_p + g^y b^*_p + g^t c^*_p + g^u d^*_p + g^v e^*_p
\]

where the coefficients \(g \in G\). From (14.2) follows now:

(16.4) \[
\beta^p = \mathbf{P}(G a^*_p) \times \mathbf{P}(G b^*_p) \times \mathbf{P}(G c^*_p) \times \mathbf{P}(G d^*_p) \times \mathbf{P}(G e^*_p).
\]

Hence \(C^p\) is a cycle when and only when \(v^i = 0\) and \(u^k = 0\) \((k \text{ unsummed})\). Therefore in the group symbolism of the introduction to the present chapter:

(16.5) \[
\beta^p = \mathbf{P}(G a^*_p) \times \mathbf{P}(G b^*_p) \times \mathbf{P}(G c^*_p) \times \mathbf{P}(G d^*_p) \times \mathbf{P}(G e^*_p).
\]
On the other hand \( \mathfrak{g}^p \) is the set of all chains

\[
C^p = \sigma^p a^p + \gamma^i t^i b^i_i.
\]

Since \( \gamma^i t^i \) (i unsummed) is merely any element of \( G(t^p) \) we have:

\[
\mathfrak{g}^p = \mathcal{P}(Ga^p) \times \mathcal{P}(G(t^p)b^p_i).
\]

The topology of the factor \( Ga^p \) is governed by that of \( G \) and the isomorphism \( Ga^p \cong G \). The topology of \( G(t^p)b^p_i \) is its relative topology as a subgroup of \( Gb^p_i \cong G \). Therefore \( \mathfrak{g}^p \) is obtained by merely replacing \( G(t^p)b^p_i \) by its closure \( G(t^p)b^p_i \) as a subgroup of \( Gb^p_i \), and this is the same as \( G(t^p)b^p_i \) (closure in \( G \)).

With this meaning of the symbols clearly before us we have then

\[
\mathfrak{g}^p = \mathcal{P}(Ga^p) \times \mathcal{P}(G(t^p)b^p_i).
\]

If we combine with (16.4) and recall that \( \mathfrak{g} = \mathfrak{g}/\mathfrak{g} \), we have:

\[
\mathfrak{g}^p(X, G) \cong \mathcal{P}(G/G(t^p)b^p_i) \times \mathcal{P}(Gt^p) \times \mathcal{P}(G(t^p)b^p_i),
\]

or finally in equivalent form:

\[
\mathfrak{g}^p(X, G) = \mathcal{P}(G^*(t^p)b^p_i) \times \mathcal{P}(Gt^p) \times \mathcal{P}(G(t^p-1)b^p_i).
\]

We have thus obtained a basic decomposition of the homology groups over any coefficient group \( G \).

Some simple conclusions may immediately be drawn from the relations just obtained notably:

(16.10) If there are no torsion coefficients for the dimension \( p \) then a \( p \)-cycle \( \sim 0 \) is a bounding cycle. Hence if there are no torsion coefficients \( "\sim 0" \) and \( "\text{bounding}" \) are equivalent. \( \text{(See 9.)} \)

For the second product at the right in (16.6) is then absent, hence \( \mathfrak{g}^p = \mathfrak{g}^p \), which is (16.10).

(16.11) If there are no torsion coefficients for the dimensions \( p \) and \( p - 1 \) the \( p \)th homology group over \( G \) reduces to the "Betti" part \( \mathcal{P}(Gt^p) \). Hence if there are no torsion coefficients all the homology groups reduce to their Betti parts.

(16.12) If in \( X : p \leq \dim x \leq q \) then: (a) no \( q \)-cycle \( \sim 0 \) unless it is zero; (b) every \( p \)-chain is a cycle.

Noteworthy special case: In a simplicial complex or in a polyhedral complex, every zero-chain is a zero-cycle.

Since there are no \( (q + 1) \)-chains different from 0, we have \( \mathfrak{g}^{p+1}(X, G) = 0 \) which is (a). Owing to the absence of \( (p - 1) \)-chains different from 0 we have \( FC^p = 0 \) whatever \( C^p \) and this is (b).

17. Some noteworthy coefficient-groups.

(17.1) Division-closure groups. For these groups the \( G(m) \) are all closed (II, 20.9) and so from (16.7):

(17.2) When \( G \) is a division-closure group, \( \mathfrak{g}^p = \mathfrak{g}^p \), and hence: (a) \( "\sim 0" \) \( \leftrightarrow \) "bounding"; (b) \( \mathfrak{g}^p = \mathfrak{g}^p/\mathfrak{g}^p \) (Steenrod [a]).

Noteworthy special cases: \( G \) is compact or discrete.
(17.3) $G$ is a discrete field. The groups $\mathbb{G}^p(X, G), \ldots, \mathbb{S}^p(X, G)$ are then finite-dimensional vector spaces over $G$ and hence discrete (II, 25.6). The dimension $R^p(X, G)$ of $\mathbb{S}^p(X, G)$ is known as the $p$th Betti number over $G$. Let $\pi$ be the characteristic of $G$. It will be recalled that $\pi$ is such that $\pi g = 0$ for every $g \in G$, and is a prime number or zero. Among the special fields of characteristic $\pi > 0$ is found $\mathbb{Z}_\pi$, the field of the residues mod $\pi$, and it is a subfield of every field $F$ of characteristic $\pi$. The corresponding chains, $\ldots$, mod $\pi$ and the associated groups and Betti numbers are written $\mathbb{G}^p(X, \pi), \ldots, R^p(X, \pi)$. We shall show in substance that for most purposes $\mathbb{Z}_\pi$ may replace $G$.

We will make use of (II, 36). In the notations there utilized and since $\mathbb{Z}_\pi$ is a subfield of $G$ we recognize immediately that $\mathbb{G}^p(X, G) = G \mathbb{G}^p(X, \pi)$. Furthermore $\mathbb{G}^p(X, \pi)$ is defined by means of $FC^p = 0$, where the coefficients of $FC^p$ are reduced mod $\pi$. The group $\mathbb{G}^p(X, G)$ is defined by the same relation save that the values $FC^p$ for the chains over $G$ are obtained by linear extension from the values for the chains mod $\pi$. Both the groups $\mathbb{G}^p(X, G)$ and $\mathbb{G}^p(X, \pi)$ are spanned by the chains $FE_i^{p-1}$ taken mod $\pi$ (i.e., with coefficients reduced mod $\pi$). We have therefore the exact situation of (II, 36.8). By that result then $\mathbb{S}^p(X, G)$ is isomorphic with the vector space over $G$ spanned by $\mathbb{S}^p(X, \pi)$, and $\mathbb{S}^p(X, G)$, $\mathbb{S}^p(X, \pi)$ have the same dimension, or

\[(17.4) R^p(X, G) = R^p(X, \pi). \]

We have obtained (17.4) without utilizing the reduction (16.9). We may also use the latter for the same purpose, and it will lead to an expression for $R^p(X, \pi)$ in terms of the integral Betti numbers and torsion coefficients.

Referring to (16.9), $R^p(X, G)$ is equal to the number of products effectively present at the right:

(a) When $\pi$ does not divide $t^p$, $G(t^p) = G$, and the corresponding term is absent. When $\pi$ divides $G(t^p) = 0$ and there is a term $\mathbb{G}^p$. Therefore the first product at the right in (16.9) consists of $\theta_\pi^p$ isomorphs of $G$, where $\theta_\pi^p$ is the number of $t^p$ with $\pi$ as a prime factor.

(b) The second product in (16.9) consists of $R^p$ isomorphs of $G$.

(c) When $\pi$ divides $t^p$, $G(t^p) \cong G$, otherwise it is zero. Therefore the third product in (16.9) consists of $\theta_\pi t^{p-1}$ isomorphs of $G$.

Thus $\mathbb{S}^p(X, G)$ is the product of $R^p + \theta_\pi t^{p-1} + \theta_\pi^p$ isomorphs of $G$. Hence

\[(17.5) R^p(X, G) = R^p + \theta_\pi t^{p-1} + \theta_\pi^p. \]

Since this value depends only on $\pi$, (17.4) follows. We write explicitly

\[(17.6) R^p(X, \pi) = R^p + \theta_\pi t^{p-1} + \theta_\pi^p. \]

The case $\pi = 0$, i.e., $G$ of characteristic zero, is not exceptional. The field $\mathbb{Z}_\pi$ is then to be replaced by the rational field $\mathbb{R}$. The corresponding chains, $\ldots$, are said to be rational. We verify directly that in (16.9) the second product alone remains, thus yielding for $G$ of characteristic zero:
(17.7) \[ R^p(X, G) = R^p(X, \mathbb{R}) = R^p(X). \]

This shows that the integral Betti numbers themselves may also be defined as the dimensions of the homology groups over a field, namely the rational field $\mathbb{R}$, or for that matter any field of characteristic zero. An incidental and frequently convenient result is that $\{ e_\alpha \}$ is a base for the rational $p$-cycles with respect to homology.

To sum up we have proved for finite complexes, a theorem which will recur in a number of instances later, and is formulated for later reference in a more general form than immediately required:

(17.8) **Universal theorem for fields.** The homology groups over a field $G$ of characteristic $\pi$ are vector spaces over $G$, and their dimensions, the Betti numbers over $G$, are equal to the corresponding Betti numbers mod $\pi$.

**Complementary result.** A maximal set of $p$-cycles mod $\pi$ independent with respect to homology is likewise a maximal independent set for any field of characteristic $\pi$.

These properties hold likewise for $\pi = 0$, the cycles mod $\pi$, and their Betti numbers being then the rational cycles and Betti numbers.

Complements (for finite complexes only): (a) The Betti numbers are all finite, and the rational Betti numbers are the numbers $R^p$ (integral Betti numbers) previously defined. (b) The Euler-Poincaré formula holds for all fields $G$:

(17.9) \[ \sum (-1)^p R^p(X, G) = \sum (-1)^p \alpha_\pi. \]

(Immediate consequence of (15.2) and (17.6).)

**Historical note.** The special theory mod 2 played an important role in earlier topology, as a reference to Veblen [V] will show.

(17.10) $G = \mathbb{R}$, the group of the reals mod 1. This time $\mathbb{P}(m) = \mathbb{P}$, hence $\mathbb{P}^*(m) = 0$. Therefore

(17.11) \[ \mathfrak{B}^*(X, \mathbb{P}) \cong P(\mathbb{P} e_\gamma) \times P(\mathfrak{B}(d^{-1}e) d e). \]

The first term is a toroidal group, which is the direct product of $R^p$ isomorphs of $\mathfrak{B}$. Since $\mathfrak{B}(d^{-1}e) \cong \mathfrak{B}^*(d^{-1}e)$, (II, 20.14) the second term in (17.11) $\cong \mathfrak{T}^{p-1}(X)$. Therefore

\[ \mathfrak{B}^*(X, \mathbb{P}) \cong P(\mathbb{P} e_\gamma) \times \mathfrak{T}^{p-1}(X). \]

This proves:

(17.12) The $p$th homology group of $X$ mod 1 is $\cong$ the product of an $R^p$-dimensional toroidal group by the $(p - 1)$-dimensional torsion group of $X$ (which is a finite group).
18. **Universal coefficient-groups.** A group $G_\theta$ is called a universal coefficient-group for $X$, whenever given the full set $\{\mathfrak{P}(X, G_\theta)\}$ and an arbitrary $G$ it is possible to determine in terms of the groups of this set and of $G$ all the groups $\{\mathfrak{P}(X, G)\}$.

When the integral homology groups and $G$ are known, so are the Betti numbers and torsion coefficients and hence also an isomorph of the product at the right in (16.9), and finally $\mathfrak{P}(X, G)$ itself. Similarly, given all the homology groups mod 1 and the group $G$, we learn from (17.11) the values of the numbers $R^p$, and also the $\mathfrak{P}^p(X)$, hence the torsion coefficients. Consequently, we may again determine the terms in (16.9) and hence the groups over $G$. Therefore

(18.1) The groups $\mathfrak{P}$, $\mathfrak{P}$ of the integers and of the reals mod 1 are both universal coefficient-groups for finite complexes.

(18.2) A n. a. s. c. for two finite complexes to have the same homology groups over every $G$ is that they have the same integral homology groups, or else the same homology groups mod 1, and so, in the last analysis, that they have the same Betti numbers and torsion coefficients.

19. The following properties which are often useful, are ready consequences of the general theory:

(19.1) If $X = \bigcup X_i$, where the $X_i$ are disjoint complexes, then:

(19.2) $\mathfrak{P}(X, G) = \bigoplus \mathfrak{P}(X_i, G)$ (every $G$),

(19.3) $R^p(X, G) = \sum R^p(X_i, G)$ ($G$ a field).

The second relation follows from the first, so we merely prove (19.2). Every chain $C^p$ of $X$ over $G$ is of the form

(19.4) $C^p = \sum C_i^p$,

where $C_i^p$ is a chain of $X_i$ over $G$. N. a. s. c. for $C^p$ to be a cycle, or to be $\sim 0$, are, respectively, that every $C_i^p$ be a cycle, or be $\sim 0$. From this to (19.2) is but a step.

(19.5) If $X^{p+1}$ is the $(p + 1)$-section of $X$ then for every $r \leq p$:

(19.6) $\mathfrak{P}(X, G) = \mathfrak{P}(X^{p+1}, G)$ (every $G$)

(19.7) $R^p(X, G) = R^p(X^{p+1}, G)$ ($G$ a field).

For $\mathfrak{P}(X, G)$ depends solely on the elements of $X^{p+1}$.

§5. APPLICATION TO SOME SPECIAL COMPLEXES

20. **Simplicial complexes.** Let $K = \{\sigma\}$ be a finite simplicial complex. The vertices of $K$ will be designated by $A$ with possible supplementary indices. If $\sigma^p = A_0 \cdots A_p \in K$ then (10.4ac):

(20.1) $F\sigma^p = \sum (-1)^q A_0 \cdots A_{q-1} A_{q+1} \cdots A_p, \quad p > 0;

(20.2) FA = 0.$
In particular for $p = 1$

(20.3) \[ F_0 = A_1 - A_0. \]

(20.4) **Definition.** If $C^0 = g^i A_i$ is a zero-chain then $\sum g^i$ is a function of $C^0$ known as the Kronecker index of $C^0$ and denoted by $\text{KI}(C^0)$. As we shall see later (28, 46) this number is a special case of a more general numerical function with the same designation.

We will now prove a series of properties relating connectedness and the zero-cycles. They are so interlocked that they will have to be proved more or less together. The following notations will be used:

- $\{K_i\} = \text{the components of } K$;
- $\{A \alpha\} = \text{the vertices of } K_i$; however, the vertex $A \alpha$ will be written $A_i$.

(20.5) A n. a. s. c. for two vertices $A_i, A'$ to be in the same component is $A_i \sim A'$.

(20.6) $C^0 \sim 0 \iff \text{KI}(C^0) = 0$.

(20.7) Every zero-cycle over any $G$ satisfies a relation

\[
C^0 \sim g^i A_i, \quad g^i \epsilon G.
\]

Moreover a relation

\[
g^i A_i \sim 0, \quad g^i \epsilon G,
\]

implies that every $g^i = 0$.

(20.8) For every group $G$:

\[
\mathcal{S}^0(K, G) \cong P(GA_i).
\]

In particular when $G = 3$, the group of the integers, then $\mathcal{S}^0(K, 3)$ is isomorphic with the free group on the generators $A_i$. This group has no elements of finite order and so there are no torsion coefficients for the dimension zero.

(20.9) The number of components of $K$ is the zero-dimensional Betti number $R^0(K, G), G \text{ any field. Hence } R^0(K, G) \text{ is independent of } G \text{ and it will be designated by } R^0(K)$.

(20.10) If $K$ is connected and $A$ any vertex then every $C^0$ satisfies the relation:

\[
C^0 \sim \text{KI}(C^0)A.
\]

Therefore when $K$ is connected $C^0 \sim 0 \iff \text{KI}(C^0) = 0$.

(a) Suppose first $G = 3$. Let $A, A'$ be any two vertices of $K_i$. Since $K_i$ is a component there is a finite set of elements of $K_i: A = \alpha_1, \ldots, \alpha_r = A'$ in which any two consecutive elements are incident (2.3). If $\sigma_j$ is not a vertex it contains a vertex $A'_{j-1}$ and a vertex $A'_{j+1}$, hence also the one-simplex $A'_{j-1}A'_{j+1}$. It follows that in the sequence joining $A$ to $A'$ we may replace $\alpha_j$ by the set of simplexes: $A'_{j-1}A'_{j+1}$, $A'_{j-1}A_{j+1}'$, and still have consecutive elements incident. Proceeding thus we will arrive at a set of the same type and of form $A \beta_1 = A, \alpha_1, A \beta_2, \ldots, A \beta_r = A'$. Hence $F \sigma_1^i = A \beta_1 - A \beta_r$ and so if $C^0 = \sum \sigma_i^i$ then $FC^0 = A' - A \sim 0$. This proves (20.5) as regards necessity.
If $C^0 \sim 0$ and $G = \mathcal{I}$ we have $C^0 = F(g^0 \sigma_1)$, hence $K(I(C^0)) = g^0 K(I(F \sigma_1)) = 0$ by (20.3). This is (20.6) for the present situation.

If $C^0$ is an integral chain we have $C^0 = \sum C^0_i$, $C^0_i \subset K_i$, and $C^0_i = g^0 A_i$. By the necessity part of (20.5) for $G = \mathcal{I}$ we have $A_i \sim A_i$, hence $C^0_i \sim \sum g^0 A_i = K(I(C^0)) A_i$, from which (20.7a) follows. Furthermore this also proves (20.10) for integral chains.

Suppose that (20.7b) holds with the $g^i$ integers and not all zero. Then $g^i A_i = FC^i = F \sum C^i_i$, $C^i_i \subset K_i$. Since chains in different components cannot cancel out we have $FC^i_i = g^i A_i$ (i unsummed). Hence by (20.6) already proved for $G = \mathcal{I}$: $K(I(FC^i_i)) = 0 = g^i$. This proves (20.7b) for $G = \mathcal{I}$. From this follows also the sufficiency proof of (20.5). For suppose $A \sim A'$ and $A_i$, $A_j$ in different components say $K_i$, $K_j$, $i \neq j$. The two vertices may then be chosen as $A_i$, $A_j$ and we would have $A_i \sim A_j$ which is ruled out. Therefore $A$, $A'$ are in the same component, and the proof of (20.5) is completed.

It follows from (20.7) for $G = \mathcal{I}$ that $\mathcal{I}(K, \mathcal{I})$ is isomorphic with the free group on the generators $\{A_i\}$ and this is merely another way of stating (20.8) for $G = \mathcal{I}$. This is as much as may be obtained for $G = \mathcal{I}$.

(b) Suppose now that $G$ is any group. We first notice that (20.9) is a consequence of (20.8) and so requires no further consideration. For all but (20.8) the only property required is that "$\sim 0" \leftrightarrow "bounding"" for the zero-cycles, and this follows from (16.10) and the fact that there are no $\sigma_i^0$ (20.8 for $G = \mathcal{I}$).

Regarding (20.8), if we go back to the derivation of (12.4) we verify that, since there are no $\sigma_i^0$, $\{c_i^j\}$ is merely any set of zero-cycles such that $\mathcal{I}(K, \mathcal{I})$ is isomorphic with the free group on the generators $c_i^j$. Therefore we may choose $\{c_i^j\} = \{A_i\}$, and so (20.8) follows from (16.11). This completes the proofs of all our propositions.

21. Complexes with cyclic and acyclic properties. An important and simple property of many noteworthy complexes is to have all the homology groups for certain dimensions vanishing or merely isomorphic with the coefficient-groups. Among these are found, for example, simplexes and their boundaries.

(21.1) Definitions. The complex $X$ is said to be cyclic [acyclic] in the dimension $p$ over $G$ if $\mathcal{I}(X, G) \cong G$ [ = 0]. It is said to be $(p, \ldots, q)$-cyclic [to be $(p, \ldots, q)$-acyclic, cyclic] over $G$ if it is cyclic over $G$ in the dimensions $p, \ldots, q$ and acyclic over $G$ in the other dimensions [acyclic over $G$ in the dimensions $p, \ldots, q$, in all dimensions]. If $X$ say, is cyclic in the dimension $p$ over $G$ for every $G$ it is merely said to be "cyclic in the dimension $p," and similarly for the other properties.

As in similar instances when $G = \mathcal{I}$, $\mathcal{I}$, $\mathfrak{A}$, $\mathfrak{A}$ we will say "integrally acyclic," "acyclic mod $n,"" and "rationally acyclic," and similarly for the other concepts.

We have at once from (16.9):

(21.2) A n.a.s.c. for a finite complex to be acyclic is that all the Betti numbers and torsion coefficients vanish.
(21.3) A n.a.s.c. for a finite complex to be \((p, \cdots, q, q')\)-cyclic is that all the torsion coefficients vanish, that the Betti number \(R^s = 1\) for \(s = p, \cdots, q\) and \(R^s = 0\) otherwise.

A convenient result is:

(21.4) If \(\dim X = n + 1 \geq 2\), and \(X\) is zero-cyclic and with a single \((n + 1)\)-element \(x^{n+1}\), then its \(n\)-section \(X^n = X - x^{n+1}\) is \((0, n)\)-cyclic, and all its \(n\)-cycles are of the form \(\delta^n = \delta x^{n+1}\).

By (19.5) it is only necessary to show that \(X^n\) is cyclic in the dimension \(n\), and this will follow if we can show that every \(n\)-cycle of \(X\) is of the form \(\delta^n\), \(\delta^n = \delta x^{n+1}\). Referring to the canonical bases (14), since there is only one \((n + 1)\)-element \(x^{n+1}\) it is the single \(e_i^{n+1}\) on hand and there are no \(d_i^{n+1}\). Therefore at the right in (16.6), only the first product is present and it is \(G\delta^n\). By (16.7) we find then \(\beta^n(X, G) = \beta^n(X, G) = G\delta^n\). Since \(X\) is acyclic in the dimension \(n\) we must have \(\beta^n(X, G) = \beta^n(X, G) = G\delta^n\). Hence every \(n\)-cycle is a \(G\delta^n\), and (21.4) follows.

(21.5) A closed connected simplicial complex is cyclic in the dimension zero (20.10).

22. The simplex, its closure and boundary. Let \(\sigma^n\) be an \(n\)-simplex. We will consider the groups of \(\sigma^n\), Cl \(\sigma^n\), \(\mathcal{B}\sigma^n\). It is often convenient to call \(\sigma^n\) an open simplex, Cl \(\sigma^n\) a closed simplex.

(22.1) Groups of \(\sigma^n\). The simplex \(\sigma^n\) itself is (trivially) \(n\)-cyclic.

(22.2) Groups of Cl \(\sigma^n\). We will show that it is zero-cyclic. For \(n = 0\) this is the same as (22.1) so we assume \(n > 0\). If \(A, A'\) are any two vertices then \(AA'\) is a one-simplex of Cl \(\sigma^n\) and so Cl \(\sigma^n\) is connected and hence cyclic in the dimension zero (21.5). Let now \(\sigma^n = A\sigma^n - 1\), and Cl \(\sigma^n - 1 = \{\sigma_i^{n-1}\}\), \(p > 1\), and \(FA\sigma_i^{n-1} = \sigma_i^{n-1} - AF\sigma_i^{n-1}\), \(p > 1\); hence

\[
\begin{align*}
FA\sigma_i^{n-1} &= C_i^{p-1} - AFC_i^{p-1}, \\
FA\sigma_i^0 &= C_i^0 - KI(C_i^0) \cdot A.
\end{align*}
\]

Let now \(\gamma^n, p > 0\), be a cycle of Cl \(\sigma^n\), and suppose first \(p > 1\). We have \(\gamma^n = AC\sigma_i^{p-1} + C_i^p, C_i^{p-1}\) and \(C_i^p \subset Cl \sigma_i^{n-1}\). Hence \(FY^n = C_i^{p-1} - AFC_i^{p-1} + FC_i^p = 0\). Since the middle term alone contains the vertex \(A\) we have \(FC_i^{p-1} = 0, C_i^{p-1} = -FC_i^p\) and hence \(\gamma^n = FAC_i^p\). When \(p = 1\) we obtain: \(FY^i = C_i^0 - KI(C_i^0)A + FC_i^1 = 0\), hence \(KI(C_i^0) = 0, C_i^0 = -FC_i^1\), and the conclusion is again the same. Therefore \(\gamma^n \sim 0\) for every \(p > 0\) and so Cl \(\sigma^n\) is zero-cyclic.

(22.4) Groups of \(\mathcal{B}\sigma^n\). By (21.4) when \(n > 1\), \(\mathcal{B}\sigma^n\) is \((0, n - 1)\)-cyclic. Since \(\mathcal{B}\sigma^1\) consists of two points \(A, B\) its sole homology group different from 0 is the one for the dimension zero and it is the product for any given \(G\) of two isomorphs of \(G\).


(23.1) Let \(X_1\) be a closed subcomplex of \(X\) and \(X_0 = X - X_1\) its open complement. The pair \((X_0, X_1)\) in the order named will be referred to as a dissection of \(X\). Our present purpose is to compare the groups of the \(X_1\) with those of
$X$ itself. We will denote by $F_i$ the boundary operator for $X_i$. If $C$ is a chain of $X$ we have $C = C_0 + C_1, C_i \subset X_i$, and we will call $C_i$ the chain $C$ reduced mod $X_j$ ($j \not= i$), or merely "$C$ mod $X_j$." If $\varphi(C)$ is a function whose range and values are chains of $X$, we have

$$\varphi(C) = \varphi_0(C) + \varphi_1(C), \quad \varphi_i(C) \subset X_i,$$

and we will call $\varphi_i(C)$: the function $\varphi$ reduced mod $X_j$ ($j \not= i$).

(23.2) The groups of $X_1$. If $x \in X_1$ then $X_1 \subset X_1$, and hence $F_1 = F_i X_1$. It follows that a cycle $\gamma$ of $X_1$ is also a cycle of $X$ and that if $C_1$ bounds in $X_1$ it also bounds in $X$.

Since the elements of $X_1$ are among those of $X$ we have an injection $\mathcal{G}_0(X_1, G) \rightarrow \mathcal{G}_0(X, G)$ in the sense of (II, 8.6). We may consider this as a simultaneous operation on all the chain-groups of $X_1$ into those of $X$, and denote it briefly as $\eta: X_1 \rightarrow X$, as if it were an operation on $X_1$ to $X$. This operation is called an injection of $X_1$ into $X$. We notice the obvious property: $F_1 = \eta F_1$. As a consequence of this, $\eta$ maps, respectively, $\mathcal{B}_0(X_1, G), \mathcal{B}_0(X_1, G)$ into $\mathcal{B}_0(X, G), \mathcal{B}_0(X, G)$, and since $\eta$ is continuous also $\mathcal{B}_0(X_1, G)$ into $\mathcal{B}_0(X, G)$. Hence (II, 5.4) $\eta$ induces a homomorphism $\eta: \mathcal{S}_0(X_1, G) \rightarrow \mathcal{S}_0(X, G)$. We will set $\eta \mathcal{S}_0(X_1, G) = \mathcal{S}_0(X_1, G), \eta \mathcal{S}_0(X_1, G) = \mathcal{S}_0(X_1, G)$.

(23.3) Since $\eta \mathcal{S}_0(X_1, G) = \mathcal{S}_0(X_1, G)$, the group $\mathcal{S}_0(X_1, G) = \mathcal{S}_0(X_1, G)/\mathcal{B}_0(X_1, G)$ may be viewed as the group of the cycles of $X_1$ as to bounding in $X$. Taking now integral chains it will be seen that the considerations of (12, 13) are still applicable when the $p$-chains are restricted to $X_1$ and the $(p + 1)$-chains are still chains of $X$. They will lead to a system (14.2) for the particular dimension $p$ (but not simultaneously for all dimensions), with the $(p + 1)$-chains $\delta_{0}^{p+1}, \delta_{i}^{p+1}$ chains of $X$ and the rest chains of $X_1$. The reductions and the other results of (16), of (17) (all but 17.8b), and of (18), follow automatically.

(23.4) The groups of $X$ mod $X_1$. The important operation is now the reduction $\pi$ of the chains of $X$ mod $X_1$. That is to say, if $C = C_0 + C_1, C_i \subset X_i$, then $\pi C = C_0$. Since $\mathcal{S}_0(X, G) = \mathcal{S}_0(X_0, G) \times \mathcal{S}_0(X_1, G), \pi$ is a collection of open homomorphisms: $\mathcal{S}_0(X, G) \rightarrow \mathcal{S}_0(X_0, G)$, (II, 6.2). We call $\pi$ the projection of $X$ into $X_0$ and denote it symbolically as $\pi: X \rightarrow X_0$. We verify here: $\pi F C = F_0 \pi C$ or $\pi F = F_0 \pi$, and we conclude as before that $\pi$ induces a collection of homomorphisms $\hat{\pi}: \mathcal{S}_0(X, G) \rightarrow \mathcal{S}_0(X_0, G)$. This time a cycle $\gamma_X$ of $X_0$ is not a cycle of $X$ but merely a chain of $X_0$ whose boundary is in $X_1$, and $\gamma_X \sim 0$ in $X_0$ means that $\gamma_X +$ a chain of $X_1 \sim 0$ in $X$. For this reason $\gamma_X$ is called a relative cycle or a cycle of $X$ mod $X_1$, and the groups of $X_0$ correspondingly written $\mathcal{S}_0(X, X_1, G), \cdots$. The term absolute cycle is sometimes applied to the cycles of $X$ itself. Thus the cycles of $X_1$ are absolute cycles, those of $X_0$ are relative cycles.

(23.5) We will now make certain identifications in accordance with (II, 8.6, ..., 8.9). First $\mathcal{S}_0(X_1, G)$ is identified with $\eta \mathcal{S}_0(X_1, G)$ in accordance with (II, 8.8) and thus becomes a closed subgroup of $\mathcal{S}_0(X, G)$. We will say
that two chains \( C^p, C^p' \in \mathfrak{C}(X, G) \) are congruent mod \( X_1 \) if they are congruent mod \( \mathfrak{C}(X_1, G) \) in the sense of (II, 5.1). We will now identify all the chains congruent to a given chain mod \( X_1 \), first with their coset mod \( \mathfrak{C}(X_1, G) \) (II, 5.1), then with the representative of the coset in \( \mathfrak{C}(X_0, G) \), thus obtaining in particular the topological identification of \( \mathfrak{C}(X, G) / \mathfrak{C}(X_1, G) \) with \( \mathfrak{C}(X_0, G) \) (II, 8.9). All these identifications will be assumed throughout the sequel in all similar instances.

Hereafter a chain mod \( X_1 \) over \( G \) is then merely a chain given to within a chain of \( X_1 \) over \( G \). The identification of the chains causes the identification of \( \mathfrak{B}^p(X_0, G) \) with the group of the cosets of \( \mathfrak{C}(X, G) \) mod \( \mathfrak{C}(X_1, G) \) consisting of the chains \( \mathfrak{C} \) over \( G \) such that \( F\mathfrak{C} \subset X_1 \). Under our identification such a chain is also to be described henceforth as a cycle mod \( X_1 \) and it is known only to within a chain of \( X_1 \).

Similarly \( \mathfrak{C}(X_0, G) \) is identified henceforth with the group of the cosets of the chains \( \mathfrak{C}(X, G) \) mod \( \mathfrak{C}(X_1, G) \) such that \( \mathfrak{C} = FC^{p+1} + D^p, D^p \subset X_1 \). The chain \( \mathfrak{C} \) is also to be described as a bounding cycle mod \( X_1 \) and is again only known to within a chain in \( X_1 \). For the same reasons \( \mathfrak{C} \sim 0 \) mod \( X_1 \) is now understood to mean that \( \mathfrak{C} - D^p \in \mathfrak{B}^p(X_0, G), D^p \subset X_1 \).

(23.6) Remark. We have temporarily denoted by \( F \) the boundary operator of \( X_1 \). However if we return to our previous custom and designate by \( F \) the boundary operator of any complex whatever then we have \( \pi F = F\pi, \pi F = F\pi \). Thus both \( \pi \) and \( \pi \) commute with \( F \). The general class of the operations with this property will come strongly to the fore in the next chapter under the designation of “chain-mapping” (IV, 9).

24. Circuits. An absolute \( n \)-circuit or merely an \( n \)-circuit is an \( n \)-complex \( X \) with the following properties: (a) \( \Gamma^n = \sum x^n_1 \) is an \( n \)-cycle mod 2; (b) no proper closed subcomplex of \( X \) possesses such a cycle, i.e., \( X \) is irreducible with respect to (a). Property (b) implies in particular: \( X = |\Gamma^n| \) (notation of 7.3).

When \( X \) is simplicial, (a) means that every \( \sigma^{n-1} \) is the face of an even number of \( \sigma^n \).

(24.1) If an \( n \)-circuit \( X \) has integral \( n \)-cycles different from 0 then: (a) their group is infinite cyclic; (b) if \( D^n \) is any integral \( n \)-cycle different from 0, then \( |D^n| = X \).

Let \( D^n = a'x^n_1 \neq 0 \) be an integral \( n \)-cycle. We show first that \( |D^n| = X \). If the \( a' \) have a common factor \( p \), \( D'^n = (1/p)D^n \) is likewise an \( n \)-cycle and as \( |D'^n| = |D^n|, D'^n \) may replace \( D^n \). Therefore we may suppose the \( a' \) relatively prime and hence one of them, say \( a' \), to be odd. Let \( b' = 0, 1 \) according as \( a' \) is even or odd. Then \( b'x^n_1 \) is a cycle mod 2 and hence \( b'x^n_1 = \Gamma^n, b'^1 = 1 \). Therefore every element of \( D^n \) is also an element of \( \Gamma^n \) and hence \( |D^n| = |\Gamma^n| = X \).

Suppose that there are integral \( n \)-cycles different from 0. Their group \( \mathfrak{B}^p(X) \) is free (11) and its dimension \( d > 0 \). If \( d > 1 \) a suitable combination \( D^n \) of the base elements will lack some \( x^n \), which contradicts \( |D^n| = X \). Therefore \( d = 1 \) and \( \mathfrak{B}^p(X) \) is infinite cyclic.
Under the same conditions \( \mathcal{B}(X) \) will have a base consisting of a single \( \Delta^n \), called a basic \( n \)-cycle. The only other basic \( n \)-cycle is \( -\Delta^n \).

(24.2) The \( n \)-circuit is called orientable when it possesses integral \( n \)-cycles, non-orientable otherwise.

A simple \( n \)-circuit (sometimes called an \( n \)-pseudo-manifold) is an \( n \)-circuit in which every \( (n - 1) \)-element is a face of precisely two \( n \)-elements. The simple \( n \)-circuit may be orientable or not.

If \( Y \) is a closed subcomplex of \( X \) and \( X - Y \) is an \( n \)-circuit, \( X \) is called a relative \( n \)-circuit, or an \( n \)-circuit mod \( Y \). This may be combined with "orientability" or the "simple-circuit" property. In the relative circuits the \( n \)-cycles \( \Gamma^n \), \( \Delta^n \) are cycles of \( X \) mod \( Y \).

Let \( X \) denote an \( n \)-circuit (absolute or relative). From the definition we infer that its \( nth \) homology group \( \mathcal{H}^n(X, 2) \) is cyclic, i.e., consists of 0 and a single element \( \Gamma^n \). Then \( X \) is orientable whenever its integral homology group \( \mathcal{H}^n(X) \) is cyclic, non-orientable when \( \mathcal{H}^n(X) = 0 \). When the circuit is relative, the homology groups are those of \( X \) mod \( Y \).

(24.3) Examples. \( \mathcal{B}^{n+1} \), \( n \geq 1 \), is an absolute orientable \( n \)-circuit. Take a rectangle \( ABCD \), match \( C \) with \( D \), \( B \) with \( A \). There results the so-called Möbius strip. If \( [ABCD : AB] = 2 \), the resulting complex is a non-orientable \( n \)-circuit mod \( (AD \cup BC) \).

(24.4) Simplicial simple \( n \)-circuit. For these important circuits the defining properties may be given a more elementary form in accordance with:

(24.5) If \( K - L \) is simplicial, n.a.s.c. for \( K \) to be a simple \( n \)-circuit mod \( L \) are:

(\( \alpha \)) every simplex \( \sigma \in K - L \) is a face of \( n \sigma \);

(\( \beta \)) every \( \sigma^{n-1} \) is a face of two and only two \( \sigma^n \);

(\( \gamma \)) the set \( M \) of the \( \sigma^{n-1} \) and \( \sigma^n \) of \( K - L \) is connected.

Notice that \( M \) is the complement of the \( (n - 2) \)-section of \( K - L \) and so it is an open subcomplex of \( K - L \).

When \( K - L \) is a simple \( n \)-circuit both \( \alpha \) and \( \beta \) hold by definition. As for \( \gamma \) if \( \{M_i\} \), \( i = 1, 2, \cdots, r \), are the components of \( M \) then \( \Gamma^n = \sum \Gamma^n_i \), \( \Gamma^n_i \subset M_i \), and \( \Gamma^n \) is an \( n \)-cycle mod \( (L, 2) \). Hence if \( K - L \) is an \( n \)-circuit we must have \( r = 1 \), or \( \gamma \) holds. Conversely, suppose that \( \alpha \), \( \beta \), \( \gamma \) hold. In view of \( \beta \), \( \sum \sigma^n_i = \Gamma^n \) is a cycle mod \( (L, 2) \) so property \( \alpha \) holds. Suppose that a proper closed subcomplex of \( K - L \) contained another such cycle \( \Gamma^n \).

Owing to \( \alpha \), \( \Gamma^n \) must lack at least one \( n \)-simplex say \( \sigma^n_i \). Let \( \sigma^n_j \) be present in \( \Gamma^n \). Since \( K - L \) is connected in view of \( \alpha \) there is a sequence which under proper labelling may be put in the form \( \sigma^n_1, \sigma^{n-1}_2, \sigma^n_3, \cdots, \sigma^n_r \), where consequent terms are incident. Now owing to \( \beta \), and since \( \Gamma^n \) is a cycle mod \( (L, 2) \), if \( \sigma^n_i \) is a face of \( \Gamma^n \), so must \( \sigma^{n-1}_{i-1} \) be, and hence likewise \( \sigma^{n-2}_{i-2} \). Consequently \( \sigma^n _1 \) must be a face of \( \Gamma^n \), and this contradiction proves that \( \Gamma^n \) cannot exist, or property \( \beta \) holds also. Therefore \( K - L \) is an \( n \)-circuit, and in view of \( \beta \) it is simple. This proves (24.5).

(24.6) Example. The sphere \( S^n = \mathcal{B}^{n+1} \), \( n \geq 0 \), is a simplicial, simple, orientable \( n \)-circuit.
§6. DUALITY THEORY FOR FINITE COMPLEXES

25. (25.1) Let \( X = \{ x_i \} \) be a finite complex and \( X^* = \{ x'_i \} \) its dual. We shall compare the various groups of the two complexes.

Since \( X^* \) is a finite complex it has all the general properties of finite complexes. However, it is convenient to adopt a terminology referring the relations in \( X^* \) back to \( X \). A \((-p)\)-chain or \((-p)\)-cycle of \( X^* \) is called a \( p \)-cochain or \( p \)-cocycle of \( X \), and denoted by \( C_p, \gamma_p \). Their groups are written \( \Delta_p(X, G) \), \( \beta_p(X, G) \), those of the bounding cocycles \( \delta_p(X, G) \). The \((-p)\)-dimensional homology groups of \( X^* \) are called the \( p \)-dimensional cohomology groups of \( X \), written \( \Sigma_p(X, G) \), and the corresponding Betti numbers and torsion coefficients are written \( R_p(X, \pi) \), \( t'_p \). For reasons of euphony we will sometimes say: dual Betti numbers, groups, etc. In substance then in the notations the dimension \((-p)\) in \( X^* \) is indicated by the subscript \( p \).

All the necessary modifications are obvious enough and need not be discussed. Notice, for later reference, that the basic boundary relations for the cochains are

\[
F g_i x'_i = \sum_i g_i[x'_i : x'_{i+1}]x'_{i+1}.
\]

The boundary of \( C_p \) is then a \( C_{p+1} \) whose dimension is that of \( C_p \) decreased by one. The "dimensional" behavior is thus the same as for chains.

Referring to (23), and in the same notations, we may also introduce new types of absolute or relative cocycles. They are: the absolute cocycles of \( X \) and the cocycles of \( X \) mod \( X_0 \) (cycles of \( X^* \) mod \( X^*_0 \)).

(25.3) Let \( K \) be simplicial. The notations being those of (10.4) we will call \( \gamma_0 = \sum A^i \) the fundamental zero-cocycle of \( K \) (27.7a). The coboundary relations are then \( FC_p = \gamma_0 C_p \), under the usual incidence number scheme (5.2), and \( FC_p = C_p \gamma_0 \), under the scheme of (5.7). This is a mere restatement of (10.4d) in the "co-terminology."

26. Instead of considering the "co-theory" as a theory of a different collection of elements from those of \( X \), some authors prefer to view it as a theory of the elements of \( X \) with \(< \) reversed. It is then necessary to introduce, side by side with \( F \), a second operator \( F^* \), the coboundary operator, defined by

\[
F^*(g^i x^i) = \sum_i g'[x_i^i : x_{i+1}^i]x_{i+1}^i,
\]

which raises the dimensions by one unit, instead of lowering them like \( F \). Thus Whitney proceeds in that manner and writes \( \partial, \delta \) for \( F, F^* \). The operator \( F^* \) is a homomorphism \( \Sigma^p \to \Sigma^{p+1} \) with similar properties to those of \( F \), the cocycles are the chains of \( X \) whose coboundary vanishes, etc.

In the present work we shall definitely consider the elements of \( X^* \) as distinct from those of \( X \) with the notations and terminology indicated in (25).
DUALITY THEORY FOR FINITE COMPLEXES

To justify our choice we may anticipate and consider cartesian products of chains and cochains as in (IV, §2). If we write down expressions such as \( C^r \times C^s, C^r \times C^s \), we know by inspection the rules for calculating the appropriate boundary chains \( F(C^r \times C^s), F(C^r \times C^s) \), the operator \( F \) being the same throughout. However, if we adopted the alternate procedure with \( F, F^* \), we should have to write all these expressions \( C^r \times C^s \), and choose each time one of four possible operators. Three factors would impose a choice between eight operators.

It may be pointed out also that our convention merely represents adherence to those employed for many years in projective geometry and related doctrines, whereby contravariant and covariant elements are represented by distinct symbols. This is in keeping with the fact that they undergo distinct transformations.

27. It is evident that all the results of (§3) are applicable to \( X^* \), i.e., to cochains, etc. Let \( \alpha_p, \cdots \) have the same meaning for \( X^* \) as \( \alpha^p, \cdots \), for \( X \). Evidently \( \alpha_p = \alpha^p \), and from (25.1) follows that if \( \eta^*(p) \) is the \((−p)\)th incidence matrix for \( X^* \), then \( \eta^*(p + 1) = (\eta(p))^t \) (the prime means the transpose). Therefore \( \rho_{p+1} = \rho^p \) and the torsion coefficients \( t^{(p)}_{p+1} \) are the same as the \( t^p \).

Since the subscripts are the negatives of the dimensions we obtain in place of (15.1)

\[
R_p = \alpha_p - \rho_{p+1} - \rho_p = \alpha^p - \rho^p - \rho^{p-1} = R^p.
\]

Hence \( \mathfrak{T}_{p+1}, \mathfrak{B}_p \) are abstractly the same as \( \mathfrak{T}^p, \mathfrak{B}^p \). This proves the following theorem which is the analogue of Poincaré's initial duality theorem for manifolds (V, 33.1) and as far as Betti groups go, is the duality theorem of [L, 286] (duality theorem for pseudo-cycles):

(27.1) First duality theorem. The \( p \)th Betti and dual Betti groups are isomorphic, and likewise the \( p \)th torsion and \((p + 1)\)st dual torsion groups, and

\[
R_p = R^p, \quad t^{(p)}_{p+1} = t^p.
\]

We state also explicitly the convenient property:

(27.3) When \( X \) is torsion-free so is \( X^* \) and the integral \( p \)th homology and cohomology groups are isomorphic with one another as well as with the \( p \)th Betti group of \( X \).

(27.4) The Betti numbers and torsion coefficients of a finite complex determine all its homology and cohomology groups.

Let us define \( X \) as \( p \)-cocyclic, \( \cdots \) whenever \( X^* \) is \((-p)\)-cyclic, \( \cdots \). Then we have by (21.3):

(27.5) Whenever \( X \) is \((p, \cdots, q)\)-cyclic or acyclic it is also \((p, \cdots, q)\)-cocyclic or acocyclic, and conversely.

(27.6) Let \( X = \{ x \} \) be such that \( p \leq \dim x \leq q \). Then: (a) no \( p \)-cocycle \( \sim 0 \) unless it is zero; (b) every \( q \)-cochain is a cocycle (10.12).

(27.7) Let \( K = \{ \sigma \} \) be a simplicial complex with vertices \( \{ A_i \} \) and duals \( \{ A^i \} \). Then:

(a) \( \gamma_0 = \sum A^i \) is a cocycle;

(b) if \( K \) is connected every zero-cocycle is of the form \( q\gamma_0 \);
(c) if $K$ is a simple $n$-circuit then every $\sigma_i^i$ is an $n$-cocycle and $\sigma_i^i \sim \pm e_i^i$ for all $i$, hence every $n$-cocycle $\sim g\sigma_i^i$.

**Proof of (a).** We have $F\eta_i^i = \eta_i^i A_i^i$, where $\sum_j \eta_j^i = 0$. Hence $FA^i = \eta_i^i e_i^i$, and so $F\gamma_0 = F \sum_i A^i = \sum_i e_i^i \sum_j \eta_i^j = 0$. Therefore $\gamma_0$ is a cocycle.

**Proof of (b).** When $K$ is connected then $R_0 = R^0 = 1$ and there are no $\xi_i$. It follows that $K$ is cocyclic in the dimension $0$. Consequently every zero-cocycle is of the form $\rho_0$ where $\rho_0$ is an integral cocycle. Suppose $\delta_0 = x_i^i A_i^i$, and let $\gamma_0 = y_0 I$. We have then $\sum_i A_i^i = y_0 x_i^i A_i^i$, and so $y_0 x_i^i = 1$. Hence $x_i = y = \pm 1$, $\delta_0 = \pm \gamma_0$, from which (b) follows.

**Proof of (c).** Let $K$ be a simple $n$-circuit and let $K_{n-2}$ be its $(n-2)$-section. By (24.5) $K - K_{n-2}$ is connected. It follows that if $\sigma_0^i$, $\sigma_0^{i+1}$ are any two $n$-simplexes of $K$ there is a sequence $\sigma_0^i = \sigma_0^{i+1}, \sigma_0^{i+1}, \cdots, \sigma_0^{n-1}, \sigma_0^n = \sigma_0^i$, in which consecutive elements are incident. Consequently this holds equally regarding $\sigma_0^i, \sigma_0^{i+1}; \cdots, \sigma_0^n$. Since $K$ is a simple circuit: $[\sigma_0^{i-1}: \sigma_0^i] = \pm 1 = \pm [\sigma_0^{i-1}: \sigma_0^{i+1}]$. Since the only elements of $[\sigma_0^n]$ incident with $\sigma_0^{i-1}$ are $\sigma_0^i$, and $\sigma_0^{i+1}$, we have $F\sigma_0^{i-1} = (\sigma_0^i \pm \sigma_0^{i+1}) \sim 0$, or $\sigma_0^i \sim \pm \sigma_0^{i+1}$, and finally $\sigma_0^n \sim \pm \sigma_0^i$.

(27.8) Example. Consider the sphere $S^n = \mathbb{R}^{n+1}, \sigma^{n+1} = A_0 \cdots A_{n+1}, n > 0$. Since $S^n$ is $(0, n)$-cyclic it is also $(0, n)$-cocyclic. Its zero-cocycles are all of the form $g \sum A_i$. We have seen (24.6) that $S^n$ is a simple $n$-circuit and so its $n$-cocycles are all $\sim gA^1 \cdots A^n$.

### 28. Kronecker index of chains and cochains.

Further progress will rest upon an extension of the concept of Kronecker index. The connection with the earlier concept will be made in (46).

(28.1) Definition. Let $\beta(p)$ be as in the Introduction. Then the Kronecker index of the couple $x_p^i$, $x_p^j$ is the number

$$KI(x_p^i, x_p^j) = \beta(p) \delta_i^j \text{(Kronecker delta)},$$

and the Kronecker index of $x_p^i$, $x_p^j$ is

$$KI(x_p^i, x_p^j) = \beta(-p) \delta_i^j = (-1)^p KI(x_p^i, x_p^j).$$

(28.4) We have just specified values for the Kronecker index whenever $X, X^*$ are so oriented that $[x_p^i : x_p^{i+1}] = [x_p^{i-1} : x_p^j]$. In order to allow for arbitrary reorientations we reorient the convention that if $X, X^*$ are reoriented by means of orientation functions $\alpha(x_p^i), \alpha^*(x_p^j)$ then

$$KI(x_p^i, x_p^j) = \alpha(x_p^i) \alpha^*(x_p^j) \beta(p) \delta_i^j = (-1)^p KI(x_p^i, x_p^j).$$

**Remark.** In [L, 165] the analogous definition of the index was given by means of (28.2) but without the factor $\beta(p)$, thus causing dissymmetry under dualization. To pass from the present to the earlier definition it is merely necessary to reorient $X^*$ by $\alpha(x_p) = \beta(-p)$.

(28.5) We shall now choose two groups $G, H$ paired to a third $J$ and with a multiplication $gh$. We define $hg = gh$, so that $H, G$ are formally paired to $J$
with the same multiplication gh. Thus the two groups \( G, H \) are paired to \( J \) in one or the other order and with a multiplication independent of the order of the pairing. We shall briefly describe this relationship by the statement "\( G, H \) are commutatively paired to \( J \)."

Suppose that we have a chain and cochain over \( G \) and \( H \):

\[
C^p = g^i x_i^p, \quad g^i \in G; \quad C_p = h_a x_a^p, \quad h_a \in H.
\]

The Kronecker index of \( C^p, C_p \), written \( \text{KI}(C^p, C_p) \) is an element of \( J \) defined by:

\[
\text{KI}(C^p, C_p) = g^i h_a \text{KI}(x_i^p, x_a^p) = \beta(p) g^i h_a.
\]

Similarly with the terms in reverse order we define

\[
\text{KI}(C_p, C^p) = \beta(-p) g^i h_a.
\]

(28.8) **Interpretation.** A noteworthy interpretation, very close to the initial reason for introducing the index, is to consider that the dual elements \( x_i^p, x_a^p \) cross one another when \( i = j \), and do not cross one another when \( i \neq j \). It will be convenient to say that \( C^p, C_p \) have a crossing at \( x_i^p, x_a^p \) whenever both \( g^i \neq 0, h_a \neq 0 \). We agree to count this crossing with the weight \( \beta(p) g^i h_a \) (i unsummed) and so the index (28.6) may be interpreted as a mode of counting the crossings suitably weighted. If \( G = H = J \) = the ring of the integers, the weights become multiplicities in a reasonable sense. Viewed in this manner the index has for example played an important role in the author's work on Algebraic Geometry.

(See [L, VIII, §4].)

29. The Kronecker index will now be utilized as a basis for deriving the duality relations between the chain- and cochain-groups. We shall use the following notations:

The chains and cochains over \( G \) and \( H \) are denoted by \( C^p \), \( C_p \), the cycles and cocycles over \( G \) and \( H \) by \( \gamma^p \), \( \gamma_p \) and their homology and cohomology classes by \( \Gamma^p, \Gamma_p \). We shall also denote by \( \mathfrak{C}^p, \mathfrak{B}^p, \mathfrak{K}^p, \mathfrak{S}^p \) the groups of chains, cycles, bounding cycles and homology groups over \( G \), and by \( \mathfrak{C}_p, \mathfrak{B}_p, \mathfrak{S}_p, \mathfrak{K}_p \) the same for the cochains, \( \cdots \) over \( H \). The group \( H \) is assumed discrete.

As a preliminary we prove the important relation:

\[
\text{KI}(FC^{p+1}, C_p) = (-1)^p \text{KI}(C^{p+1}, FC_p)
\]

which is the analogue of Formula 20 of [L, 169]. If \( C^{p+1} = x_i^{p+1}, C_p = x_a^p \), both sides of (29.1) become \( \beta(p) [x_i^{p+1} : x_a^p] \), so (29.1) holds. Since the two sides are bilinear in \( x_i^{p+1}, x_a^p \), (29.1) holds in all cases.

(29.2) **The index obeys the commutation rule**

\[
\text{KI}(C_p, C^p) = (-1)^p \text{KI}(C^p, C_p).
\]

(29.4) \( \text{KI}(C^p, C_p) \) is a group multiplication for \( \mathfrak{C}^p, \mathfrak{C}_p \) which pairs them to \( J \).

Since \( Gx_i^p, Hx_a^p \) are respective isomorphs of \( G, H \) they are paired to \( J \) with the multiplication \( \beta(p) gh \). Since \( \mathfrak{C}_p \) is discrete

\[
\mathfrak{C}^p = PGx_i^p, \quad \mathfrak{C}_p = PHx_a^p,
\]

(29.4) follows from (II, 16.1).
(29.6) \( \gamma' \) or \( \gamma_p \sim 0 \rightarrow \text{KI}(\gamma', \gamma_p) = 0. \)

An equivalent formulation is

(29.7) \( B_p, B' \) annul \( \bar{B'}, \bar{B} \).

It is only necessary to prove the property of the pair \( \bar{B'}, B_p \), and hence (II, 15.4) that \( B' \) annuls \( B_p \), or that \( \gamma' = \text{FC}^{p+1} \) annuls \( B_p \), and this follows immediately from (29.1)_p, since it yields:

\[ \text{KI}(\gamma', \gamma_p) = (-1)^p \text{KI}(\text{FC}^{p+1}, 0) = 0. \]

(29.8) \( \text{KI}(\gamma', \gamma_p) \) depends solely upon the classes \( \Gamma^p, \Gamma_p \) (29.6).

(29.9) Definition. The fixed value of \( \text{KI}(\gamma', \gamma_p) \) under (29.8) is called the Kronecker index of the classes \( \Gamma^p, \Gamma_p \) written \( \text{KI}(\Gamma^p, \Gamma_p) \).

(29.10) \( \text{KI}(\Gamma^p, \Gamma_p) \) is a group multiplication for \( \bar{B'}, \bar{B} \), and obeys the commutation rule (29.3), (with \( \Gamma \) in place of \( C \)).

Except for commutation (29.10) is a consequence of (29.4), and (II, 15.6), while the commutation rule follows from (29.2).

(29.11) If \( G, H \) are \( J \)-orthogonal so are \( \bar{C}, \bar{C} \).

For \( G \bar{x}_p, Hx_p \) are then \( J \)-orthogonal and so (29.11) follows from (29.5) and (II, 16.1).

(29.12) If \( G, H \) are \( J \)-orthogonal, \( B_p \) is the annihilator of \( \bar{B'} \) and likewise \( \bar{B} \) of \( \bar{B'} \).

It is sufficient to prove the property of the pair \( B_p, \bar{B'} \). We have just shown that every \( \gamma_p \) annuls \( \bar{B'} \), so it is only necessary to prove the converse, or that if \( \gamma_p \) annuls \( \bar{B'} \) it is a cocycle. If \( \gamma_p \) annuls \( \bar{B'} \) it annuls \( \bar{B} \) and so by (29.1):

\[ \text{KI}(\text{FC}^{p+1}, \gamma_p) = (-1)^p \text{KI}(\text{FC}^{p+1}, F\gamma_p) = 0. \]

Thus \( F\gamma_p \in C_{p+1} \) annuls \( \bar{C}^{p+1} \), and so by (29.11)_p+1, \( F\gamma_p = 0 \) or \( \gamma_p \) is a cocycle.

30. Duality theorems. The situation which will now be faced will recur again and again in a more or less similar form. It is therefore best to introduce at the outset a systematic terminology designed especially to avoid undue repetition later.

(30.1) Definition. The pair \( (G, H) \) in the order named, is said to form a normal couple whenever one of the following two possibilities arises:

(a) \( G \) is compact, \( H \) is discrete and they are dually paired by a commutative multiplication \( gh \) to \( B \). In particular then they are orthogonal and each \( \cong \) the character-group of the other.

(b) \( G = H = J \) is a discrete field and the multiplication \( gh \) is merely the multiplication of the field \( J \). Notice that \( G \) may be viewed as a linearly compact (one-dimensional) vector space over \( J \), and \( H \) as a (one-dimensional) discrete vector space over \( J \), dually paired under the multiplication \( gh \), which is merely the multiplication of the field \( J \) (II, 32.5).
We may now state the

(30.2) SECOND DUALITY THEOREM FOR FINITE COMPLEXES. If \( G, H \) is a normal couple then the \( p \)th homology and cohomology groups \( \mathfrak{H}(X, G), \mathfrak{H}_p(X, H) \) are dually paired (to \( \mathfrak{P} \) or to the discrete field \( J \) when \( G = H = J \)) and with the class Kronecker index as the group multiplication.

This is the duality theorem of [L, 286] with the all-important Pontrjagin group duality complement.

Since \( G, H \) are dually paired to \( \mathfrak{P} \) or \( J \) as the case may be, so are their isomorphs \( Gx_i^p, Hx_i^p \) and with the Kronecker index as the multiplication. Hence the same holds for \( \mathfrak{C}^p, \mathfrak{S}_p \) (II, 20.7, 33). Since \( G, H \) are dually paired, \( \mathfrak{S}_p \) and \( \mathfrak{S}_p^* \) are one another’s annihilators in \( \mathfrak{C}_p, \mathfrak{C}^p \) (29.12; II, 20.5, 33), and likewise for \( \mathfrak{C}^p, \mathfrak{S}^*_p \) (\( \mathfrak{S}^*_p \)). Therefore \( \mathfrak{S}_p^* = \mathfrak{S}^*/\mathfrak{S}^*p \) and \( \mathfrak{S}_p = \mathfrak{S}_p/\mathfrak{S}^*_p \) are likewise orthogonal to \( \mathfrak{P} \) or \( J \) as the case may be (II, 15.6) and hence dually paired (II, 20.6, 33). Since \( \Gamma^p, \Gamma_p \) are merely the cosets of \( \gamma^p, \gamma_p \) mod \( \mathfrak{S}^*_p, \mathfrak{S}^*_p \), the multiplication of the dual pairing is the one described under (II, 15.5a) and it is precisely the class Kronecker index. This proves the theorem.

31. Dual categories. The preceding theorem will serve as a pattern for a number of similar theorems occurring later. In order to facilitate their description and minimize repetition, we introduce the convenient concept of dual categories.

Let \( A, B \) be two collections of cycles and cocycles of all the different dimensions over various groups of coefficients. For the missing dimensions the groups are taken to be zero. Let it be possible to define the groups \( \mathfrak{S} \) and hence the homology and cohomology groups \( \mathfrak{S} = \mathfrak{S}/\mathfrak{S}_p \), likewise the Kronecker index \( \text{KI}(\gamma^p, \gamma_p) \) with the same properties other than orthogonality as in (29). When the coefficient-group is a field \( J \) it is assumed that the corresponding groups \( \mathfrak{S}, \mathfrak{S} \) are vector spaces over \( J \), and in particular satisfy the basic convention (II, 22.2). Under our assumptions one may define a class index \( \text{KI}(\Gamma^p, \Gamma_p) \).

If \( (G, H) \) is any normal couple and \( \mathfrak{H}_p(G), \mathfrak{H}_p(H) \) are the corresponding homology and cohomology groups, we shall say that the cycles of \( A \) and the cocycles of \( B \) [the cocycles of \( B \) and the cycles of \( A \)] are:

- **dual categories** whenever the groups \( \mathfrak{H}_p(G), \mathfrak{H}_p(H) \) are dually paired (to \( \mathfrak{P} \) or to the discrete field \( J \) when \( G = H = J \)) and with the class Kronecker index as the group multiplication;

- **weak dual categories** whenever the groups are defined only for \( G = H = J \) = a discrete field, and are vector spaces orthogonal to \( J \) with the class Kronecker index as the multiplication. Whenever the dimensions of the paired spaces are finite their pairing is again a full dual pairing of vector spaces (II, 34).

Since orthogonality to \( \mathfrak{P} \) or a discrete field \( J \) results in each case in the dual pairing (II, 20.6, 33) we may say that: (a) the characteristic property of dual categories is orthogonality to \( \mathfrak{P} \) or \( J \); (b) weak dual categories are those where only orthogonality to \( J \) may take place.
In the terminology just introduced (30.2) assumes the form:
(31.1) The cycles and cocycles of a finite complex in one or the other order are
dual categories.

As a further application we also have:
(31.2) Let $X, X_0, X_1$ be as in (23). Then the cycles of $X$ mod $X_1$ and the
cocycles of $X_0$ in one or the other order are dual categories.

A similar statement may be made for the cocycles of $X$ mod $X_0$ and the
cycles of $X_1$, but it is merely the expression of (31.1) for $X_1$ itself. Notice
also that when $X_0 = \emptyset$ and $X = X_1$, (31.2) reduces to (31.1).

32. Several noteworthy properties of dual categories are immediate con-
sequences of properties of vector spaces.

We suppose then that $A, B$ are dual categories of any sort and take the groups
over a discrete field $J$. The formulation is given so as to include possible infinite-
dimensional groups which may occur later. The Betti and dual Betti numbers
have their usual significance of dimension of the homology and cohomology
groups.

(32.1) The $p$th Betti and dual Betti numbers over $J$ are finite and equal or
else both infinite (II, 25.9d).

When these numbers are finite, in particular for a finite complex, (32.1)
gives the full content of (30.2) for the groups over the field $J$.

(32.2) If the cycles $\gamma_i^p$, ($i = 1, 2, \ldots, r$) are independent with respect to homolo-
gy, there can be selected cocycles $\gamma^i_p$, ($j = 1, 2, \ldots, r$) such that

\begin{equation}
KI(\gamma_i^p, \gamma^j_p) = \delta_{ij}.
\end{equation}

For this is true for the classes (II, 25.9b), and so by (29.8) for $\gamma^p, \gamma_p$.

(32.4) If the Betti numbers are finite and $\{\gamma_i^p\}, \{\gamma^j_p\}$ are maximal independent
sets (with respect to $\sim$), then

\begin{equation}
|KI(\gamma_i^p, \gamma^j_p)| \neq 0.
\end{equation}

Since the $\Gamma_i^p$ are independent, by (II, 25.9a) classes $\Gamma_i^p$ may be chosen such
that

\begin{equation}
|KI(\Gamma_i^p, \Gamma^j_p)| \neq 0,
\end{equation}

which, in view of (29.8) yields (32.4).

33. Returning to the duality theorem for finite complexes, in view of its
importance, and also for later purposes, we shall indicate another proof based
on the comparison of canonical bases (14).

Let us pass from the bases $\{x_i^p\}$ for the integral chains to new bases $\{e_i^p\}$
by simultaneous transformations

\begin{equation}
x_i^p = \lambda_i^p e_i^p, \quad \lambda^p = \| \lambda_i^p \| \text{ unimodular.}
\end{equation}

It will be convenient to designate by $\lambda = \| \lambda_i^p \|$ the matrix $(\lambda^p)^{-1}$, i.e., such
that $\lambda_i^p \lambda^p_k = \delta_k (p \text{ unsummed})$. Since $\lambda$ is also unimodular,

\begin{equation}
x_i^p = \lambda_i^p e_i^p
\end{equation}
is a simultaneous transformation from the bases \( \{x^i_p\} \) to the new bases \( \{e^i_p\} \). If we let \( x^i_{p+1} \) = \( \eta^i_{p+1} \), then:

\[
(33.3) \quad F x^{i+1} = \eta^{i+1}_{i} x^{i}, \\
(33.4) \quad F x^{i}_{p} = \eta^{i}_{p} x^{i}_{p+1},
\]

the matrix in (33.4) being the transpose of the matrix in (33.3). Using (33.1) and (33.2) we now obtain

\[
(33.5) \quad F e^{i+1} = \zeta^{i+1}_{i} e^{i}, \\
(33.6) \quad F e^{i}_{p} = \zeta^{i}_{p} e^{i}_{p+1},
\]

where \( \zeta^{i}_{p} = \lambda^{p+1}_{p} \eta^{i}_{p} \lambda^{i}_{p+1} \); again the matrix of (33.6) is the transpose of the matrix of (33.5).

By an elementary calculation:

\[
(33.7) \quad KI(e^{i}_{p}, e^{j}_{p}) = \beta(p) b^{i}_{p}.
\]

In other words the index is invariant under simultaneous application of (33.1) and (33.2).

Suppose in particular that (33.1) is the transformation to the canonical bases \( \{a^{i}_{p}, \cdots, e^{i}_{p}\} \) of (14) and let the corresponding new bases \( \{e^{i}_{p}\} \) for the co-cycles be \( \{a^{i}_{p}, \cdots, e^{i}_{p}\} \). In other words if \( e^{i}_{p} = a^{i}_{p}, \cdots \) then \( e^{i}_{p} = a^{i}_{p}, \cdots \).

Formula (14.2) specified the form of the diagonal matrix \( \zeta^{i}_{p} \) of (33.5) so that

\[
(33.8) \quad F a^{i+1}_{p} = 0, \quad F b^{i+1}_{p} = 0, \quad F c^{i+1}_{p} = 0, \\
F d^{i+1}_{p} = \zeta^{i}_{p} b^{i}_{p}, \quad F e^{i+1}_{p} = a^{i}_{p}.
\]

Applying (33.6) we have immediately

\[
(33.9) \quad F a^{i}_{p} = e^{i+1}_{p}, \quad F b^{i}_{p} = \zeta^{i}_{p+1} b^{i+1}_{p}, \quad F c^{i}_{p} = 0, \quad F d^{i}_{p} = 0, \quad F e^{i}_{p} = 0,
\]

where \( \zeta^{i}_{p+1} = \zeta^{i}_{p} \), and \( i \) is not summed in (33.8), (33.9).

Furthermore

\[
(33.10) \quad KI(a^{i}_{p}, a^{j}_{p}) = \cdots = KI(e^{i}_{p}, e^{j}_{p}) = \beta(p),
\]

and all the other indices will be zero. Thus we have proved:

\[
(33.11) \quad \text{At the same time as the bases for the chains are reduced to the canonical form (14.2) those for the cochains may be reduced to the canonical form (33.9) with indices related as stated. Notice that in (33.9) the analogues of } a^{i}_{p}, \cdots, e^{i}_{p} \text{ are } e^{i}_{p}, \cdots, a^{i}_{p}. \]

34. The application to the duality theorem is immediate. Suppose that we have two groups of coefficients \( G \) and \( H \) for the homology and cohomology groups, respectively. Then the direct decomposition (16.9) and the result of (33.9) yield:

\[
(34.1) \quad S^{p}(X, G) \cong P(G^*(t^{p}_{i})b^{p}_{i}) \times P(Gc^{p}_{i}) \times P(G[t^{p-1}_{i}]d^{p}_{i}),
\]
\[ \Phi_p(X, H) \cong \mathbf{P}(H[t_{p+1}]b_p^*) \times \mathbf{P}(HC_p^*) \times \mathbf{P}(H^*(t_p^*)d_p^*). \]

Referring to (II, 20.12), or else directly if \( G = H = J \), a field, we find that when \((G, H)\) is a normal couple then \( G^*(t_p^*)b_p^* \) and \( H[t_{p+1}]d_p^* \) are dually paired with the Kronecker index as the group multiplication. Similarly each group in (34.1) is dually paired with one and only one of the groups in (34.2). Hence (II, 20.7, 33) \( \Phi^p(X, G) \) and \( \Phi_p(X, H) \) are likewise dually paired with the Kronecker index as the group multiplication, and this is (30.2).

§7. LINKING COEFFICIENTS. DUALITY IN THE SENSE OF ALEXANDER

35. The Kronecker index may be considered as the algebraic analogue of the intuitive concept of "multiplicity of intersection," for instance of two plane curves, in geometry. Another closely related geometric concept is that of linking coefficient, of two curves \( C, D \) in a three-space \( \mathbb{R}^3 \), which describes the "algebraic" number of times each twists around the other. We shall show that under certain conditions such numbers may be introduced in complexes, and as we shall see later (VII, 9) in certain topological spaces.

Much of the argument will refer to finite complexes which are \((p - 1, p)\)-acyclic. Let \( X \) be such a complex, and \((G, H)\) a normal couple. If \( \gamma^p \) is a cycle over \( G \) and \( \gamma_p \) a cocycle over \( H \), we have \( \gamma^p \sim 0 \) and so since \( G, H \) are division-closure groups \( \gamma^p = FC^p \) (17.2). Suppose also that \( \gamma^p = FC^p \). Therefore \( F(C^p - C^p) = 0 \) and \( C^p - C^p \), being a \( p \)-cycle, is \( \sim 0 \). Hence by (29.6):

\[ KI(C^p, \gamma_p) = KI(C^p, \gamma_p). \]

Thus the index at the left is independent of the \( C^p \) bounded by \( \gamma^p \), and its value is known as the linking coefficient of \( \gamma^p, \gamma_p \) written \( \text{Lk}(\gamma^p, \gamma_p) \). One must keep in mind that it is only defined for \( \gamma^p, \gamma_p \) over a normal couple \( G, H \).

Since cycles and cocycles are dual categories \( X \) is also \((p - 1, p)\)-acyclic. This enables us to interchange their role and so define a linking coefficient \( \text{Lk}^*(\gamma^p, \gamma_p) \). However (29.3) yields at once \( \text{Lk}^* = (-1)^{p-1} \text{Lk} \), so except for a fixed change in sign, their values are equal.

36. The duality theorems which have been given so far relate merely the groups of a complex to one another. The linking coefficients will enable us to give full expression to duality theorems of a different type introduced by J. W. Alexander. They may be described at this stage, as relating under certain conditions the groups of a closed subcomplex to those of the complement. What is commonly known as Alexander's duality theorem is a duality theorem for topological complexes immersed in spheres. However the general intent is always the same, and we shall refer to the whole class of similar propositions as "duality theorems of the type of Alexander."

37. (37.1) Theorem. Let \( X \) be \((p - 1, p)\)-acyclic and let \( X_1 \) be a closed subcomplex of \( X \) and \( G \) compact or a field. Then there subsists the isomorphism

\[ \Phi^{p-1}(X_1, G) \cong \Phi^p(X, X_1, G). \]
If \( \gamma^p \) is a cycle of \( X \) mod \( X_1 \), \( \delta^{p-1} = F \gamma^p \) is an absolute cycle of \( X_1 \). Here \( F \) is the boundary operator for \( X \) itself. Thus \( F \) induces a homomorphism: \( \mathfrak{B}(X, X_1, G) \to \mathfrak{B}^{p-1}(X_1, G) \). Suppose \( \gamma^p \sim 0 \) mod \( X_1 \). Since \( G \) has the division-closure property, \( \gamma^p = F \sigma^{p+1} + D^p, D^p \subset X_1 \), and hence \( \delta^{p-1} = FD^p \), or \( \delta^{p-1} \sim 0 \) in \( X_1 \). Therefore \( F \) maps \( \mathfrak{B}(X, X_1, G) \to \mathfrak{B}^{p-1}(X_1, G) \) and hence (II, 5.4) \( F \) induces a homomorphism \( \varphi: \mathfrak{B}(X, X_1, G) \to \mathfrak{B}^{p-1}(X_1, G) \). To prove (37.1) we merely need to show that \( \varphi \) is an isomorphism.

(a) \( \varphi \) is a mapping of \( \mathfrak{B}(X, X_1, G) \) onto \( \mathfrak{B}^{p-1}(X_1, G) \). Since \( X \) is acyclic in the dimension \( p-1 \) and \( G \) is a division-closure group every \( \delta^{p-1} \) is an \( F \gamma^p \), so (a) holds.

(b) \( \varphi \) is univalent. It is required to show that \( F \gamma^p = \delta^{p-1} \sim 0 \) in \( X_1 \) \( \Rightarrow \gamma^p \sim 0 \) mod \( X_1 \) in \( X \). Since \( G \) is a division-closure group if \( \delta^{p-1} \sim 0 \) in \( X_1 \) there is a \( D^p \) in \( X_1 \) such that \( \delta^{p-1} = FD^p \) and as a consequence \( \gamma^p - D^p \) is a cycle of \( X \). Since \( X \) is acyclic in the dimension \( p \) we have \( \gamma^p - D^p \sim 0 \) in \( X \) or \( \gamma^p \sim 0 \) mod \( X_1 \) in \( X \) which proves (b).

The group \( G \) may be compact or else a discrete field. Suppose \( G \) compact. The groups \( \mathfrak{S} \) over \( G \) are then compact also. By (a), (b) \( \varphi \) is a mapping which is an isomorphism in the algebraic sense of one compact group into another. It follows that \( \varphi \) is topological and hence it is an isomorphism. When \( G \) is a discrete field the groups \( \mathfrak{S} \) are finite-dimensional vector spaces, hence discrete and so \( \varphi \) is again an isomorphism. This proves (37.1).

38. Let again \( G, H \) be a normal couple and \( (X_0, X_1) \) a dissection of \( X \). If \( \delta^{p-1} \) is a cycle of \( X_1 \), we have \( \delta^{p-1} \sim F \gamma^p \), \( \gamma^p \) a cycle of \( X_0 \). We may therefore introduce \( \text{Lk}(\delta^{p-1}, \gamma_\rho) \) and we have

\[
(38.1) \quad \text{KI}(\gamma^p, \gamma_\rho) = \text{Lk}(\delta^{p-1}, \gamma_\rho) = \text{Lk}(F \gamma^p, \gamma_\rho).
\]

It is obvious that, \( \text{Lk} \) takes a fixed value when \( \delta^{p-1}, \gamma_\rho \) vary in fixed classes \( \Delta^{p-1}, \Gamma_\rho \) of \( X_1 \) and \( X_0 \), and this value is by definition the class linking coefficient \( \text{Lk}(\Delta^{p-1}, \Gamma_\rho) \). From (37.2) we deduce:

\[
(38.2) \quad \text{KI}(\Gamma^p, \Gamma_\rho) = \text{Lk}(\Delta^{p-1}, \Gamma_\rho) = \text{Lk}(F \Gamma^p, \Gamma_\rho),
\]

where \( F \Gamma^p \) denotes the homology class of \( F \gamma^p \) in \( X_1 \). From the duality theorem (30.2) and (38.2) follows then:

(38.3) Duality theorem. Let \( X \) be \( (p-1, p) \)-acyclic, and let \( (X_0, X_1) \) be a dissection of \( X \), with \( X_0 \) open and \( X_1 \) closed. Given any normal couple \( (G, H) \), the groups \( \mathfrak{S}^{p-1}(X_1, G) \) and \( \mathfrak{S}_p(X_0, H) \) are dually paired with the class linking coefficient as the group multiplication.

(38.4) Obvious remark. In (38.3) the two groups \( G, H \) may be interchanged.

Coupling (38.3, 38.4) with (30.2) for \( X_1 \) we find:

(38.5) Under the same conditions as in (38.3) we have \( \mathfrak{S}^{p-1}(X_1, G) \cong \mathfrak{S}_p(X_0, G) \) for any \( G \) which is compact, discrete or a field (Kolmogoroff [b]; see Alexandroff [f]).
In the special case where $G = H$ = a field of characteristic $\pi$ we have:

(38.6) Under the same conditions as in (38.3):

\[ R^{n-1}(X_1, \pi) = R_p(X_0, \pi). \]

39. We shall now consider certain important related special cases.

(39.1) $X$ is acyclic. (38.3) holds then for all $p$.

(39.2) $X$ is simplicial and zero-cyclic. Then (38.3) holds for $p > 1$. Since $X$ is zero-cyclic, it is connected, and a zero-cycle $\gamma^0 \sim 0$ in $X$ when and only when $K(\gamma^0) = 0$ (20.10).

The homology classes of the zero-cycles in $X_1$ which are $\sim 0$ in $X$ form a subgroup $\mathcal{H}(X_1, G)$ of $\mathcal{H}(X_1, G)$, and the same argument goes through as before provided that $\mathcal{H}(X_1, G)$ is replaced by $\mathcal{H}_0(X_1, G)$. Now if $\gamma^0$ is any zero-cycle of $X_1$, and if $A_i$ are vertices one on each component of $X_1$, then (20.7a):

\[ \gamma^0 \sim g_i^i A_i = K(\gamma^0)A_1 + g_i^i(A_i - A_i) = K(\gamma^0)A_1 + \delta^0, \]

where $K(\delta^0) = 0$. Since $A_i \sim 0$ in $X_1$ if $\Gamma^0$ is the class of $A_i$ then

\[ \mathcal{H}(X_1, G) \cong \mathcal{H}(X_1, G) / \Gamma^0. \]

Thus in (38.3) in the present instance $\mathcal{H}(X_1, G)$ must be replaced by $\mathcal{H}(X_1, G) / \Gamma^0$. In these and similar expressions later $\Gamma^0$ represents the subgroup of the classes of the cycles $g_1 A_1$.

(39.3) $X$ is simplicial, n-dimensional, and $(0, n)$-cyclic. Suppose first $n > 1$. For $1 < p < n$, the situation is as under (39.1), and for $p = 1$ as under (39.2).

Let $p = n$. Since dim $X = n$ and $X$ is n-cyclic: $\mathcal{H}(X, G) \cong G_\gamma^\gamma$, where $\gamma^\gamma$ is a basic integral n-cycle and so (37b) must be replaced by

\[ (\delta^0 \sim 0 \text{ in } X) \rightarrow (\gamma^0 \sim g_i^i \text{ mod } X_1 \text{ in } X). \]

As a consequence in place of (37.2) we have, if $\Gamma_\gamma^\gamma$ is the class of $\gamma_\gamma^\gamma$ (basic class):

(39.4)

\[ \mathcal{H}(X, G) / \Gamma_\gamma^\gamma \cong \mathcal{H}^{n-1}(X_1, G), \]

and the factor-group at the left must replace $\mathcal{H}^n(X, X_1, G)$.

Finally if $n = 1$, we must combine the operation under (39.2) with the one just described and as they cancel, (38.3) is applicable as it stands.

To sum up we may state:

(39.5) Theorem (38.3) is valid when $X$ is: (a) acyclic for all dimensions $p$; (b) zero-cyclic and simplicial for all $p$, provided that $\mathcal{H}(X_1, G)$ is replaced by $\mathcal{H}(X_1, G) / \Gamma^0$, where $\Gamma^0$ is the class of a vertex of $X_1$; (c) $(0, n)$-cyclic and simplicial for all $n$ provided that $\mathcal{H}(X, G)$, $\mathcal{H}(X, X_1, G)$ are replaced by $\mathcal{H}(X_1, G) / \Gamma^0$, $\mathcal{H}(X_1, G) / \Gamma^0$, except that (38.3) applies as it stands for $n = 1$.

(39.6) The explicit Betti number relations are:

(a) $X$ acyclic: (38.6a) for all $p$;

(b) $X$ simplicial and zero-cyclic:

\[ R^{n-1}(X_1, \pi) = R_p(X_0, \pi) + \delta_1^0. \]

(c) $X$ simplicial, $(0, n)$-cyclic and dim $X = n$:...
\[ R^{p-1}(X_1, \pi) = R_p(X_0, \pi) + \delta_p^p - \delta_p^p. \]

For Betti numbers mod 2 the last formula is the analogue for complexes of Alexander's original result for manifolds.

(39.7) **Examples.** An augmented closed n-simplex, in the sense defined later in (42), is acyclic (42.6), and so (38.3) is valid for such a complex and all \( p \). The ordinary closed n-simplex is zero-cyclic and falls under (39.2) (second case of 39.5). Finally \( \mathcal{E}^{n+1}, n > 0, \) is \((0, n)\)-cyclic and \( n \)-dimensional, thus falling under (39.3) (third case of 39.5).

§8. **HOMOLOGY THEORY OF INFINITE COMPLEXES**

40. In endeavoring to carry over to infinite complexes the theory developed so far, serious difficulties arise in defining groups \( \mathcal{B}, \mathcal{F} \), of any sort, unless the complexes are at least star- or closure-finite. The simplest situation is of course when they are locally finite. Fortunately these types include all the types of interest in topology and certainly all those for which any general results are known. We shall therefore confine our attention to star-, closure-, and locally finite complexes.

Let then \( X = \{x\} \) be infinite and of one of the three types just mentioned. This time we may introduce two kinds of chain- or cochain-groups:

- \( \mathcal{C}^*(X, G) = \mathcal{P}(Gx_0^*) \), the group of the infinite chains over any \( G \);
- \( \mathcal{C}^f(X, G) = \mathcal{P}^*(Gx_0^f) \), the group of the finite chains over a discrete \( G \);

and the similar cochain groups \( \mathcal{C}_p(X, G), \mathcal{C}^f_p(X, G) \).

Referring now to (II, 8.4) we have the following situations.

(a) \( X \) is star-finite. Then \( F \) defines for every \( p \) and \( G \) a chain-homomorphism \( \mathcal{C}^*(X, G) \to \mathcal{C}^{p-1}(X, G) \). When \( G \) is discrete \( F \) defines in addition homomorphisms \( \mathcal{C}^p(X, G) \to \mathcal{C}^{p+1}(X, G) \).

(b) \( X \) is closure-finite. The situation is the same for \( X^\ast \) as previously for \( X \), i.e., with cycles and cocycles interchanged. We have then homomorphisms of the groups of finite chains over a discrete \( G \) and in addition \( F \) defines homomorphisms \( \mathcal{C}_p(X, G) \to \mathcal{C}_{p+1}(X, G) \) (any \( G \)).

(c) \( X \) is locally finite. Then \( F \) defines the four types of homomorphisms considered under (a) and (b).

We notice also that when \( G \) is a discrete field all the groups \( \mathcal{C} \) under discussion are vector spaces and so they fall under the fundamental convention (II, 22.2) for such spaces.

In any one of the three cases just considered the groups \( \mathcal{B} \) may be defined as in (7, 8, 9) and likewise the groups \( \mathcal{F} \) as the factor-groups \( \mathcal{F} = \mathcal{B}/\mathcal{F} \) (\( \mathcal{F} = \mathcal{F} \) for the groups of finite chains). We may therefore state the comprehensive

(40.1) **Theorem.** When \( X \) is star-finite [closure-finite] the homology [cohomology] groups of the infinite cycles [cocycles] of \( X \) over any \( G \) may be introduced in the same manner as for finite complexes. When \( X \) is locally finite this holds for both the infinite cycles and cocycles. In all three cases this holds also for the finite cycles and cocycles over a discrete \( G \).
Complementary Remarks. (40.2) We call attention to the fact that many definitions given for finite complexes are directly applicable to certain infinite complexes. In particular:

(a) When $X$ is star-finite we may introduce as before the following concepts: dissections and related groups for infinite cycles (23), the cyclic or acyclic types of (21) corresponding to infinite cycles, and also the circuits of various kinds (24) which are now described in terms of groups of infinite $n$-cycles.

(b) When $X$ is closure-finite the dissections and the cocyclic and acyclic types may be introduced.

(c) When $X$ is locally finite there may be introduced all the concepts mentioned under (a) and (b).

(40.3) Suppose $G$ compact. Then if $X$ is star-finite the groups of infinite chains $\mathfrak{C}^p(X, G)$, $\mathfrak{S}^p(X, G)$, $\mathfrak{F}^p(X, G)$ which may then be introduced are all compact: the second as a closed subgroup of the first, and the third as the image of $\mathfrak{C}^{p+1}(X, G)$ under $F$. As a consequence $\mathfrak{F}^p(X, G)$ is closed in $\mathfrak{C}^p(X, G)$, or $\mathfrak{F}^p(X, G) = \mathfrak{S}^p(X, G)$. Hence $\mathfrak{C}^p(X, G) = \mathfrak{S}^p(X, G)/\mathfrak{F}^p(X, G)$ and it is also compact. Similarly of course for a closure-finite $X$ and the groups $\mathfrak{C}_p$, $\mathfrak{S}_p$, $\mathfrak{F}_p$, $\mathfrak{S}_p$.

(40.4) When $G$ is a linearly compact field the groups, $\mathfrak{C}$, $\mathfrak{S}$, $\mathfrak{F}$ are linearly compact and the same argument goes through as is seen by reference to (II, 27.2, $\cdots$, 27.5). The groups $\mathfrak{F} = \mathfrak{S}/\mathfrak{F}$ are found this time to be linearly compact.

(40.5) Betti numbers. They are defined in the same way as before, as the dimensions of the vector spaces $\mathfrak{S}^p(X, J)$ or $\mathfrak{F}^p(X, J)$, $J$ a discrete field. We may notice here and now that the universal theorem for fields (17.8) is valid for the case under consideration. For $\mathfrak{S}^p(X, J)$, $\mathfrak{F}^p(X, J)$ are spanned here also by $\mathfrak{S}^p(X, \pi)$, $\mathfrak{F}^p(X, \pi)$ and so the asserted property is a direct consequence of (II, 36.8).

(40.6) Alternate definition of the homology groups. Let $X$ be star-finite. Besides the topologized homology group $\mathfrak{H}^p = \mathfrak{S}^p/\mathfrak{F}^p$ one may consider the purely formal algebraic factor-group $\mathfrak{H}^p = \mathfrak{S}^p/\mathfrak{F}^p$ (or even more generally $\mathfrak{H}^p = \mathfrak{S}^p/\mathfrak{F}^p$ where $\mathfrak{F}^p$ is a subgroup of $\mathfrak{S}^p$ such that $\mathfrak{F}^p \subset \mathfrak{S}^p \subset \mathfrak{F}^p$). This would amount to taking $\mathfrak{H}^p$ untopologized. As stated in (40.3) and (40.4) the two concepts are algebraically equivalent when $G$ is compact or a field. In other cases, however, (for instance for integral chains) it may very well happen that $\mathfrak{F}^p = \mathfrak{S}^p$ and that also $\mathfrak{F}^p/\mathfrak{S}^p$, $\mathfrak{F}^p/\mathfrak{F}^p$ are essentially different. The latter and likewise the group $\mathfrak{F}^p/\mathfrak{S}^p$ (taken discrete) have been considered recently to advantage by Eilenberg [a] and Steenrod [b] (Appendix A).

(40.7) Universal coefficient-groups. It has been proved by Čech [d] that the group of the integers is universal for the homology groups of the finite cycles of a locally finite complex. A complete description of all the groups of such complexes has just been obtained by Eilenberg and MacLane [a] (Appendix A).

(40.8) The analogue of the question considered in (23.3) is of interest later. We suppose $X$ infinite, $Y$ a finite closed subcomplex and consider the groups of the cycles of $Y$ over a discrete $G$ reduced with respect to bounding in $X$. Here again we readily arrive at (14.2)
for a single dimension \( p \), except that the \((p + 1)\)-chains in (14.2) are to be replaced by finite cycles of \( X \mod Y \). Let \( \sum_p \) denote the system thus obtained. Let also \( M \) be a finite closed subcomplex of \( X \) which includes \( Y \) and all the \( d_r^{p+1}, e_r^{p+1} \). If we reduce the cycles of \( Y \) with respect to bounding in \( M \) we still obtain \( \sum_p \), for we have already utilized all the relations of bounding in \( M \). Similarly if \( M \) is replaced by any other closed finite subcomplex \( M_1 \supset M \). Since the groups in question for any \( G \) depend solely upon \( \sum_p \), (23.3), the groups of \( Y \) reduced with respect to bounding in \( M \) and \( M_1 \) must be the same. From this we conclude that the groups relative to bounding in \( X \) and \( M \) are the same. For otherwise \( Y \) must contain a cycle \( \gamma^* \sim 0 \) in \( X \) but \( \sim 0 \) in \( M \). Hence if \( \gamma^* = FC^{r+1}, C^{r+1} \) finite, and \( M_1 \) is any finite closed subcomplex containing \( M \) and \( C^{r+1} \), the reductions relative to bounding in \( M \) and \( M_1 \) cannot yield the same groups, a contradiction proving our statement. We conclude then:

(40.9) If \( Y \) is a finite closed subcomplex of the complex \( X \), then the homology groups of the cycles of \( Y \) reduced relative to finite bounding in \( X \) are the same as those reduced relative to bounding in a certain finite closed subcomplex \( M \) containing \( Y \). Hence in particular the remarks of (23.3) are applicable to the groups in question. Thus they have finite Betti numbers and the group of the integers is a universal coefficient-group (18).

41. Duality. Let \( X \) be star-finite, and \( G, H \) commutatively paired to \( J \). We consider the group \( \mathcal{G}^\circ \) of the infinite chains over \( G \). Since \( X \) is star-finite we may introduce the infinite cycles over \( G \); they form a subgroup \( \mathcal{B}^\circ \) of \( \mathcal{G}^\circ \), likewise the infinite bounding cycles over \( G \) with group \( \mathcal{B}^\circ \subset \mathcal{B}^\circ \). Therefore the homology groups of the infinite cycles over \( G \) are \( \mathcal{H}^\circ = \mathcal{B}^\circ / \mathcal{B}^\circ \).

Regarding the cocycles, since \( X \) need not be closure-finite, only finite cocycles may be allowed, and groups over a discrete \( H: \mathcal{C}^\circ, \mathcal{B}^\circ, \mathcal{B}^\circ, \mathcal{C}^\circ = \mathcal{B}^\circ / \mathcal{B}^\circ \).

It is hardly necessary to observe that the index \( KI(C^\circ, C^\circ) \) may be defined as in (28). Indeed it may even be defined when both \( C^\circ, C^\circ \) are infinite (\( H \) being then any topological group) provided that \( \gamma \) have a finite number of crossings.

When \( X \) is closure-finite the situation is the same with cycles and cocycles interchanged.

We are now in position to state

(41.1) The properties of the Kronecker index given in (29) are valid for infinite cycles [cocycles] and finite cocycles [cycles] in a star-finite [closure-finite] complex \( X \).

For the proofs loc. cit. apply without modification.

We may now repeat for \( X \), and also for \( X^* \) when \( X \) is closure-finite, the argument of (30) and thus obtain

(41.2) Duality theorem for star- or closure-finite complexes. When \( X \) is star-finite [closure-finite] the infinite cycles [cocycles] and the finite cocycles [cycles] are dual categories. When \( X \) is locally finite both types of dual categories are present.

(41.3) Linking coefficients. The full argument and definitions of (§7) may be extended to locally finite complexes, and in particular:

(41.4) Theorem. The duality theorems (38.3, 39.5) of the Alexander type, hold for locally finite complexes.
§9. AUGMENTABLE AND SIMPLE COMPLEXES

42. Let \( K = \{ \sigma \} \) be a simplicial complex, \( \{ A_i \} \) its vertices, \( \{ A^i \} \) their duals. Upon examining the argument in (5.1) we readily verify that \( K \) does not cease to be a complex if we increase it by a new null-simplex \( \epsilon \) such that: (a) \( \epsilon \) is a face of every \( \sigma \); (b) \( \dim \epsilon = -1 \); (c) \( [A_i : \epsilon] = [\epsilon : A_i] = 1 \), \( [\sigma^p : \epsilon] = [\epsilon : \sigma^p] = 0 \) for \( p > 0 \). The complex \( K_\epsilon = K \cup \epsilon \) thus obtained is said to be \( K \) augmented (A. W. Tucker [a]). The chief differences between \( K \) and \( K_\epsilon \) are embodied in the properties:

(42.1) A finite zero-chain \( C^0 \) is a cycle of \( K_\epsilon \) when and only when its Kronecker index \( \text{KI}(C^0) = 0 \).

For in \( K_\epsilon \) we have \( \text{FC}^0 = \text{KI}(C^0) \epsilon \).

As a noteworthy special case:

(42.2) The differences \( A_i - A_j \) are integral cycles of \( K_\epsilon \) but \( A_i \) is not.

Let \( \{ K_i \} \) be the components of \( K \) and \( B_i \) a vertex of \( K_i \). A one-chain \( C^i \) of \( K \) is likewise one of \( K_\epsilon \), and whether considered as in \( K \) or \( K_\epsilon \) its boundary \( FC^i \) is the same. It follows that (20.7) holds for \( K_\epsilon \) and finite chains. If \( \gamma \) is a zero-cycle of \( K_\epsilon \) we have \( \text{KI}(\gamma) = 0 \) and hence by (20.7a):

\[
\gamma^i \sim g^i(B_i - B_i) + \text{KI}(\gamma)B_i \sim g^i(B_i - B_i).
\]

By (20.7b) also

\[
g^i(B_i - B_i) \sim 0 \rightarrow g^i = 0.
\]

From this we deduce the analogue of (20.8) for \( K_\epsilon \):

(42.4) \( S^5(K_\epsilon, G) \cong S^5(K, G)/GB_1 \),

where for simplicity \( B_i \) is identified with its class. As a special case of (42.3) if \( K \) is connected \( \gamma^0 \sim 0 \), and hence:

(42.5) If \( K \) is connected then \( K_\epsilon \) is acyclic in the dimension zero.

(42.6) \( (\text{Cl} \sigma^p)_\epsilon \) is acyclic and \( (\text{Bd} \sigma^p)_\epsilon \) is \((n-1)\)-cyclic (22.2, 22.4, 42.5).

(42.7) \( K^\circ(K) = K^\circ(K_\epsilon) + 1, \) (42.4).

43. Let now \( X = \{ x \} \) be any closure-finite complex with \( \dim x \geq 0 \). Is it possible to "augment" \( X \), i.e., to increase it by a \((-1)\)-dimensional element \( \epsilon \), which is to be a face of every \( x \), and with incidence numbers \( \lambda_i = [x^i : \epsilon] = [\epsilon : x^i] \) not all zero and \( [\epsilon^p : \epsilon] = [\epsilon : \epsilon^p] = 0 \) for \( p > 0 \)? If so \( X \) is said to be augmentable and the new complex \( X_\epsilon = X \cup \epsilon \) is called \( X \) augmented. In order that \( X_\epsilon \) be a complex it must fulfill conditions K1234 of (1), or which is equivalent, its dual \( X_\epsilon^* \) must fulfill them. The first three are automatically satisfied and so only \( K4 \) is in question. By (8.3) it reduces to requiring that if \( \epsilon^* \) is the dual (one-dimensional) of \( \epsilon \), then \( \text{FF} \epsilon^* = 0 \). Since \([x^i : \epsilon] = [\epsilon^* : x^i] = \lambda_i \), this
is equivalent to requiring $\Lambda_\alpha z_0^\alpha = 0$, i.e., that $\gamma_0 = \lambda_\alpha z_0^\alpha$ be an integral zero-cocycle, a result due to Tucker [a]. The particular zero-cocycle arising in the augmentation is called the fundamental zero-cocycle, and we shall say that "$X$ is augmentable, or augmented, with fundamental zero-cocycle $\gamma_0"$.

44. All the possible modes of augmenting $X$ correspond to its different non-trivial integral zero-cocycles, i.e., to the nonzero elements of $\mathcal{Z}_0(X)$. Since $\dim X^* \leq 0$: $\mathcal{Z}_0(X) = \mathcal{Z}_0(X)$ (27.6a). Hence the integral zero-cocycles form a free group of rank $R_0 = R^d$. Therefore

(44.1) A n. a. s. c. for augmentability is that the Betti number $R_0 \neq 0$. Each mode of augmenting is uniquely determined by an integral zero-cocycle $\gamma_0 = \lambda_\alpha z_0^\alpha$, and in $X_\alpha = X \cup \epsilon$, $\lambda_\alpha = [z_0^\alpha : \epsilon]$.

An equivalent condition of augmentability is $FFC_0 = 0$ for every finite chain $C_0$ over $G$ in $X_\alpha$. If $FC_0 = g^i z_0^i$, this yields as n. a. s. c.:

(44.2) \[ \lambda_0 g^i = \text{KI}(FC_0, \gamma_0) = 0, \]

where the index is an element of $G$. Therefore

(44.3) The n. a. s. c. for augmentability in (44.1) is equivalent to requiring the existence of a non-trivial integral zero-cocycle $\gamma_0$ such that $\text{KI}(FC_0, \gamma_0) = 0$ for all finite $C_0$.

45. Suppose $X$ augmented and with the fundamental zero-cocycle $\gamma_0$. If $\gamma$ is a closed subcomplex of $X$ we may write $\gamma_0 = \gamma_0' + \gamma_0''$ where $\gamma_0'$ is in $Y^*$ and $\gamma_0''$ has no element in $Y^*$. We shall say that $Y^*$ meets $\gamma_0$ when $\gamma_0' \neq 0$, and we shall call $\gamma_0'$ for the present the intersection of $Y^*$ with $\gamma_0$. Since $\gamma_0'$ is a cocycle of $Y$ we have:

(45.1) Let $X$ be augmentable with fundamental cocycle $\gamma_0$. Then every closed subcomplex $Y$ such that $Y^*$ meets $\gamma_0$ is also augmentable and with a fundamental cocycle $\gamma_0'$ which is the intersection of $\gamma_0$ with $Y^*$.

When $Y$ is finite and augmentable as stated it may be augmented with $\gamma_0'$ as fundamental cocycle. We shall denote in any case by $Y_\alpha$ the new augmented complex $Y \cup \epsilon$, when $\gamma_0' \neq 0$, and $Y$ itself when $\gamma_0' = 0$. Notice that $Y_\alpha$ may depend a priori upon the cocycle $\gamma_0$ chosen as fundamental for $X$. In fact the significance of the choice of $\gamma_0$ as fundamental cocycle lies in a sense in that it provides a uniform procedure for augmenting the finite subcomplexes of $X$.

46. Returning for a moment to the simplicial complex $K$, let $\delta_0 = \sum A_1^i$. Then $FFC_0 = g^i A_0^i$, $C_0$ finite,

(46.1) \[ \sum g^i = \text{KI}(C_0, \delta_0). \]

Therefore the Kronecker index as a sum of coefficients is in fact also a "chain-cochain" index. Furthermore if $FC_0 = g^i A_0^i$, $C_0$ finite,

(46.2) \[ \sum g^i = 0 = \text{KI}(FC_0, \delta_0). \]

Therefore $K$ is augmentable with $\gamma_0 = \delta_0$, i.e., with unity as the new incidence numbers $[A_1 : \epsilon]$. This is precisely the way in which $K_\alpha$ has been obtained in (42). We also know that if $K$ is connected, $R^d = 1$, so that all the zero-cocycles are
of the form $g_0$. Therefore when $K$ is finite and connected it can only be augmented in essentially one way and with incidence numbers $\lambda_i$ all equal. Their common value $\lambda$ is the only "indeterminate" in the augmentation.

Referring also to (28.8) we have the following noteworthy interpretation for $\text{Kl}(C^0) =$ the number of weighted crossings of $C^0$ with the cocycle $\delta^0$.

47. The preceding properties suggest the following extension of simplicial complexes:

\[(47.1) \text{Definition. A complex } X = \{x\} \text{ is said to be simple whenever: (a) } X \text{ is closure-finite; (b) } X \text{ is augmentable and this with a fundamental zero-cocycle which in a suitable orientation of } X \text{ is given by } \gamma_0 = \sum x^0_0; (c) \text{ every } (\text{Cl } x)_a \text{ is acyclic.} \]

We agree first of all to orient $X$ so that $\gamma_0 = \sum x^0_0$. Then in $X$ the Kronecker index $\text{Kl}(C^0, \gamma_0)$, $C^0$ finite, is the sum of the coefficients of $C^0$, and it will be denoted once more by $\text{Kl}(C^0)$.

Notice that when $X$ is simplicial with vertices $\{A_i\}$ then $\gamma_0 = \sum A^i$. Thus for a simplicial complex the fundamental cocycle in the sense of (25.3) is the same as the fundamental cocycle of (47.1).

If $X$ is simple and $Y$ is a closed subcomplex of $X$, then $X - Y$ is called an open simple complex. By contrast $X$ or $Y$ are also called closed simple complexes.

How close the approximation is to simplicial complexes is attested by the following properties:

\[(47.2) \text{Simplicial and polyhedral complexes are simple.} \]

For simplicial complexes it is a consequence of (44). For polyhedral complexes the proof will be given later (IV, 28.2).

\[(47.3) \text{Every closed subcomplex of a simple complex is simple.} \]

\[(47.4) \text{When } X \text{ is simple every } p\text{-element has at least one } (p - 1)\text{-face, hence at least one vertex, and every one-face has two vertices (Whitney [d]).} \]

Suppose $Fx^p = 0$, $p > 0$. Then $x^p$ is a $p$-cycle of $(\text{Cl } x)_a$, hence $x^p \sim 0$ in $(\text{Cl } x)_a$, and since both have the dimension $p$, $x^p = 0$, which is absurd. Therefore $Fx^p \neq 0$, so that $x^p$ must have at least one $(p - 1)$-face, and therefore step by step it is shown to have at least one vertex.

Suppose that $x^i$ has only one vertex $x^0$ so that $Fx^i = gx^0$. Since $(\text{Cl } x^i)_a$ is augmented, $\text{Kl}(Fx^i) = g = 0$, hence again $x^i$ must be a cycle which we have just ruled out. Suppose on the other hand that $x^i$ has three vertices $x^0, x^1, x^2$. Then $x^0 - x^1$ and $x^1 - x^2$ are cycles of $(\text{Cl } x^i)_a$ and hence $x^0 - x^1 = gFx^i$, $x^1 - x^2 = hFx^i$, where $g, h$ are distinct nonzero integers. As a consequence $h(x^1 - x^2) = g(x^0 - x^1)$, $g = h$, $x^2 - x^1$, a contradiction.

\[(47.5) \text{When } X \text{ is finite and satisfies (47.1b) then } "X \text{ is zero-cyclic}" \text{ and } "X_a \text{ is acyclic}" \text{ are equivalent. Hence in (47.1) } "\text{every } (\text{Cl } x)_a \text{ is acyclic}" \text{ may be replaced by } "\text{every } \text{Cl } x \text{ is zero-cyclic}". \]

Let $X$ be zero-cyclic and let $C^0 = g'x^0$ be a zero-cycle of $X_a$. Then in $X_a: FC^0 = 0 = \text{Kl}(C^0)$. Since $X$ is zero-cyclic $x^1 - gx^0 \sim 0$ for some $g$. Hence $\text{Kl}(x^1 - gx^0) = (1 - g) = 0$, $g = 1$, $x^1 - x^1 \sim 0$. Hence $C^0 = C^1 - \text{Kl}(C^0)x^1 = g'(x^0 - x^1) \sim 0$ in $X$ and hence also in $X_a$. Therefore $X_a$ is acyclic. The converse is immediate.
A one-dimensional simple complex $X^1$ is simplicial.

By (47.4) $X^1$ has the ordering relations of a simplicial complex. Let $A$, $B$ be the vertices of $x^i$. If $Fz^i = gA + hB$ we have $K(Fz^i) = g + h = 0$, $g = -h$, $Fz^i = g(A - B)$. Also by (47.5) $A \sim kB$, or $A - kB = Fmx^i = mg(A - B)$. Hence $mg = 1$, $g = \pm 1$. Therefore $[x^i:A] = -[x^i:B] = \pm 1$. Thus $x^i$ has the incidence numbers of a simplicial one-complex and (47.6) follows.

(47.7) Properties (20.5, \ldots, 20.10) hold for a simple complex, provided that the zero-dimensional homology groups and Betti numbers are those of the finite zero-cycles.

For the elements of $X$ are all connected with those of its one-section $X^i$ which is simplicial, and the proof of (20.5) refers solely to $X^i$ and its finite zero-cycles.

(47.8) When $X$ is connected and simple all the modes of augmenting $X$ are essentially unique, in the sense that all the possible fundamental cocycles are merely the multiples of a single cocycle (Whitney [d]).

For by (41.2) $R_0 = R_j^0 = 1$. It follows that every integral cocycle $\delta_0 = \sum \lambda_0 z_0^i$ is a rational multiple of $\gamma_0 = \sum z_0^i$, and hence it is an integral multiple: $\delta_0 = \lambda \sum z_0^i$.

(47.9) Properties (42.1, \ldots, 42.5, 42.7) hold for any simple complex $X$.

For they depend solely upon the one-section $X^i$ of $X$ and $X^i$ is simplicial (47.6).

(47.10) When $X$ is simple, $(\Omega x^p)_a, p \geq 1$, is $(p - 1)$-cyclic, and all its $(p - 1)$-cycles are of the form $gFz^p$.

For $p > 1$ this follows from (21.4, 47.5), and for $p = 1$ from the fact that $x^1$ is a simplex (47.6).