CHAPTER VI
NETS OF COMPLEXES

The passage from finite complexes to infinite complexes or topological spaces necessitates some limiting process, and the theory of nets will provide the necessary mechanism. In its general form it may be viewed as abstracted from the Čech homology theory for topological spaces (Čech [a]), which will be adopted as the basic theory in (VII). An important special type of net, the sequential spectrum, was already utilized by Alexandroff [a], for compacta, likewise for infinite complexes and compacta in [L, VII].

A close parallel will be found between nets and finite complexes and we shall have here also open and closed subnets, their projections and injections. In the general net there are no chains and so the operations bear directly upon the cycles. However for a spectrum chains may again be introduced, the similarity with complexes being greatly increased thereby.

By combining the subnets there will be obtained a noteworthy complementary mechanism, the web, which will have important applications in (VII).

The general theory of nets and webs will be applied to infinite complexes, and in particular to a type which we have termed metric. Such complexes will be shown to have a special "metric" homology theory, which includes the well known Vietoris theory for compacta, but has other applications as well.

General references: Alexandroff [a, f], Čech [a, b], Chevalley [a], Freudenthal [b], Lefschetz [L4, XVII], Steenrod [a].

§1. DEFINITION OF NETS AND THEIR GROUPS

1. A net is a collection of complexes with special relations. A good point of departure is therefore a suitable type of infinite product of complexes. Since these infinite products have as yet but few applications we will not dwell upon them very long.

(1.1) Consider then a system \( \{X_\lambda \} \) of finite complexes indexed by \( \Lambda = \{\lambda\} \) and let \( \mathfrak{X} = \mathbf{P}X_\lambda \) be the product of the complexes as sets of elements. It is not our purpose to turn \( X \) into a complex in the sense of (III, 1)—such a complex would be, in fact, irrelevant here. We may, however, introduce the groups of chains, \( \cdots \), of \( \mathfrak{X} \) over a given coefficient group \( G \) in the following way. Write \( \mathfrak{G}_x^p, \mathfrak{B}_x^p, \mathfrak{H}_x^p \) for \( \mathfrak{G}(X_\lambda, G), \cdots \), where the groups are as in (III, 7, 8). Define now the groups \( \mathfrak{G}^p(\mathfrak{X}, G), \mathfrak{B}^p(\mathfrak{X}, G), \mathfrak{H}^p(\mathfrak{X}, G) \) as

\[
\mathfrak{G}^p(\mathfrak{X}, G) = \mathbf{P}\mathfrak{G}_x^p, \quad \mathfrak{B}^p(\mathfrak{X}, G) = \mathbf{P}\mathfrak{B}_x^p, \\
\mathfrak{H}^p(\mathfrak{X}, G) = \mathbf{P}\mathfrak{H}_x^p.
\]
By (I, 12.5) we have then
\[ \tilde{\mathcal{F}}^*(x, G) = \mathcal{P}_{\tilde{\mathcal{F}}}^*, \]
and we readily show that
\[ (1.3) \quad \mathcal{F}^*(x, G) = \mathcal{B}^*(x, G) / \tilde{\mathcal{F}}^*(x, G) = \mathcal{P}_{\mathcal{F}}^*. \]

There is an obvious parallel development in the direction of "weak" products whose details are omitted.

2. The next step is essentially analogous to the passage from products of groups to inverse or direct systems (II, 13, 14).

(2.1) DEFINITION. A net \( X \) is a system of finite complexes \( \{ X_\lambda \} \) indexed by a directed set \( \Lambda = \{ \lambda; \succ \} \) and with the following properties:

N1. When \( \lambda \succ \mu \) there exist one or more chain-mappings, also called "projections," \( \pi_\mu^\lambda : X_\lambda \to X_\mu \).

N2. When \( \lambda \succ \mu \succ \nu \) and \( \pi_\mu^\nu \), \( \pi_\mu^\nu \) are projections so is \( \pi_\mu^\nu \).

N3. Any two projections \( \pi_\mu^\lambda, \pi_\nu^\lambda, \lambda \succ \mu \), are homologous (\( \pi_\mu^\lambda \gamma_\lambda \sim \pi_\mu^\lambda \gamma_\lambda \) for every cycle \( \gamma_\lambda \) of \( X_\lambda \)).

(2.2) Let \( N^* \alpha \) denote \( N \alpha \) with \( \succ \) replaced by \( \prec \). If \( X \), still indexed by \( \Lambda = \{ \lambda; \succ \} \), has properties \( N^* \alpha \) it is known as a cone. In one or the other case a convenient designation is \( X = \{ X_\lambda ; \pi_\mu^\lambda \} \). Unless otherwise stated, in such a designation \( X \) will be understood to be a net.

(2.3) Let \( X_\lambda^* \) denote as usual the dual of \( X_\lambda \), and \( \pi_\mu^* : X_\mu^* \to X_\lambda^* \), the dual of \( \pi_\mu^\lambda \). Then if one of \( X = \{ X_\lambda ; \pi_\mu^\lambda \}, X^* = \{ X_\lambda^* ; \pi_\mu^* \} \) is a net the other is a cone.

(2.4) Special designations. The net \( X \) is called

- simplicial if the \( X_\lambda \) and the \( \pi_\mu^\lambda \) are simplicial;

- simple if the \( X_\lambda \) and the \( \pi_\mu^\lambda \) are simple and in addition the \( \pi_\mu^\lambda \) have simple carriers;

- sequential if \( \Lambda = \{ 1, 2, \cdots \} \);

- a spectrum if the \( \pi_\mu^\lambda \) are unique;

- a sequential spectrum if the net is both sequential and a spectrum.

Evidently a simplicial net is simple.

The dimension of \( X \), written \( \text{dim } X \), is \( \sup \text{ dim } X_\lambda \).

3. Since there may occur multiple projections \( \pi_\mu^\lambda, \pi_\mu^* \), it will be necessary for the groups \( \tilde{\mathcal{G}}, \tilde{\mathcal{F}} \) to have recourse to the mechanism of (II, 13.7, 14.8). Not so, however, as we shall see, for the groups \( \mathcal{G} \). Since the cycles and cocycles, and not their classes, are the elements usually arising in the applications their properties will be examined in some detail.

As a consequence of N3 we have:

(3.1) Any two projections \( \pi_\mu^\lambda, \pi_\mu^* \in \{ \pi_\mu^\lambda, \pi_\mu^* \} \), \( \lambda \succ \mu \), induce the same simultaneous homomorphism \( \tilde{\pi}_\mu^\lambda : \tilde{\mathcal{F}}_\mu^* \to \tilde{\mathcal{F}}_\lambda^* \in \{ \tilde{\pi}_\mu^\lambda : \tilde{\mathcal{F}}_\mu^* \to \tilde{\mathcal{F}}_\lambda^* \} \) so that there is a unique inverse (direct) system \( \{ \tilde{\mathcal{F}}_\lambda^* ; \tilde{\pi}_\mu^\lambda \} \) \( \{ \tilde{\mathcal{F}}_\mu^* ; \tilde{\pi}_\mu^\lambda \} \) in the sense of (II, 13.1) \((\text{II}, 14.1)\).

We have then limit-groups of the two systems and we lay down:
(3.2) Definitions. $\mathfrak{N}^p = \lim \{ \mathfrak{N}_r; \mathfrak{N}_r \}$ and $\mathfrak{N}_p = \lim \{ \mathfrak{N}_r; \mathfrak{N}_r^* \}$ are, respectively, the $p$th homology and $p$th cohomology groups of the net over $G$. The elements $\Gamma^p$, $\Gamma^q$ of the two groups are the homology and cohomology classes over $G$; the terms will be justified presently.

An element of $\mathfrak{N}^p$ is a collection $\Gamma^p = \{ \Gamma^p_x \}$ where $\Gamma^p_x \in \mathfrak{N}^p_x$, and $\lambda > \mu \rightarrow \pi^p_\lambda \Gamma^p_x = \Gamma^p_{\lambda x}$. The $\Gamma^p_x$ are the coordinates of $\Gamma^p$. Furthermore referring to (I, 38,2) if $u_0$ is any open set in $\mathfrak{N}^p_x$, and $U_1 = \{ \Gamma^p_x \mid \Gamma^p_x \in u_0 \}$, the aggregate $\{ U_1 \}$ is a base for $\mathfrak{N}^p$.

(3.3) Let now a $p$-cycle $\gamma^p$ of $X$ over $G$ be defined as follows: $\gamma^p = \{ \gamma^p_x \}$, where $\gamma^p_x \in \mathfrak{N}^p_x$ and $\lambda > \mu \rightarrow \pi^p_\lambda \gamma^p_x \sim \gamma^p_\mu$ in $X_x$. The $\gamma^p_x$ are the coordinates of $\gamma^p$. Define $\gamma^p = 0$ when every $\gamma^p_x = 0$, and if $\gamma^p = \{ \gamma^p_x \}$, set $\gamma^p + \gamma^p = \{ \gamma^p_x + \gamma^p_\mu \}$. As a consequence $\mathfrak{N}^p = \{ \gamma^p \}$ is a group, the group of the $p$-cycles over $G$. The topology in $\mathfrak{N}^p$ is assigned by means of a subbase as follows: take any open set $U_1$ in $\mathfrak{N}^p_x$ and let $V_1 = \{ \gamma^p \mid \gamma^p_x \in U_1 \}$. The collection $\{ V_1 \}$ is chosen as a subbase for $\mathfrak{N}^p$.

(3.4) $\mathfrak{N}^p$ is a closed subgroup of $\mathfrak{N}^p(X, G) = \mathbf{P} \mathfrak{N}^p$, the group of (1.2) (II, 13.7.a).

Among the cycles of the net are found the collections $\delta^p = \{ \delta^p_x \mid \delta^p_x \sim 0 \}$ known as bounding cycles. The terminology is justified by (3.5). If $\gamma^p - \gamma^p$ is a bounding cycle we write the usual homology $\gamma^p \sim \gamma^p$. Evidently $\mathfrak{N}^p = \{ 0 \} = \mathfrak{N}^p(X, G)$ (group of (1.2) is a subgroup of $\mathfrak{N}^p$.

(3.5) $\mathfrak{N}^p$ is closed in $\mathfrak{N}^p$.

For $\mathfrak{N}^p$ is closed in $\mathfrak{N}^p(X, G)$ (I, 13.7.c).

Henceforth $\mathfrak{N}^p$ is identified with $\mathfrak{N}^p / \delta^p$, $\Gamma^p$ being identified with the coset of $\gamma^p$ mod $\delta^p$. It is also called the homology class of $\gamma^p$.

(3.7) When $G$ is compact so are $\mathfrak{N}^p$, $\mathfrak{N}^p$ and $\mathfrak{N}^p$.

For $\mathfrak{N}^p(X, G)$ is then compact and (3.7) is a consequence of (II, 13.7.d).

(3.8) Let $G$ be a division-closure group and $G_0$ a discrete group isomorphic with $G$ in the algebraic sense. Then the groups $\mathfrak{N}^p$, $\mathfrak{N}^p$, $\mathfrak{N}^p$ over $G$ are isomorphic in the algebraic sense with the corresponding groups over $G_0$. In other words, when $G$ is a division-closure group its topology may be disregarded without modifying $\mathfrak{N}^p$, $\mathfrak{N}^p$, $\mathfrak{N}^p$ algebraically.

Let $\gamma^p = \{ \gamma^p_x \}$ be a cycle over $G$. Owing to the division-closure property (III, 17.2): (a) $\lambda > \mu \rightarrow \pi^p_\lambda \gamma^p_x - \gamma^p_\mu = FC^p_x \mu$; (b) $\gamma^p \sim 0 \leftrightarrow \gamma^p_x = FC^p_x \mu$; the chains $C^p_x$, $C^p_x \mu$ are chains over $X_x$, $X_\mu$ over $G$. Since chains over $G$ are likewise chains over $G_0$, $\gamma^p$ is a cycle over $G_0$ and conversely, if $\gamma^p$ is a cycle over $G_0$ it is likewise a cycle over $G$. If $\gamma^p \sim 0$ as a cycle over one of $G$, $G_0$ then also $\gamma^p \sim 0$ as a cycle over the other. Since the identification of $\gamma^p$ as a cycle over $G$ with itself as a cycle over $G_0$ manifestly defines an isomorphism in the algebraic sense of the groups $\mathfrak{N}^p$, $\mathfrak{N}^p$ over $G$ with the corresponding groups over $G_0$, (3.8) follows.

(3.9) If $\gamma^p = \{ \gamma^p_x \}$ is a cycle and $\gamma^p \sim 0$ for all elements of $\{ \nu \}$ cofinal in $\lambda$ then $\gamma^p \sim 0$.

Given $\lambda$ there is a $\nu > \lambda$ and so $\gamma^p_\nu \sim \pi^p_\nu \gamma^p_\mu \sim 0$. Hence $\gamma^p \sim 0$. 

(3.10) If dim $X = n$ is finite then all the groups $\mathcal{D}^n_p$, $p > n$, are zero (2.4, 3.3).

(3.11) Essential cycles. Following Čech, $\gamma^p_\mu$ is defined as essential for $X_\mu$ whenever $\lambda > \mu$ implies the existence in $X_\lambda$ of a $\gamma^p_\mu$ such that $\pi^*\gamma^p_\lambda \sim \gamma^p_\mu$ in $X_\mu$. Under these conditions $\Gamma^p_\mu$ is an essential element of $\mathcal{D}^p(X_\mu)$ in the sense of (II, 27.11).

(3.12) If $G$ is a field then for every $\mu$ there is a $\lambda_0$ such that if $\gamma^p_\mu \in \mathcal{Z}^p(X_\mu)$ then $\pi^*\lambda^p \gamma^p_\mu$ is an essential cycle for $X_\mu$ and hence the $\mu$ coordinate of a cycle of $X$ (Čech [a]; II, 27.13).

By (II, 27.13) there exists for given $\mu$ and every $p \leq \dim X_\mu = n$ an index $\lambda_p \leq \mu$ such that $\pi^*\lambda^p \Gamma^p_\mu$ is essential, and hence such that $\pi^*\lambda^p \gamma^p_\mu$ is essential. Since $n$ is finite we may choose a $\lambda_0 > \lambda_1, \ldots, \lambda_n$, and then every $\pi^*\lambda_0 \gamma^p_\mu$ (all $p$) will be essential.

(3.13) Betti and Alexandroff numbers. If $G$ is a field the $p$th Betti number of $X$ over $G$ is $R^p(X, G) = \dim \mathcal{D}^p(X, G)$, when the latter is finite, and $R^p(X, G) = \infty$ otherwise.

Since the $X_\mu$ are finite so are their Betti numbers. Hence the Alexandroff numbers of $X$ (i.e., of the $\mathcal{D}^p(X, G)$) are equal to the corresponding Betti numbers of the net.

4. (4.1) We now pass to the cocycles $\gamma_p$ and their classes $\gamma_p$ over a discrete group $G$. The groups $\mathcal{D}^p_p$ over $G$ have already been defined. Any $\Gamma_p$ is a collection of elements $\Gamma^p_\lambda$, the representatives of $\Gamma^p_\mu$, such that if $\Gamma^p_\mu$ and $\Gamma^p_\nu$ exist then for some $\lambda > \mu$, $\nu$ we have $\pi^*\lambda \Gamma^p_\mu = \pi^*\lambda \Gamma^p_\nu$. A $p$-cocycle $\gamma_p$ over $G$ is now a maximal collection of cocycles $\gamma^p_\lambda$ over $G$, the representatives of $\gamma^p_\mu$, such that if $\gamma^p_\mu$ and $\gamma^p_\nu$ exist then for some $\lambda > \mu$, $\nu$ we have $\pi^*\lambda \gamma^p_\mu = \pi^*\lambda \gamma^p_\nu$ in $X_\lambda$. The cocycles $\gamma^p_0 = 0$ are representatives of a unique $\gamma_p$ denoted by $0$. If $[\gamma^p_\mu], [\gamma^p_\nu]$ are the representatives of $\gamma^p_\mu, \gamma^p_\nu$ choose corresponding to $\mu, \nu$ any $\lambda > \mu, \nu$. Then one may show that the cocycles $[\pi^*\lambda \gamma^p_\mu + \pi^*\lambda \gamma^p_\nu]$ are representatives of a unique cocycle written $\gamma_p + \gamma'_p$. Under these conditions $\mathcal{Z}^p = [\gamma_p]$ is a group, the group of the cocycles over $G$, and it is taken discrete. The cocycles $\delta_p$ which have a representative $\delta^p_\mu$ such that for some $\lambda > \mu$ we have $\pi^*\lambda \delta^p_\mu \sim 0$ are known as bounding cocycles. If $\gamma_p - \gamma'_p$ is a bounding cocycle we write the usual homology $\gamma_p \sim \gamma'_p$. Evidently $\mathcal{Z}^p = [\delta_p]$ is a subgroup of $\mathcal{Z}^p$.

(4.2) $\mathcal{D}^p \cong \mathcal{Z}^p/\mathcal{B}_p$ (II, 14.8a).

We may now identify $\mathcal{D}^p_p$ with $\mathcal{Z}^p_p/\mathcal{B}_p$ and thus consider $\Gamma_p$ as the coset of $\gamma_p \bmod \mathcal{B}_p$. It is also called the cohomology class of $\gamma_p$.

(4.3) If $G$ is a field we define the dual Betti numbers as $R_p(X, G) = \dim \mathcal{D}^p(X, G)$, when the latter is finite, and $R_p(X, G) = \infty$ otherwise. The "dual" Alexandroff numbers of $X$ (i.e., those of the $\mathcal{D}^p_p(X, G)$) are, as in (3.13), the same as the dual Betti numbers.

(4.4) If dim $X = n$ is finite then all the groups $\mathcal{D}^p_p$, $p > n$, are zero (2.4).

(4.5) $\gamma_p$ or $\Gamma_p$ have representatives for some $[\mu]$ cofinal in $[\lambda]$.

For if say $\Gamma^p_\mu$ is a representative of $\Gamma_p$ then so is every $\pi^*\lambda \Gamma^p_\mu$, $\lambda > \mu$.

(4.6) Cones. All the results obtained for nets carry over to cones with
cycles and cocycles interchanged. In particular in cones the discrete groups are those of the cycles.

5. Application to connectedness in simplicial nets. We shall find here again the same general relations between connectedness and the zero-dimensional groups as in simplicial and simple complexes (III, 20, 47.7). In point of fact we could develop the same considerations for simple nets, but the simplicial case is sufficient for the topological applications.

Let then \( X = \{ X_\lambda ; \sigma^\lambda \} \) be simplicial and let \( \{ X_\lambda \} \) be the components of \( X_\lambda \). Since each contains a vertex and homologous vertices are mapped by \( \sigma^\lambda \) into homologous vertices \( \sigma^\lambda \) is connected and hence in some \( X_{\sigma^\lambda} \). We shall say that the component \( X_{\sigma^\lambda} \) is essential whenever if \( \lambda > \mu \) there exists an \( X_\lambda \), such that \( \sigma^\lambda X_\lambda \subset X_\mu \). A component of \( X \) is a collection \( X' = \{ X_\lambda' \} \), where \( X_\lambda' \) is a component of \( X_\lambda \), (one of the sets \( X_\lambda \)), and where \( \lambda > \mu \to \sigma^\lambda X_\lambda \subset X_\mu' \). Evidently \( X' \) itself is a net, (a subset of \( X \) in the terminology of 12). We shall denote by \( \rho \) the cardinal number of the components of \( X \).

(5.1) The component \( X' \) is uniquely determined by its coordinates \( X' \) for \( \{ \tau \} \) cofinal in \( \{ \lambda \} \).

For \( X_\lambda' \) is uniquely determined by \( X_\mu' \), \( \nu > \lambda \).

(5.2) If \( X_1', \ldots, X_i' \) are distinct components of \( X \) (\( \tau \) finite), with \( X_\nu = \{ X_\lambda \} \), then for some \( \mu \) the \( X_\nu' \) are distinct components of \( X_\mu' \).

Take any pair \( X_\nu', X_i' \), \( i \neq j \). For some index \( \mu(i, j) \) we must have \( X_{\mu(i, j)}' \neq X_{\mu(i, j)}' \). Therefore whatever \( \nu > \mu(i, j) \) necessarily \( X_\nu' \neq X_j' \), for otherwise their projections \( X_{\mu(i, j)}' \), \( X_{\mu(j, i)}' \) in \( X_{\mu(i, j)} \) would coincide. Since the number of indices \( \mu(i, j) \) is finite there is a \( \mu > \) every \( \mu(i, j) \), and the \( X_\nu' \) will then all be distinct.

(5.3) The Betti numbers \( R^\rho(X; G) \), over any field \( G \), are all equal to \( \rho \) when \( \rho \) is finite, and infinite otherwise.

Their common value is designated by \( R^\rho(X) \) or \( R^\rho_0 \), and known as the zero-dimensional Betti number of the net.

The notations remaining the same choose a vertex \( A_\mu \) on \( X_\mu \). Since \( \sigma^\mu A_\mu \) is a vertex of \( X_\mu \) in the same component as \( A_\mu \), we have \( \sigma^\mu A_\mu \sim A_\mu \). Therefore \( \gamma^\rho_{\mu} = [A_\mu] \) is an integral zero-cycle of \( X_\mu \) and hence of \( X \). Similarly \( [gA_\mu] \), \( g \in G \), is a zero-cycle of \( X_\nu \) over \( G \) and it is denoted by \( g\gamma^\rho_{\mu} \).

The \( \gamma^\rho_{\mu} \) are independent. For suppose \( g\gamma^\rho_{\mu} \sim 0 \) and choose \( \mu \) as in (5.2). We must have \( gA_\mu \sim 0 \) in \( X_\mu \), which implies \( g = 0 \), since the \( A_\mu \) are in distinct components of \( X_\mu \). It follows that \( R^\rho(G) \geq \rho \), so if \( \rho = \infty \) likewise \( R^\rho(G) = \infty \).

There remains then to dispose of the finite case.

We assume then \( \rho \) finite and \( \tau = \rho \), so that \( \{ X_\mu \} \) is now a maximal set of distinct components of \( X \). Choose again \( \mu \) as in (5.2), so that now the \( \rho \) components \( X_\mu' \) of \( X_\mu \) are distinct, and of course essential. Moreover clearly no other component of \( X_\mu \) may be essential. It follows that for \( \nu > \mu \) there are exactly \( \rho \) essential components of \( X_\nu \). For there can be no more, and there are at least that many, else there would be fewer than \( \rho \) in \( X_\mu \). The \( \rho \) essential components of \( X_\mu \) must be the \( X_\mu' \) and they are thus distinct. If \( \delta^\rho = \{ \delta^\rho_{\mu} \} \) is a zero-cycle
we have then \( \delta^0 \sim g_t^0 A_\nu \) and \( \pi^\nu g_t^0 A_\nu \sim g_t^0 A_\nu \sim g_t^0 A_\nu^\nu \). Since the \( A_\nu \) are independent \( g_t^0 = g_t^0 \) is independent of \( \nu \). Thus \( \delta^0 \sim g_t^0 A_\nu \). Therefore \( \delta^0 - g_t^0 \gamma^0 \) has its \( \nu \) coordinate \( \sim 0 \), and so (3.9), \( \delta^0 \sim g_t^0 \gamma^0 \). It follows that \( \mathcal{R}^0(G) \leq \rho \), and finally

\[
\mathcal{R}^0(G) = \rho.
\]

We have also proved the complementary property:

(5.4) When \( \mathcal{R}^0 \) is finite \( \{ \gamma^0 \} \), where \( \gamma^0 = [\gamma^0] \) is a zero-cycle of \( X \), is a base for the zero-cycles over any group \( G \) and so

(5.5)

\[
S^0(X, G) = P(G \gamma^0).
\]

More generally:

(5.6) When \( \mathcal{R}^0 \) is finite,

\[
S^0(X, G) = P S^0(X', G).
\]

For if \( \gamma^0 = [\gamma^0] \) is any cycle over \( G \) we have \( \gamma^0 = \sum \gamma^0 \), \( \gamma^0 \subset X^0 \), and \( \gamma^0 \sim 0 \rightarrow \gamma^0 \sim 0 \) in \( X^0 \). From this and the fact that \( \{ \nu \} \) is clofain in \( \{ \lambda \} \) follows readily that \( \gamma^0 = [\gamma^0] \) is a cycle of \( X^0 \) with \( \gamma^0 = \sum \gamma^0 \), and also that \( \gamma^0 \sim 0 \) in \( X \) when and only when every \( \gamma^0 \sim 0 \) in \( X^0 \). From this to (5.6) is but a step.

§2. DUALITY AND INTERSECTIONS

6. For the duality relations a Kronecker index is required. Keeping nets and conets together for the present let the groups \( G, H \) be commutatively paired to \( J \), and let \( \Gamma^u, \Gamma_\nu \) be over \( G, H \). If \( \lambda > \mu \) and \( \Gamma_p, \Gamma_\nu \) in the net, \( \Gamma^u \) in the conet have \( \lambda, \mu \) representatives we find from (IV, 10.5) with \( \tau = \pi^u \), the permanence relation for the index:

(6.1)

\[
KI(\Gamma^u, \Gamma_\nu) = KI(\Gamma^u, \Gamma_\nu^u).
\]

Their common value is defined as the class index \( KI(\Gamma^u, \Gamma_\nu) \). One may likewise introduce \( KI(\Gamma_p, \Gamma^u) \) and one finds from the commutation rule (III, 29.3) for the index in \( X^0 \):

(6.2)

\[
KI(\Gamma^u, \Gamma_p) = (-1)^u KI(\Gamma_p, \Gamma^u).
\]

If \( \gamma^0 \in \Gamma^u, \gamma_p \in \Gamma_p \) we define

(6.3)

\[
KI(\gamma^0, \gamma_\nu) = (-1)^u KI(\gamma_\nu, \gamma^0) = KI(\Gamma^u, \Gamma_\nu).
\]

Clearly \( KI(\gamma^0, \gamma_\nu) = KI(\gamma^0, \gamma_\nu^u) \) for the \( \lambda \) for which \( \gamma_p \) has a representative when \( X \) is a net, or \( \gamma^0 \) has one when \( X \) is a conet.

Referring to (II, 16.4, 17.7), and the known pairing of the groups \( \mathcal{F}^0, \mathcal{S}^0_p \) and \( \mathcal{F}_p^0, \mathcal{S}_p^0 \) (III, 29.4, 29.10) we find;

(6.4) \( KI(\Gamma^u, \Gamma_p) \) is a multiplication pairing \( S^0(X, G), S_p(X, H) \) to \( J \), and similarly for \( KI(\gamma^0, \gamma_p) \) and \( \mathcal{F}^0(X, G), \mathcal{S}_p(X, H) \).
Referring now to (II, 20.7, 33) and (III, 31.1) we have the basic

(6.5) Duality theorem. The cycles and cocycles of a net [cocycles and cycles of a conet] are dual categories.

7. Application to the Betti numbers. It follows from (6.5) that for any field $J$ we have the duality relation for the Betti numbers:

$(7.1) \quad R^p(X, J) = R_p(X, J)$.

We extend the universal theorem for fields (III, 17.8) to nets and prove:

$(7.2) \quad$ The Betti numbers depend solely upon the characteristic of the field.

Owing to (7.1) it is sufficient to consider only one of $R^p, R_p$. We will assume that $X$ is a net and consider $R_p$. For a conet the reasoning would be the same with $R^p$ and cycles in place of $R_p$ and cocycles. To prove (7.2) we merely need to show that if $J_1$ is an extension of $J$ we have

$(7.3) \quad R_p(X, J_1) = R_p(X, J)$.

Let then $\{ \Gamma^I_p \}$ be a base for $\mathcal{S}_p(X, J)$ and $\{ \Gamma^J_p \}$ a maximal subset of elements of this base with linearly independent representatives $\{ \Gamma^J_p \}$ for a given fixed $\lambda$. Since every $\Gamma^I_p \in \mathcal{S}_p(X, J)$ represents a cohomology class of $X$ linearly dependent upon the $\{ \Gamma^J_p \}$, a base may be obtained for $\mathcal{S}_p(X_1, J)$ consisting of $\{ \Gamma^J_p \}$ and of a set $\{ \Gamma^J_p \}$ whose elements are representatives of the zero of $\mathcal{S}_p(X, J)$. It is now a consequence of (III, 17.8) that if $\Delta_p \in \mathcal{S}_p(X, J_1)$ has the representative $\Delta^*_p$ then

\[ \Delta^*_p = \alpha^I \Gamma^I_p + \beta^J \Gamma^J_p; \quad \alpha^I, \beta^J \in J_1. \]

Since the $\Gamma^J_p$ are representatives of the zero of $\mathcal{S}_p(X, J)$ and finite in number, there exists a $\nu > \lambda$ such that $\pi^\lambda \Gamma^J_p = 0$ and hence $\pi^\lambda(\beta^J \Gamma^J_p) = \beta^J \pi^\lambda \Gamma^J_p$ is in the zero of $\mathcal{S}_p(X, J_1)$, or

\[ \Delta_p = \alpha^I \Gamma^I_p. \]

Thus every $\Delta_p$ depends upon the $\Gamma^I_p$. On the other hand a non-trivial relation of the form

\[ \alpha^I \Gamma^I_p = 0, \quad \alpha^I \in J_1, \]

means that for some $\lambda$ we have

\[ \alpha^I \Gamma^I_p = 0. \]

By (III, 17.8) the $\Gamma^I_p$ are linearly dependent elements of $\mathcal{S}_p(X_1, J)$. Hence the $\Gamma^I_p$ are linearly dependent elements of $\mathcal{S}_p(X, J)$, which is ruled out since they are elements of a base for this vector space. This contradiction proves (7.3) and hence (7.2).

Since $R^p(X, J)$ depends solely upon the characteristic $\pi$ we designate henceforth by $R^p(X, \pi)$ its value for all fields of characteristic $\pi$ and similarly for
$R_p(X, J)$ and $R_p(X, \pi)$. For the Betti numbers over the rationals (characteristic zero) we write as usual $R^p(X)$, $R_p(X)$.

8. **Intersections.** The only permanence relations available are those based upon (V, 14.1) for simple complexes. *We must therefore assume $X$ to be simple, or as a special case simplicial.* The groups being as before let $\Gamma^p = \{ \Gamma_q \}$, $\Gamma_\pi = \{ \Gamma^p_\pi \}$ be over $G, H$. Then (V, 14.1ab) yield the permanence relations for intersections:

\begin{equation}
\Gamma^p_\lambda \cdot \Gamma^p_\mu = \pi^p_\lambda(\Gamma^p_\lambda \cdot \Gamma^p_\mu), \quad \lambda > \mu,
\end{equation}

\begin{equation}
\Gamma^p_\lambda \cdot \Gamma^p_\mu = \pi^p_\lambda(\Gamma^p_\lambda \cdot \Gamma^p_\mu).
\end{equation}

It is understood of course that $\lambda, \mu$ are such that, wherever need be, representatives are available and this may always be assumed to be the case for some cofinal subset of $\{ \lambda \}$.

Suppose now that $X$ is a net and let $H$ be discrete. By (II, 17.2) and (8.1) the set $\{ \Gamma^p_\lambda \}$ defines a class of $(p - q)$-cycles of $X$ over $J$ which is by definition the intersection of $\Gamma^p_\lambda$ and is written $\Gamma^p_\lambda \cdot \Gamma_\pi$. Similarly when $G, H, J$ are discrete by (II, 17.4) and (8.2) $\{ \Gamma^p_\lambda \}$ defines a class of $(p + q)$-cochains of $X$ over $J$ which is the intersection of $\Gamma^p_\lambda \cdot \Gamma_\pi$ and is written $\Gamma^p_\lambda \cdot \Gamma^p_\mu$. There are obvious modifications for conets which we leave to the reader. From these definitions follows immediately:

(8.3) Theorem (V, 8.8) holds for intersections in simple and a fortiori in simplicial nets and conets.

§3. FURTHER PROPERTIES OF NETS

9. **Partial nets.** Let $X = \{ X_\lambda; \pi^p_\lambda \}$ be a net [conet] and let $\{ \mu \} \subset \{ \lambda \}$. Then $X' = \{ X_\lambda; \pi^p_\mu \}$ is likewise a net [conet], said to be a *part* of $X$ or a *partial* net [conet] of $X$. When $\{ \mu \}$ is cofinal in $\{ \lambda \}$ then $X'$ is said to be *cofinal* in $X$. In the statement to follow we shall take cycles over $G$, cocycles over $H$, where $G, H$ are commutatively paired to $J$. In the net case the groups of cycles are topologized and those of the cocycles discrete, while in the conet case it is the other way around. The classes in $X, X'$ are, respectively, denoted by $\Gamma$, $\Gamma'$.

(9.1) *If* $X'$ *is a partial net [conet] of* $X$ *then there are the simultaneous homomorphisms of (II, 13.3, 14.5):*

\begin{equation}
\tau^* \cdot \delta^p(X, G) \to \delta^p(X', G),
\end{equation}

\begin{equation}
r^* \cdot \delta^p(X', H) \to \delta_p(X, H),
\end{equation}

and there subsists the permanence relation for the index:

\begin{equation}
\text{KI}(\tau \Gamma^p, \Gamma'_\pi) = \text{KI}(\Gamma^p, r^* \Gamma'_p).
\end{equation}

*Furthermore if there are intersections in $X$ and hence in $X'$ then:*

\begin{equation}
(\tau \Gamma^p) \cdot \Gamma'_\mu = \tau(\Gamma^p \cdot r^* \Gamma'_\mu),
\end{equation}

\begin{equation}
r^* \Gamma'_\mu \cdot r^* \Gamma'_\mu = r^*(\Gamma'_p \cdot \Gamma'_\mu).
\end{equation}
(9.7) It is a consequence of (9.6) that $\tau^*$ induces a homomorphism of the cohomology rings of $X'$ into the corresponding rings for $X$.

(9.8) When $X'$ is cofinal in $X$ then $X$ and $X'$ have the same homology theory.

(9.9) Remark. Let $X$, $X'$ be nets and $G$, $H$ a normal couple. Since $S^p(X, G)$, $S_p(X, H)$ are then dually paired with the Kronecker index as the group multiplication (6.5), the class $\tau^*\Gamma_p$ is uniquely determined by the values of $K_I(\Gamma_p, \tau^*\Gamma'_p)$ for all $\Gamma_p$. Therefore in view of (9.4), $\tau^*$ is uniquely determined by $\tau$. Similarly in the net case $\tau$ is uniquely determined by $\tau^*$. Thus as regards the homology and cohomology groups $\tau$ and $\tau^*$ are related like dual chain-mappings of complexes. For this reason we shall say that $\tau, \tau^*$ are dual.

The relations (9.2), (9.3), (9.4) are the analogues of the permanence relations for the index and intersections under chain-mappings (IV, 10.4; V, 14.1).

The proofs are very simple. Assume $X$ to be a net. Then if $\Gamma^p = \{ \Gamma^p_x \in S^p(X, G) \}$ the subcollection $\{ \Gamma^p \}$ of $\{ \Gamma^p \}$ consists of the coordinates of a $\Gamma^p \in S^p(X', G)$ and by (II, 13.3) $\Gamma^p \rightarrow \Gamma'^p$ defines a simultaneous homomorphism $\tau$ (9.2). Similarly if $\Gamma^p \in S_p(X', G)$ has for representatives $\{ \Gamma^p_x \}$ then the latter are also representatives of $\Gamma^p \in S_p(X, G)$ and by (II, 14.5) $\Gamma^p \rightarrow \Gamma_p$ defines a simultaneous homomorphism $\tau^*$ (9.3). Thus the coordinates of $\Gamma^p$ include those of $\tau^*\Gamma^p$ and the representatives of $\Gamma^p$ are also representatives of $\tau^*\Gamma^p$. Since the Kronecker index $K_I(\Gamma^p, \Gamma_p)$ in $X$ is defined by its values for any coordinate $\lambda$ such that $\Gamma^p_\lambda$ exists we may choose a coordinate $\lambda$ for which $\Gamma^p_\lambda$ exists. Then the two sides in (9.4) become both equal to $K_I(\Gamma^p, \Gamma_p)$ and so they are equal. A similar argument applies to (9.5), (9.6). This proves (9.1) for nets. The modifications needed for nets are obvious.

If $[\mu]$ is cofinal in $[\lambda]$, by (II, 13.3, 14.5) $\tau$ and $\tau^*$ become isomorphisms and (9.8) follows.

Application. Suppose that $[\lambda]$ has an upper bound $\lambda_0$, that is to say, there is a last complex $X_{\lambda_0}$ in $X$. Then $\lambda_0$ is in fact cofinal in $[\lambda]$. The homology theory of $X_{\lambda_0}$ as a “net” is obviously the same as its homology theory as a complex. Therefore (9.8) yields here:

(9.10) If $[\lambda]$ has an upper bound $[\lambda_0]$, the homology theory of $X$ is the same as the homology theory of its last comp.: $X_{\lambda_0}$.

10. Augmentation.

(10.1) Let our usual net be simple. If $\gamma^0 = \{ \gamma^0 \}$ then since $\pi^0_\lambda, \lambda > \mu$, is simple we have $K_I(\pi^0_\lambda \gamma^0) = K_I(\gamma^0_\lambda)$ and since $\pi^0_\lambda \gamma^0 \sim \gamma^0_\lambda$, also $K_I(\pi^0_\lambda \gamma^0) = K_I(\gamma^0_\lambda)$. Thus $K_I(\gamma^0_\lambda) = K_I(\gamma^0_\lambda)$. If $\mu, \nu$ are any two indices there is a $\lambda > \mu, \nu$ and then $K_I(\gamma^0_\lambda) = K_I(\gamma^0_\nu) = K_I(\gamma^0_\lambda)$. Thus $K_I(\gamma^0_\lambda)$ is independent of $\lambda$ and its fixed value is defined as the index $K_I(\gamma^0)$.

(10.2) Let $\gamma^0$ be the fundamental zero-cocycle of $X$ (the sum of the duals of the vertices of $X_\lambda$). According to (IV, 10.8) the constancy of $K_I(\gamma^0)$ implies for $\lambda > \mu$: $\pi^0_\lambda \gamma^0 = \gamma^0_\lambda$. Hence $\gamma^0 = \{ \gamma^0 \}$ is a zero-cocycle of $X$. We call it the fundamental zero-cocycle of the net. From $K_I(\gamma^0) = K_I(\gamma^0)$, and $K_I(\gamma^0, \gamma^0) = K_I(\gamma^0_\lambda, \gamma^0_\lambda)$, follows $K_I(\gamma^0) = K_I(\gamma^0, \gamma^0)$. 

(10.3) Let us now augment $X_\lambda$ to $X_{\lambda\alpha}$ by the addition of a $(-1)$-element $\epsilon_\alpha$, then extend $\nu^\lambda_\alpha$, $\lambda > \mu$, to a chain-mapping $\tau^\lambda_\alpha: X_{\lambda\alpha} \to X_{\alpha\alpha}$ by imposing $\tau^\lambda_\alpha \epsilon_\alpha = \epsilon_\alpha$ (IV, 9.10). It is an elementary matter to verify that properties $N$ of (2.1) continue to hold and so $X_\alpha = \{ X_{\lambda\alpha}; \tau^\lambda_\alpha \}$ is still a net. It is known as $X$ augmented.

(10.4) We observe that a zero-cycle of $X: \gamma^\alpha = \{ \gamma^\alpha \}$ is also a zero-cycle of $X_\alpha$ when and only when $\gamma^\alpha$ is a cycle of $X_{\lambda\alpha}$, i.e., when and only when $N\text{I}(\gamma^\alpha) = N\text{I}(\gamma^\alpha) = 0$. From this follows readily as in (III, 42), with $1^\alpha$ as the class of $\gamma^\alpha$:

\[(10.6) \quad S^\alpha(X_\alpha, G) \cong S^\alpha(X, G)/G1^\alpha,\]

\[(10.7) \quad R^\alpha(X) = R^\alpha(X_\alpha) + 1.\]

(10.8) To sum up, we may say that as regards augmentation in simple nets the situation is the same as for simple complexes (see notably (III, 42.1, 42.5, 42.7)).

11. Products of nets. Let $X = \{ X_\lambda; \pi^\lambda_\alpha \}, Y = \{ Y_\mu; \omega^\mu_\alpha \}$ be two nets and $\Lambda = \{ \lambda; > \}, \mu = \{ \mu; > \}$ their indexing sets. The product $\Lambda \times M = \{ (\lambda, \mu) \}$ ordered by $(\lambda, \mu) > (\lambda', \mu') \iff \lambda > \lambda', \mu > \mu'$ is directed by $>$. Hence $\{ X_\lambda \times Y_\mu; \pi^\lambda_\alpha \times \omega^\mu_\alpha \}$ is a net known as the product of $X, Y$ and denoted by $X \times Y$.

(11.1) If $X, Y$ are both simple, spectra or countable spectra, so is $X \times Y$. However, if $X, Y$ are both simplicial and of dimension greater than 0 then $X \times Y$ is not simplicial.

(For simple nets (V, 17.8) is needed.)

11.2 Suppose now that for dimensions not exceeding $s$ both $X, Y$ have finite Betti numbers mod $\pi$, $R^p(X, \pi), R^q(Y, \pi)$, $p, q \leq s$, and that the dimensions of the elements of the $X_\lambda, Y_\mu$ are above a certain fixed $t$. Then:

\[(11.3) \quad R^p(X \times Y, \pi) = \sum_{p+q=s} R^p(X, \pi)R^q(Y, \pi).\]

If one of the numbers $R^p(X, \pi), R^q(Y, \pi), p, q \leq s$, is infinite so is $R^s(X \times Y, \pi)$. Thus the relations for the Betti numbers are the same as for products of finite complexes (IV, 6.9, 6.10). If all the Betti numbers mod $\pi$ of the two nets are finite we may write down formal Poincaré power series (analogous to the Poincaré polynomials) and (IV, 6.11, 6.12) will hold here also. All that is necessary, however, is to prove (11.3).

If $\gamma^\alpha = \{ \gamma^\alpha_\lambda \}, \delta^\alpha = \{ \delta^\alpha_\lambda \}$ are cycles of $X, Y$ mod $\pi$, we verify at once that $\{ \gamma^\alpha_\lambda \times \delta^\alpha_\mu \}$ is a $(p + q)$-cycle of $X \times Y$ mod $\pi$ which we call the product of $\gamma^\alpha, \delta^\alpha$ and denote by $\gamma^\alpha \times \delta^\alpha$. Moreover $\gamma^\alpha \times \delta^\alpha \sim 0 \iff \gamma^\alpha \times \delta^\alpha \sim 0$. If say $\gamma^\alpha \sim 0$ then some $\gamma^\alpha_\lambda \sim 0$ and hence every $\delta^\alpha_\lambda \sim 0$ (IV, 6.13a), or $\delta^\alpha \sim 0$. Thus $\gamma^\alpha \times \delta^\alpha \sim 0 \iff$ one of $\gamma^\alpha, \delta^\alpha \sim 0$. The same remarks apply with obvious modifications to cocycles.
(11.4) Assuming then that the Betti numbers at the right in (11.3) are finite we can find maximal linearly independent sets of $p$-cycles and $p$-cocycles of $X \bmod \pi$, $\{\gamma_p^i\}$, $\{\gamma_p^i\}$ where $\gamma_p^i = \{\gamma_p^i\}$, $\gamma_p^i = \{\gamma_p^i\}$ and $i = 1, 2, \ldots, R^p(X, \pi)$. Furthermore (III, 32.3) the sets may be so chosen that

$$|| KI(\gamma_p^i, \gamma_p^i) || = 1.$$ 

The substitutions $X \rightarrow Y$, $\gamma \rightarrow \delta$, $p \rightarrow q$, $\lambda \rightarrow \mu$ yield similar elements for $Y$. Hence (IV, 6.7, 6.8; V, 17.9, 17.10)

$$\{\gamma_p^i \times \delta^i\}, \quad \{\gamma_p^i \times \delta^i\}$$

are sets such that

$$| KI(\gamma_p^i \times \delta^i, \gamma_p^i \times \delta^i) | = \pm 1, \quad p + q = p' + q' = s.$$ 

From this we conclude that the elements in the respective sets (11.6) are independent. Hence if $\rho'$ is the sum at the right in (11.3) we have $R'(X \times Y) \geq \rho'$.

If $R'(X, \pi) = \infty$, $p \leq s$, the same argument with $i$ running to any integer $m$ shows that $R'(X \times Y, \pi) \geq m$, and $R'(X \times Y, \pi) = \infty$ also.

Returning now to the case where the Betti numbers at the right in (11.3) are all finite, suppose that there exists an $s$-cocycle $d_s$ of $X \times Y \bmod \pi$ independent of those in the second set (11.6). This cocycle will have a representative $d_s^0$, a cocycle of $X_s \times Y_s \bmod \pi$, which will be $\sim 0 \bmod \pi$. From the definition of the $\gamma_p^i$ we infer that a maximal linearly independent set of $p$-cocycles of $X_s \bmod \pi$ is obtained by augmenting $\{\gamma_p^i\}$ by a set $\{\gamma_p^i\}$ consisting of representatives of the zero $p$th cohomology class of $X$. Thus for each $h$, and hence for all together since their number is finite, there exists a $\lambda_0 > \lambda$ such that $\pi_{\lambda_0} \gamma_p^i \sim 0$. Similarly a maximal linearly independent set for $Y_s$ is obtained by augmenting $\{\delta^i\}$ by a set $\{\delta^i\}$ such that for some $\mu_0 > \mu$ we have $\omega_{\mu_0} \delta^i \sim 0$.

Now by (IV, 6.8) we have

$$d_s^0 \sim \sum_{p + q = s} a_{ij}(p, q) \gamma_p^i \times \delta^j + \text{terms} \gamma \times \delta, \gamma \times \delta, \gamma \times \delta.$$ 

From this follows

$$\pi_{\lambda_0} \times \omega_{\mu_0} (d_s^0 - \sum a_{ij}(p, q) \gamma_p^i \times \delta^j) \sim 0,$$

and so finally

$$d_s \sim \sum a_{ij}(p, q) \gamma_p^i \times \delta^j.$$ 

Consequently $R'(X \times Y) \leq \rho'$, hence both are equal. This proves (11.3).

12. Open or closed subnets. Dissections.

(12.1) The various concepts centering around dissections in complexes (III, 23) will now be extended to nets. The only limitation, caused by the fact that projections need not be unique, is that the operations loc. cit. will not have to bear upon the cycles. Moreover the subnets can only be introduced for nets which satisfy in place of N3 of (2.1) the more restricted condition:
N3'. if $\pi^*_\lambda$, $\pi^*_\mu$, $\lambda > \mu$, are distinct projections then they are chain-homotopic.

That N3' $\rightarrow$ N3 is a consequence of (IV, 15.2). Notice also that if N*3' is the analogue of N3 with $>$ replaced by $<$, then N*3' for $\pi^*_\lambda$ is a consequence of N3' (IV, 15.7) and N*3' $\rightarrow$ N*3 (IV, 15.2).

Suppose then that $X$ satisfies N3', hence also N*3' (relative to the $\pi^*_\lambda$) and let $(X_0, X_1)$ be a dissection of $X$, $(X_0$ open, $X_1$ closed) such that:

(a) $\lambda > \mu \rightarrow \pi^*_\lambda X_0 \subset X_1$;

(b) if $\pi^*_\lambda, \pi^*_\mu, \lambda > \mu$, are distinct projections (hence chain-homotopic) then the related homotopy operator $\mathcal{D}_\mu$ is such that $\mathcal{D}_\mu X_0 \subset X_1$.

Referring to (IV, 22) there may be introduced the induced operations $\pi^*_\lambda, \mathcal{D}_\mu$ and their duals $\pi^*_\lambda, \mathcal{D}_\mu$. Then (IV, 22), $\pi^*_\lambda, \pi^*_\mu, \mathcal{D}_\mu, \mathcal{D}_\mu$ are related like $\pi^*_\mu, \cdots$. It follows that $X_i = \{X_0; \pi^*_\lambda\}$ is a net which satisfies N123'. We call $X_0$ an open subnet of $X$, $X_1$ a closed subnet of $X$, and the pair $(X_0, X_1)$ (in that order) a dissection of $X$. The two subnets $X_0, X_1$ are also said to be complementary.

We are thus in position to introduce the cycles and cocycles of the $X_i$, their groups and Kronecker indices.

(12.2) Suppose that in addition to (a, b) we have:

(c) there is a simple set-transformation $\mathcal{T}_\mu: X_\lambda \rightarrow X_\mu, \lambda > \mu$, such that $t^*_\mu X_0 \subset X_1$, and that any two projections $\pi^*_\lambda, \pi^*_\mu$ are contiguous in $\mathcal{T}_\mu$.

Under these conditions $\mathcal{T}_\mu$ induces the set-transformations $\mathcal{T}_\mu X_0 \rightarrow X_\mu$ and their duals $t^*_\mu$. Referring then to (V, 13, 14) one may repeat the considerations of (8) regarding the intersections in $X_i$ and derive (8.3) for these intersections.

In point of fact $X_1$ is a simple net and the intersections induced in $X_1$ are recognized, by reference to (V, 13), to coincide with its intersections as a simple net.

(12.3) Since a cycle of $X_0$ [cocycle of $X_0$] is absolute and a cycle of $X_\mu$ [cocycle of $X_\mu$] is a cycle of $X_0$ mod $X_1$ [cocycle of $X_\mu$ mod $X_\alpha$] we shall naturally describe the cycles of $X_1$ [cocycles of $X_0$] as absolute, and the cycles of $X_0$ [cocycles of $X_1$] as cycles of $X_0$ mod $X_1$ [cocycles of $X_\mu$ mod $X_\alpha$], or also as relative cycles [cocycles].

Since the $X_i$ are nets (6.5) yields

(12.4) The cycles of $X_\alpha$ mod $X_1$ [of $X_1$] and the cocycles of $X_0$ [of $X_\mu$ mod $X_1$] are dual categories and they have intersections when (c) holds. The universal theorem for fields (7.2) holds for both types.

(12.5) Let $(X'_0, X'_1)$ be a second dissection of $X$ and let $X'_\mu, \cdots$ have their obvious meaning. If $X_1 \subset X_\mu$ throughout we will say that $X'_1$ is contained in $X_1$, written $X'_1 \subset X_1$. Suppose this to be the case. Since $\pi^*_\mu = \pi^*_\mu | X_\mu$, we have $\pi^*_\mu X'_0 = \pi^*_\mu X'_1 \subset X'_1$, and since $X'_1$ is a closed subcomplex of $X_\mu$, $X'_1$ is in fact a closed subnet of $X_1$. Similarly if $X_0 \subset X'_\alpha$ throughout we say that $X_0$ is contained in $X'_0$, written $X_0 \subset X'_0$. The same argument shows that $X_0$ is then an open subnet of $X'_0$ also.

(12.6) If $X$ is a spectrum or a sequential spectrum so are $X_0, X_1$, while if $X$ is simplicial or simple so is $X_1$. 


13. The comparison of the groups of $X$ with those of the subnets will yield certain important sets of homomorphisms associated with $(X_0, X_1)$.

(13.1) We choose a fixed coefficient group $G$ and denote by $\mathcal{B}_p$, $\cdots$, $\mathcal{B}_q$, $\cdots$, $\mathcal{B}_r$, $\cdots$, $\mathcal{B}_s$, $\cdots$ the groups of the cycles of $X$, $X_1$, $X_1$, $\cdots$, $X_n$ and by $\mathcal{B}_p$, $\cdots$ the same for the cocycles. In the case of the latter $G$ is assumed discrete. We will set also:

\[
\begin{align*}
\tau_\alpha & = \text{the projection } X_\alpha \to X_{\alpha} ; \\
\theta_\alpha & = \text{the injection } X_{\alpha} \to X_\alpha ; \\
\tau_\lambda^* & = \text{the injection } X_{\lambda}^* \to X_{\lambda}^* ; \\
\theta_\lambda^* & = \text{the projection } X_{\lambda}^* \to X_{\lambda}^* .
\end{align*}
\]

As we recall (IV, 10.14) $\tau_\lambda$, $\tau_\lambda^*$, likewise $\theta_\lambda$, $\theta_\lambda^*$ are dual chain-mappings.

Each of the chain-mappings $\pi_\lambda$, $\tau_\lambda$, $\cdots$ induces a simultaneous homomorphism in the corresponding homology or cohomology groups which will be denoted by $\pi_\lambda^*$, $\tau_\lambda^*$, $\cdots$.

(13.2) If $\gamma^\prime_\lambda = [\gamma^\prime_\lambda]$ is a cycle of $X_1$ then it is also a cycle of $X$. The identical transformation $\gamma^\prime_\lambda \to \gamma^\prime_\lambda$ defines a simultaneous homomorphism (imbedding) in the algebraic sense $\theta: \mathcal{B}_\lambda^\prime \to \mathcal{B}_\lambda^\prime$. If $U_\lambda$ is open in $\mathcal{B}_\lambda^\prime$ and $V_\lambda = \{ \gamma^\prime_\lambda \in U_\lambda \}$ $W_\lambda = \{ \gamma^\prime_\lambda \in U_\lambda \}$ then $W_\lambda = \theta V_\lambda$ and since $\{ V_\lambda \}$, $\{ W_\lambda \}$ are subbases for $\mathcal{B}_\lambda^\prime$, $\mathcal{B}_\lambda^\prime$, $\theta$ is a simultaneous isomorphism (of each $\mathcal{B}_\lambda^\prime$ with a subgroup of the corresponding $\mathcal{B}_\lambda^\prime$). Now $\gamma^\prime_\lambda \sim 0$ in $X_1 \to \gamma^\prime_\lambda \sim 0$ in $X_{\alpha} \to \gamma^\prime_\lambda \sim 0$ in $X \to \theta \gamma^\prime_\lambda \subset \mathcal{B}_\lambda^\prime$. Therefore $\theta$ induces a simultaneous homomorphism $\theta: \mathcal{B}_\lambda^\prime \to \mathcal{B}_\lambda^\prime$. We call $\theta$ the injection $X_1 \to X$. Its analogy with the injection in the case of complexes (III, 23.2) is obvious.

Similarly the mapping $\gamma^\lambda_\alpha \to \gamma^\lambda_\rho$ of each cocycle of $X_0$ on itself generates a simultaneous homomorphism $\tau^*: \mathcal{B}_\rho^\lambda \to \mathcal{B}_\rho$ called the injection $X_\rho^* \to X^*$ which induces a simultaneous homomorphism $\tau^*$ of the cohomology groups: $\mathcal{B}_\rho^\lambda \to \mathcal{B}_\rho^\lambda$.

(13.3) For the proper treatment of the projections it is advisable to deal with the cycles of $X_0$ and cocycles of $X_1$ in a manner similar to (III, 23.5).

As in loc. cit. we first identify a cycle $\gamma^\lambda_\alpha$ with the chains $\gamma^\lambda_\alpha + C^\lambda_\alpha$, $C^\lambda_\alpha \subset X_{\alpha} \cdot$ Thus a cycle mod $X_{\alpha} \cdot$, or cycle of $X_{\alpha}$ is now merely any chain with boundary in $X_{\alpha}$. A cycle of $X$ is then defined as a collection $\gamma^\lambda_\alpha = \{ \gamma^\lambda_\alpha \}$, where $\gamma^\lambda_\alpha$ is a chain with boundary in $X_{\alpha}$ such that $\lambda > \mu \to \pi^\lambda_\mu \gamma^\lambda_\alpha = \gamma^\lambda_\mu \sim 0$ mod $X_{\mu}$.

That is to say, $\pi^\lambda_\mu \gamma^\lambda_\alpha = \gamma^\lambda_\mu + \lambda$ of a chain of $X_{\alpha}$ is in $\mathcal{B}_\rho^\lambda$. Since $\pi^\lambda_\mu = \pi^\lambda_\mu \mod X_{\mu}$, it is easy to identify the groups of the cycles of $X$ mod $X_{\alpha}$ with those of the cycles of $X_0$ (as a net) and this justifies the appellation "cycles mod $X_1$" for the cycles of $X_0$ (12.3).

(13.4) It is an elementary matter to verify that

\[
\lambda > \mu \to \tau_\mu \pi^\lambda_\mu \sim \pi^\lambda_\mu \tau_\lambda \to \tau_\mu \pi^\lambda_\mu = \pi^\lambda_\mu \tau_\lambda .
\]

Hence (II, 13.5) if $\Gamma^\rho = \{ \Gamma^\rho \} \in \mathcal{B}_\rho$ then $\Gamma^\rho = \{ \gamma^\rho \} \in \mathcal{B}_\rho$ and $\Gamma^\rho \to \Gamma^\rho$ defines a homomorphism $\tau: \mathcal{B}_\rho \to \mathcal{B}_\rho$. 

Let now $\gamma' = [\gamma^p] \in \Gamma^p$ and set $\gamma_0' = [\tau_0 \gamma^p]$. Since $\tau_0 \gamma^p \in \Gamma^p_0$, we have $\gamma_0' \in \Gamma_0^p$. Thus $\gamma' \to \gamma_0'$ defines a homomorphism $\tau$, the projection $X \to X_0$ (II, 13.5). We notice that $\gamma' \sim 0 \to \gamma_0' \sim 0 \to \tau_0 \gamma^p \sim 0 \to \gamma_0' \sim 0$. Thus $\tau_0 \Gamma_0^p \subseteq \Gamma_0^p$. It follows in particular that $\tau$ induces $\tau_0$.

(13.5) Similar considerations are valid, with all topological arguments omitted, regarding the cocycles mod $X_0$, the projection being this time $\theta^*: X^* \to X_1^*$.

(13.6) Suppose now $G, H$ commutatively paired to $J$ and let the cycles be over $G$, the cocycles over $H$, and the groups of cocycles taken discrete. In particular $H$ is assumed discrete. If $\gamma_0^p$ has the representative $\gamma_0^{ol}$ then by (IV, 10.5):

$$\text{KI}(\gamma_0^p, \gamma_0^{ol}) = \text{KI}(\gamma_0^p, \gamma_0^p),$$

and hence from the definition of the indices in nets we obtain the relation of permanence:

$$\text{KI}(\tau_0 \gamma^p, \gamma_0^{ol}) = \text{KI}(\gamma^p, \tau_0 \gamma_0^{ol}).$$

Therefore $\tau, \tau^*$ are dual. We prove similarly that $\theta, \theta^*$ are dual with the permanence relation

$$\text{KI}(\theta \gamma^p, \gamma_0^p) = \text{KI}(\gamma^p, \theta \gamma_0^p).$$

If intersections are present the relations of permanence (V, 14.1ab) yield the analogous relations here and we shall not repeat them.

It is hardly necessary to observe that the same situation prevails regarding conets.

To sum up then we have proved:

(13.9) Theorem. With the dissection $(X_0, X_1)$ of a net or conet $X$ ($X_0$ open, $X_1$ closed) there are associated the following simultaneous homomorphisms of the groups of cycles:

(a) a projection $\tau: X \to X_0$, or reduction mod $X_1$ of the cycles of $X$;

(b) an injection $\theta: X_1 \to X$ or mapping into themselves of the groups of the cycles of $X_1$;

(c) a projection $\theta^*: X^* \to X_1^*$ or reduction mod $X_0$ of the cocycles of $X$;

(d) an injection $\tau^*: X_0^* \to X^*$ or mapping into themselves of the groups of the cocycles of $X_0$;

(e) each of $\tau, \ldots, \theta^*$ maps groups $\mathcal{G}$ into groups $\mathcal{G}$ and hence they induce simultaneous homomorphisms $\tau, \ldots, \theta^*$ of the appropriate homology or cohomology groups into one another;

(f) each of $\tau, \tau^*$, likewise each of $\theta, \theta^*$ is dual to the other;

(g) the permanence relations (13.7, 13.8) hold, and in case there are intersections, the same relations as (V, 14.1ab) for $\tau, \tau^*$ and $\theta, \theta^*$ are fulfilled.

14. Special properties of subnets when $G$ is compact or a field. In this special case it will be found possible to restore the chain-cochain relations
existing in complexes. For convenience we will denote by $C$, $γ$, $Γ$ chains, cycles, classes related to $X₀$ (thus $γ^p_0$ for a cocycle of $X_0$, $\cdots$) and $D$, $δ$, $Δ$ the same when related to $X₁$. The other notations are as in (13.1) with the addition of $C$ for groups of chains.

(14.1) If $γ^p = [γ^p]$ is a cycle of $X$ mod $X₁$, then $δ^{p-1} = [Fγ^p]$ is an (absolute) cycle of $X₁$, and it is denoted by $δ^{p-1} = Fγ^p$. This holds for any division-closure group $G$.

Let $δ^{p-1} = [δ^{p-1}_μ]$, $δ^{p-1}_A = Fγ^p_A$. Suppose $λ > μ$. We have $γ^p_A \sim ν^λ γ^p_A$ mod $X_μ$ in $X_λ$, and since $G$ is a division-closure group, $X_μ$ contains $C^{p+1}_μ$ such that

$$FC^{p+1}_μ = γ^p_μ - ν^λ γ^p_μ - D^p_μ.$$  

Since the right-hand side is an absolute cycle and $ν^λ_μ$ commutes with $F$, we have:

$$Fγ^p_μ - ν^λ γ^p_μ = δ^{p-1}_μ = ν^λ δ^{p-1}_μ = FD^p_μ,$$

or $δ^{p-1}_μ \sim ν^λ δ^{p-1}_μ$ in $X_μ$. Therefore $δ^{p-1}$ is a cycle of $X₁$.

(14.2) If $δ^{p-1} = [δ^{p-1}_μ]$ is a cycle of $X₁$ which is \sim 0 in $X$ then $δ^{p-1}$ is an $Fγ^p$ in the sense of (14.1).

The condition on $δ^{p-1}$ means that $δ^{p-1}_μ \sim 0$ in $X_μ$, and so as before $δ^{p-1} = Fγ^p_A$, where $γ^p_A$ is a cycle of $X_μ$ mod $X_μ$. It follows that $P_λ = F^{-1} δ^{p-1}_μ \neq 0$. Let also $Q_μ = \{Γ_μ | γ^p_μ \in P_λ\}$ and denote by $t_μ$ the natural projection $S^p_μ \to S^p_μ$. When $G$ is compact so is $S^p_μ$ and since $F$ is continuous $P_λ$ is likewise compact (I, 23.1). From the continuity of $t_μ$ follows then that $Q_μ$ is likewise compact (I, 23.2) and it is different from $∅$. Furthermore if $[γ^p_μ]$ are the induced projections of the homology groups then $λ > μ \mapsto ν^λ_μ Q_μ \subset Q_μ$. Thus $[Q_μ ; ν^λ_μ]$ is an inverse system of compact spaces, and so it has a limit-element $Γ^p = \{Γ^p_μ\}$. When $G$ is a field $S^p_μ$ is finite-dimensional and hence linearly compact (II, 27.7). We find then by reference to (II, 27.6) that the same conclusion may be reached.

By assumption the class $Γ^p_μ$ contains a $γ^p_μ$ such that $Fγ^p_μ = δ^{p-1}_μ$ and $γ^p = [γ^p]$ behaves as required.

(14.3) $γ^p \sim 0$ mod $X₁ \to Fγ^p \sim 0$ in $X₁$.

For $γ^p \sim 0$ mod $X₀ \to γ^p \sim 0$ mod $X_μ \to γ^p_μ - D^p_μ = FC^{p+1}_μ \to Fγ^p_μ \sim 0$ in $X_μ \to Fγ^p \sim 0$ in $X₁$.

The same propositions may be obtained for the cocycles. This time $G$ must be discrete.

(14.4) If $δ^p = [δ^p_μ]$ is a cocycle of $X$ mod $X₀$ then $γ^{p+1} = \{Fδ^p_μ\}$ is an (absolute) cocycle of $X₀$ and it is denoted by $Fδ^p$.

(14.5) If $γ^{p+1} = \{γ^{p+1}_μ\} \sim 0$ in $X$ then it is an $Fδ^p$.

(14.6) $δ^p \sim 0$ mod $X₀ \to Fδ^p \sim 0$ in $X₀$.

The proofs are very similar to those of (14.1, 14.2, 14.3) and are omitted.

15. Net duality in the sense of Alexander. We shall extend to nets the results of (III, 38, 39) for complexes. The terms “cyclic, $\cdots$" are to receive the same meaning as in (III, 21.1). The notations remain those of (13, 14).

(15.1) If $X$ is $(p - 1, p)$-acyclic and $G$ is compact or a field then
\[ \delta^{p-1}(X_1, G) \cong \delta^p(X, X_1, G) = \delta^p(X_0, G). \]

It is to be kept in mind that the compared homology groups are those of \( X^1 \) itself at the left and the groups of \( X \mod X_1 \) at the right. The latter are also the groups of the net \( X_0 \). In the notations of (13) the groups are also written less explicitly \( \delta^p \), \( \delta^q \).

To prove (15.1) we first need

(15.2) **Under the conditions of (15.1)** \( \delta^{p-1} = F\gamma^p \sim 0 \) in \( X_1 \to \gamma^p \sim 0 \mod X_1 \).

Let first \( G \) be compact. Then \( \mathcal{G}_n^p \) (group of chains of \( X_n \)) is compact. Owing to the continuity of \( F \) and of the group operations, \( R_\lambda = \gamma_\lambda^p - (F^{-1}\delta_\lambda^{p-1}) \cap \mathcal{G}_n^p \) is likewise compact and is a collection of absolute cycles. If \( \partial_\lambda \) is the natural mapping \( \mathcal{B}_n^p \to \mathcal{B}_n^p \), then \( S_\lambda = \partial_\lambda^p R_\lambda \) is again a compact set. If \( \Gamma^p = S_\lambda \) and \( \gamma^p = \gamma^p \), then

\[ \gamma^p \sim \gamma^p - D^p \]

and so

\[ \gamma^p = \gamma^p - D^p + FD^p. \]

Suppose \( \lambda > \mu \). Since \( S_\mu^p X_\lambda \subset X_{1\mu} \), and \( \gamma^p \) is a cycle of \( X_0 \), we have

\[ S_\mu^p \gamma^p = \gamma^p - D^p. \]

Since \( S_\mu^p \) commutes with \( F \), the right-hand side is a cycle and so \( FD^p = F\gamma^p = \delta^{p-1}_\mu \) and consequently \( \Gamma^p = S_\mu^p \Gamma^p \cap S_\mu \) as in (13.1). In other words \( S_\mu^p S_\mu \subset S_{1\mu} \). Once more \( \{ S_\lambda ; \gamma_\lambda^p \} \) is an inverse mapping system of compact spaces, and so it possesses a limit-element \( \{ \gamma^{p+1}_\lambda \} = \Gamma^p \). This limit-element is such that \( \Gamma^p \) contains a representative (absolute) cycle \( \gamma^p = \gamma^p + D^p \). Since \( X \) is \( p \)-acyclic, \( \gamma^p \sim 0 \) and hence \( \gamma^p \sim 0 \), \( \gamma^p \sim 0 \mod X_1 \), and finally \( \gamma^p \sim 0 \mod X_1 \). This proves (15.2) when \( G \) is compact. When \( G \) is a field the proof is the same with compactness replaced by linear compactness.

**Proof of (15.1).** Since \( X \) is \((p - 1)\)-cyclic (14.1) and (14.2) together assert that \( F\mathcal{B}_n^p = \mathcal{B}_n^{p-1} \), then (14.3) and (15.2) that this mapping induces an isomorphism \( \mathcal{B}_n^p \to \mathcal{B}_n^{p-1} \). Since the two \( \delta \) groups are closed in the corresponding \( \mathcal{B} \) groups (3.5) by (II, 5.4) \( F \) induces a homomorphism \( \tau: \mathcal{B}_n^p \to \mathcal{B}_n^{p-1} \) which by (15.2) is univalent. Thus \( \tau \) is continuous and one-one. Since the two \( \mathcal{B} \) groups are compact or linearly compact \( \tau \) is an isomorphism (I, 32.4; II, 27.8) and (15.1) is proved.

(15.3) **Linking coefficient.** Let again \( X \) be \((p - 1), p\)-acyclic and let \((G, H)\) be a normal couple (III, 30.1). Take a cycle \( \delta^{p-1} \) of \( X_1 \) over \( G \) which is \( \sim 0 \), and a cocycle \( \gamma_\rho \) of \( X_0 \) over \( H \). By (14.2) we have \( \delta^{p-1} = F\gamma^p \), where \( \gamma^p \) is a cycle \( X_1 \), and we define the linking coefficient of \( \delta^{p-1}, \gamma_\rho \) as:

\[ \text{Lk}(\delta^{p-1}, \gamma_\rho) = \text{KI}(\gamma^p, \gamma_\rho). \]

It is shown to have the same properties as for complexes (III, 35, 38), and in particular there is a class linking coefficient \( \text{Lk}(\Delta^{p-1}, \Gamma^p) \) given by
(15.5) \[ \text{Lk}(\Delta^{p-1}, \Gamma_p) = \text{KI}(\Gamma^p, \Gamma_p), \]

where \( \delta^{p-1} = F^{\gamma^p}, \gamma^p \in \Gamma^p \). From the known properties of the index we infer then:

(15.6) \[ \text{Lk defines a group multiplication for } \mathfrak{S}^{p-1}(X_1; G) \text{ and } \mathfrak{S}_p(X_0; H), \]

(15.7) From this point to the duality theorem is but a step. We continue to assume \( X \) to be \((p - 1, p)\)-acyclic. By (12.4) \( \mathfrak{S}^p(X_0, G) \) and \( \mathfrak{S}_p(X_0, H) \) are dually paired with the Kronecker index as the multiplication. Coupling this with (15.1, 15.6) we obtain (III, 38.3) for nets. It is now possible to repeat the argument leading to the results of (III, 39.1) with simple nets instead of simplicial complexes, and with the vertices, loc. cit., replaced by the cycles \( \gamma^p \) of (10.5) associated with the components of the net. We may thus state explicitly:

(15.8) **Theorem.** The duality theorems of Alexander's type for complexes (III, 38.3, 39) are valid for nets, it being understood that simplicial complexes are to be replaced by simple nets.

§4. SPECTRA

16. We have already defined spectra (2.4). The unique projections characteristic of a spectrum offer the advantage that chains and chain-groups may be introduced, thus bringing the homology theory of a spectrum one step nearer to the prototype, the homology theory of complexes. However, there is no guarantee that the homology groups based on the chains are the same as the net groups, and this question will be our major problem as regards spectra. Notice that since cocycles are determined by a single coordinate they will not concern us seriously in this connection.

A "cospectrum" may of course be introduced, but will be dispensed with as not really useful in the sequel.

Let then \( X = \{X_\lambda; \pi_\lambda^\lambda\} \) be a spectrum. Since \( \pi_\lambda^\lambda \) is unique if \( \mathfrak{S}^\lambda, \mathfrak{S}_\lambda, \mathfrak{S}^\lambda_\lambda \), \( \mathfrak{S}_\lambda^\lambda \) are the usual groups of \( X_\lambda \) over a fixed \( G \), the collection \( \{\mathfrak{S}^\lambda, \pi_\lambda^\lambda\} \) is an inverse system and it has a limit-group \( C^p \) whose elements \( c^p = \{c^\lambda_\lambda\} \) are known as the projective \( p \)-chains over \( G \). If \( c^{p+1} = \{c^{p+1}_\lambda\} \in C^{p+1} \) then \( \{F \cap c^{p+1}_\lambda\} \in C^p \) and is written \( Fc^{p+1} \). The boundary operator \( F \) thus defined is a homomorphism \( C^{p+1} \rightarrow C^p \) (II, 13.5) and clearly \( FF = 0 \). The projective cycles, bounding projective cycles and their groups \( Z^p, F^p \) are defined as for complexes.

The projective \( p \)th homology group over \( G \) is of course \( H^p = Z^p/F^p \).

The co-elements are treated in the same way save that all the systems are direct, and hence all groups discrete.

It follows immediately from the definitions that \( H_p \) is the ordinary group of \( X \) as a net, and so the projective groups of cocycles will not require any special considerations.

(16.1) The mapping \( \gamma_\lambda: C^p \rightarrow \mathfrak{S}^\lambda \) defined by \( c^p \rightarrow c^\lambda_\lambda \) is a homomorphism which commutes with \( F \) and so maps \( Z^p \rightarrow \mathfrak{S}^\lambda_\lambda, F^p \rightarrow \mathfrak{S}^\lambda \) (II, 13.4).
(16.2) $\mathbb{Z}^p$ is closed in $C^p$.

(16.3) When $G$ is compact so are the projective groups $C^p, Z^p, F^p, H^p$, and when $G$ is a field they are linearly compact vector spaces over $G$.

When $G$ is compact so are the groups $\mathbb{G}^p$, and hence also $\mathbb{C}^p$, from which follows the same for $Z^p, F^p$, and finally for $H^p$ by (II, 5.5). When $G$ is a field the groups $\mathbb{G}^p$ are finite-dimensional and hence linearly compact. The rest is then the same with "compact" replaced by "linearly compact."

From (II, 13.3) there follow now:

(16.4) If $\{\mu; \rangle \subset \{\lambda; \rangle$ then $X' = \{X{\lambda}\}$ is likewise a spectrum said to be "a part of $X." If $c^\lambda = \{c^\lambda{\mu}\}$ is a chain of $X$ then $c'' = \{c''{\lambda}\}$ is a chain of $X'$ and $c^\lambda \to c''$ defines a homomorphism $\tau$ of the projective groups, $C^p, Z^p, F^p, H^p$ of $X$ into the corresponding groups of $X'$.

(16.5) Under the same conditions if $\{\mu\}$ is cofinal in $\{\lambda\}$, $\tau$ is an isomorphism between the groups $C^p, Z^p, F^p, H^p$ of the two spectra.

17. Let $\mathcal{G}^p, \mathcal{B}^p, \mathcal{D}^p$ denote the net groups (as defined in §1), all groups in question being over a fixed $G$. When do we have $H^p = \mathcal{D}^p$? Sufficient conditions, covering all requirements later, are given by the

(17.1) Theorem. In a spectrum the projective homology groups over a group $G$ which is compact or a field are isomorphic with the corresponding net groups.

Since in the duality theorems for a net no other types of homology groups occur we have:

(17.2) Corollary. As regards the duality theorems for a spectrum of finite complexes the groups of cycles may be chosen projective throughout.

Clearly also:

(17.3) In a spectrum the group of bounding projective cycles over a group $G$ which is compact or a field is closed, i.e., we have $F^p = \overline{F^p}$ and hence $H^p = \mathcal{Z}^p / F^p$.

(17.4) Before we proceed with the proofs we shall show that

(a) Every class $\gamma^p \in \mathcal{D}^p$ contains a projective cycle $\gamma^p$.

(b) $\Gamma^p = 0 \leftrightarrow \gamma^p \in \mathcal{F}^p$.

(17.5) Suppose first $G$ compact. The groups $\mathbb{G}^p, \mathcal{B}^p, \mathcal{D}^p$ are then all compact. Let $\gamma^p = \{\gamma^p{\lambda}\} \in \mathcal{F}^p$ and set $X{\lambda} = \{X{\lambda}{\mu}\}$. Evidently $X{\lambda}{\mu} \subset \mathbb{G}^p$ for $\lambda > \mu$. Since $\mathbb{G}^p$ is compact so is $X{\lambda} = \gamma^p + X{\lambda}$. Hence $\{X{\lambda}; X{\lambda}\}$ is an inverse mapping system of compact spaces. By (I, 39.1) there exists an element $\gamma^p = \{\gamma^p{\lambda}\}$ in the limit-space of the system. Clearly $\gamma^p$ is a projective cycle and $\gamma^p \in \Gamma^p$. This proves (a) for $G$ compact.

Suppose now $\Gamma^p = 0$. The projective cycle $\gamma^p$ just obtained will then have the property that $\gamma^p \sim 0$ for every $\lambda$. Since $\mathbb{G}^p = \mathcal{B}^p$ there exist in $X{\lambda}$ chains $c{\lambda}^{r+i}$ such that $F{\lambda}^{r+i} = \gamma^p$. Let $X{\lambda}^{r+i} = \{c{\lambda}^{r+i}; F{\lambda}^{r+i} = \gamma^p\}$. If $c{\lambda}^{r+i}$ and $c{\lambda}^{r+i}$ are elements of $\mathcal{D}^{r+i}$ then $c{\lambda}^{r+i} - c{\lambda}^{r+i}$ is a $(p + 1)$-cycle. Therefore
\( x^{p+1}_\lambda = c^{p+1}_{\lambda} + \mathfrak{B}^{p+1}_\lambda \). Since \( \mathfrak{B}^{p+1}_\lambda \) is compact, \( x^{p+1}_\lambda \) is likewise compact. As before \( \{x^{p+1}_\lambda, x^n_\alpha\} \) is an inverse system of compact spaces and there is an element \( e^{p+1} = [c^{p+1}] \) in its limit-space. Clearly \( e^{p+1} \) is a projective chain and \( Fc^{p+1} = \gamma^p \). This proves (3) for \( G \) compact.

(17.6) Suppose now \( G \) a field. Since the groups \( \mathfrak{S}_\alpha \), \( \cdots \) are finite-dimensional vector spaces over \( G \), they are linearly compact, and we find by reference to (II, 27.6) that the proofs just given for \( (\alpha, \beta) \) are still valid in the present instance.

(17.7) Proof of (17.1). Let \( \tau \) denote the homomorphism \( \mathbb{Z}^p \rightarrow \mathfrak{B}^p \) in the algebraic sense which is the identity on \( \mathbb{Z}^p \). It is clear that \( \tau \mathbb{F}^p \subset \mathfrak{B}^p \). Furthermore here \( \mathbb{F}^p \) is compact or linearly compact and hence closed in \( \mathbb{Z}^p \) (I, 32.1; II, 27.5) and \( \mathfrak{B}^p \) is always closed in \( \mathbb{Z}^p \) (3.5). Therefore \( \tau \) induces a homomorphism \( \tau: \mathbb{H}^p \rightarrow \mathfrak{B}^p \) in the algebraic sense and by \( (\alpha, \beta) \) this is an isomorphism in the algebraic sense. Since \( \lim \mathfrak{B}_\alpha^p = \mathbb{Z}^p \), \( \lim \mathfrak{B}_\alpha^p = \mathbb{F}^p \), \( \lim \mathfrak{B}_\alpha^p = \mathbb{H}^p \), we find by (II, 13.6) that \( \tau \) is open and hence it is an isomorphism, proving (17.1).

18. Sequential spectra. In a sequential spectrum \( \{X_n; \mathfrak{P}^p_n\} \) (2.4) the \( \pi^{n+1}_n \) determine all the projections since N3 of (2) yields

\[
\pi^{n+1}_n = \pi^{n+1}_m \pi^{n+2}_m \cdots \pi^{n+1}_{m-k} - 1.
\]

(18.2) The homology theory of a countable net is either that of a single complex of the net or else that of a sequential spectrum.

If \( |\lambda| \) has an upper bound \( \lambda_0 \) then the homology theory of \( X \) is that of \( X_{\lambda_0} \) (9.10). In the contrary case (I, 4.4) \( |\lambda| \) has a cofinal sequence and so we may assume \( X = \{X_1, X_2, \cdots \} \). Select now a definite \( \pi^{n+1}_n \) for each \( n \) and determine \( \pi^{n+1}_n \), \( k > 1 \), by means of (18.1). The resulting net with these projections is a sequential spectrum whose homology theory is the same as for the initial net.

19. For certain sequential spectra one may strengthen (17.1) as follows:

(19.1) If the sequential spectrum \( X = \{X_n; \mathfrak{P}^{n+1}_n\} \) is such that every \( \pi^{n+1}_n \) is a mapping onto for the chain-groups, then (17.1) holds for every division-closure group \( G \).

We first prove:

(a) Every class \( \Gamma^p \subset \mathfrak{S}^p \) contains a projective cycle \( \gamma^p \).

(b) \( \Gamma^p = 0 \rightarrow \gamma^p \subset \mathfrak{F}^p \).

The first is the same as (17.4a), the second differs from (17.4b) in that \( \mathbb{F}^p \) is replaced by its closure. The proofs will be different from those of (17.4ab).

Proof of (a). Let \( \Gamma^p \subset \mathfrak{S}^p \) and \( \gamma^p = [\gamma^p_\alpha] \subset \Gamma^p \). Suppose that we have found \( \gamma^p_\alpha \), \( 1 \leq r \leq n \), such that

\[
\gamma^p_\alpha = \pi^{r+1}_\alpha \gamma^p_\alpha,
\]

\[
\gamma^p_\alpha = \gamma^p_\alpha,
\]

\[
\gamma^p_\alpha \sim \gamma^p_\alpha,
\]

for some \( s, 1 \leq s \leq n \);

(19.2)

I say that a similar set may be found for \( n + 1 \), with the same \( s \). We have in fact \( \gamma^p_n \sim \pi^{n+1}_n \gamma^p_{n+1} \) in \( X_n \). Since \( X_n \) is finite and \( G \) is a division-closure group \( X_n \) contains a chain \( c^{p+1}_n \) such that \( Fc^{p+1}_n = \gamma^p_n - \pi^{n+1}_n \gamma^p_{n+1} \). Since \( \pi^{n+1}_n \) is
a mapping "onto" there exists a chain $c^{p+1}_n$ such that $\pi^{n+1}_n c^{p+1}_n = c^{p+1}_n$. If we replace therefore $\gamma^{p+1}_n \gamma^{p+1}_n = \gamma^{p+1}_n \gamma^{p+1}_n = \gamma^{p+1}_n$, and so the same situation as before for the set $\gamma^{p+1}_n$, $\gamma^{p+1}_n$, $\gamma^{p+1}_n$. This gives an inductive construction for a projective cycle $\gamma^{p+1}$. We have from (19.2), $\gamma^{p+1}_n \sim \gamma^{p+1}_n$ for $n \geq s$ and also for $n < s$: $\pi^{p+1}_n \gamma^{p+1}_n = \gamma^{p+1}_n = \pi^{p+1}_n \gamma^{p+1}_n$ and so $\gamma^{p+1} \sim \gamma^{p+1}$. In other words we have obtained a projective cycle $\gamma^{p+1} \in \Gamma^p$ and having in addition an assigned coordinate in common with a given $\gamma^{p+1} \in \Gamma^p$. In particular this proves (\alpha).

Passing now to (\beta), let $\Gamma^p = 0$. Then $\gamma^{p+1}_n \sim 0$ in $X_n$, and as long as before $X_n$ contains $c^{p+1}_n$ such that $F c^{p+1}_n = \gamma^{p+1}_n$. Choose now $c^{p+1}_n = \pi^{p+1}_n c^{p+1}_n$, $r < n$. Suppose also that $c^{p+1}_n$, $\cdots$, $c^{p+1}_m$, $m \geq n$, have been found such that $c^{p+1}_n = \pi^{p+1}_n c^{p+1}_n$, $1 \leq r \leq m - 1$. Since $\pi^{p+1}_n$ is a mapping "onto," there exists a chain $c^{p+1}_n$ such that $c^{p+1}_n = \pi^{p+1}_n c^{p+1}_n$. We thus obtain a chain $c^{p+1}_n = \{c^{p+1}_n\}$ of $X$ itself such that $F c^{p+1}_n = \gamma^{p+1}_n$, $1 \leq r \leq n$.

Let $U_n$ be any nucleus of $B_n$, and let $U_n$ be the nucleus of $Z^p$ consisting of the $\gamma^{p+1}_n = \{\gamma^{p+1}_n\}$ such that $\gamma^{p+1}_n \in U_n$. It is clear that $\{U_n\}$ is a base for the nuclei of $Z^p$. We have just shown that for every $n$ there is a chain $c^{p+1}_n$ such that $\gamma^{p+1}_n = F c^{p+1}_n \in U_n$. Therefore $\gamma^p$ is in $F^p$. This proves (\beta).

Referring to (17.7) we find that the mapping $\tau: Z^p \to B$ there considered induces an isomorphism $\tau: H^p \to \hat{B}^p$ in the algebraic sense. Let now $U_n$ be open in $B_n$, and let $V_n$ be the aggregate of the $\Gamma^p$ containing a $\gamma^{p+1}_n = \{\gamma^{p+1}_n\}$ with $\gamma^{p+1}_n \in U_n$, $\Gamma^p$, the aggregate of the $\Gamma^p$ containing a projective cycle with the same property. The sets $\{V_n\}$, $\{V_n\}$ form bases for $\hat{B}^p$, $H^p$. Since there is a $\gamma^{p+1}_n \in \Gamma^p$ with the same coordinate $\gamma^{p+1}_n$ as any particular $\gamma^p \in \Gamma^p$ we have $V_n = V_n$ and so $\tau$ is topological. This completes the proof of (19.1).

§5. APPLICATION TO INFINITE COMPLEXES

20. Here as in (III, §8) significant results are only obtained when the complex $X = \{x\}$ is star- or closure-finite.

Suppose first $X$ star-finite and let $\{X_\lambda\}$ be the finite open subcomplexes of $X$. We order $\{\lambda\}$ by the inclusion of the $X_\lambda$, and so $\{\lambda; >\}$ is directed. Denote by $\pi^{\lambda}_n$ the projection $X_\lambda \to X_n$, $\lambda > \mu$, and by $\pi^{\mu}_n$ its dual the injection $X^{\mu}_n \to X^\lambda_n$ (IV, 10.13). The verification of the conditions $N_i$, $N^*_i$ of (2) is elementary and since the projections are unique $\Sigma = \{X_\lambda; \pi^{\lambda}_n\}$ is a spectrum.

Let $G$ be a topological group and $H$ a discrete group. Let $\mathcal{C}(X, G)$, $\mathcal{C}^p(X, G)$ be the groups of the infinite chains of $X$ and of the chains of $X_\lambda$ (necessarily finite) over $G$, both topologized. Let $\mathcal{C}^p_p(X, H)$, $\mathcal{C}_p(X, H)$ be the discrete groups of the finite cochains of $X$ and of the cochains of $X_\lambda$ (necessarily finite) over $H$. From the definition of chain-groups (II, 8) we infer at once that $\mathcal{C}(X, G)$ is the limit of the inverse system $\{\mathcal{C}^p(X, G), \pi^{\lambda}_n\}$ and $\mathcal{C}^p_p(X, H)$ the limit of the direct system $\{\mathcal{C}_p(X, H), \pi^{\lambda}_n\}$. Simpler results are obtained when $X$ is closure-finite by passing to $X^*$.
(20.1) When $X$ is star-finite [closure-finite] the collection of its finite open [closed] subcomplexes generates a spectrum [cospectrum] $\Sigma$ whose projective groups of the cycles [cocycles] over $G$ and of the cocycles [cycles] over a discrete $H$ are the same as the groups of the infinite cycles [cocycles] of the complex $X$ itself over $G$ and of the finite cocycles [cycles] of $X$ over $H$.

(20.2) When $X$ is simple and star-finite [closure-finite] the available intersections between classes of infinite cycles [cocycles] and finite cocycles [cycles] as defined in (V, §2) are readily identified with those obtained from the spectrum [cospectrum].

(20.3) If we combine (6.5) and (17.2) with (20.1) we obtain a second proof of the duality theorem (III, 41.2) for infinite complexes.

(20.4) From (17.3) we deduce immediately that in a star-finite complex and when $G$ is compact or a field then $\hat{\Delta} = \Delta$ and hence $\hat{\Sigma} = \Sigma / \Delta$.

(20.5) When $G$ is a division-closure group and $X$ is star-finite, the net groups $\hat{\Delta}(\Sigma, G)$ of the spectrum $\Sigma$ and the groups $\hat{\Delta}(X, G)$ of the infinite cycles of $X$ over $G$ are isomorphic (Steenrod [a]).

Since the groups of $X$ are the direct sums of those of its components, we may replace $X$ by any component and so assume it countable. This being the case $\Sigma = \{X_i\}$ is then countable. Let $\Sigma'$ be the corresponding elementary spectrum constructed in the proof of (18.2) and with the same homology theory. All the projections of $\Sigma$ existing between the complexes of $\Sigma'$ are now projections of $\Sigma'$ also, since $\Sigma$ is a spectrum. Therefore $\Sigma'$ is cofinal in $\Sigma$. As far as (20.5) is concerned $\Sigma'$ may clearly replace $\Sigma$. Since the projections in $\Sigma'$ are all "mappings onto," (20.5) becomes a consequence of (19.1).

(20.6) The dual of (20.5) obtained by passing to $X^*$, is readily stated and left to the reader.

§6. WEBS

21. In the sequel we shall require a weaker analogue of the lattice, the web which arises out of the relations of inclusion between sets, complexes, or nets.

(21.1) Web of sets. Let $\mathcal{A}$ be a topological space. Changing slightly our standard notations we designate its open sets by $A$, and its closed sets by $B$, with complementary indices, as $A_1, B'$, etc.

An open web of $\mathcal{A}$ is a collection $\mathcal{A} = \{A_1\}$ of open sets such that given $A_1, A_2$ there exist $A_1, A_2$ such that $A_1 \subset A_2, A_2$ and $A_2 \supset A_1, A_1$. In other words (in an obvious sense) both $\{A_1 \subset C\}$ and $\{A_1 \supset D\}$ are directed. A closed web $\mathcal{B} = \{B_1\}$ is defined in the same way with $B$ replacing $A$ throughout.

If $\mathcal{A} = \{A_1\}$ is an open web and $B_1 = \mathcal{A} - A_1$ then $\mathcal{B} = \{B_1\}$ is a closed web. Each of $\mathcal{A}, \mathcal{B}$ is said to be the complement of the other.

A partial web $\mathcal{A}'$ or $\mathcal{B}'$ of $\mathcal{A}$ or $\mathcal{B}$ (partial system in the sense of (I, 40)) is one whose sets make up a subcollection of those of $\mathcal{A}$ or $\mathcal{B}$. We say that $\mathcal{A}' = \{A'_1\}$ is cofinal [coinitial] in $\mathcal{A}$ if every $A_1$ is contained in [contains] an $A'_1$. If both occur then $\mathcal{A}'$ is said to be coterminal with $\mathcal{A}$. Similarly with $\mathcal{B}$ and $\mathcal{B}'$. 
(21.2) Web of complexes. Replace ℜ by a complex K and subsets by subcomplexes of K. There result, by manifest analogy, the open and closed webs of subcomplexes of K, or merely webs of complexes, and everything said in (21.1) carries over to this case.

(21.3) Web of nets. Let $X = \{X_\lambda ; \pi^\lambda\}$ be a net. In (12.5) there have been defined the relations of inclusion between the open subnets, those between the closed subnets, as well as complementation for open or closed subnets. Therefore we may paraphrase the definitions of (21.1) and introduce webs of open subnets or of closed subnets of X, (more simply called open or closed webs of nets). Explicitly an open web of nets is a collection $\mathfrak{N} = \{A_\lambda\}$ of open subnets of X such that given $A_\lambda, A_\mu$ there is an $A_\sigma \subseteq A_\lambda, A_\mu$ as well as an $A_\sigma \supseteq A_\lambda, A_\mu$ and similarly for closed webs. Complementary webs and the other terms are defined as in (21.1).

(21.4) Direct and inverse webs. A web $\mathfrak{N} = \{A_\lambda\}$ of any one of our types is said to be direct if $\{\lambda\}$ is ordered by the inclusions of the $A_\lambda: \lambda \succ \mu \iff A_\mu \supseteq A_\lambda$, and to be inverse if $\{\lambda\}$ is ordered in the opposite way.

(21.5) Ideal elements. These structures chiefly designed for a closer analysis of the behavior of the cycles of a complex "at infinity" were already considered in [L, 295]. Their description in terms of webs is very simple. An open ideal element in a set or complex is an open web $\mathfrak{N} = \{A_\lambda\}$ such that $\cap A_\lambda = \emptyset$, $\cup A_\lambda = \mathfrak{R}$ or K as the case may be. Similarly for a closed ideal element. If $\mathfrak{N}, \mathfrak{S}$ are complementary and one of them is an ideal element so is the other. The associated homology groups fall under the general category of the groups of webs to be taken up presently.

22. Homology theories. As we shall see, the direct and inverse webs resulting from a given web of complexes or nets generally give rise to two distinct homology theories.

(22.1) An important role is played by certain sets of homomorphisms which we shall now examine. Suppose first that we are dealing with subnets of a given net X. Let $A, A'$ be open subnets of X and let $B = X - A, B' = X - A'$. If $B' \subseteq B$ then $A \subseteq A'$. As we have seen (12.5) A is an open subnet of $A'$ and $B'$ a closed subnet of $B$. Referring then to (13.9) we have the following operations:

- a projection $\pi: A' \to A$ or reduction mod $(A' - A)$ of the cycles of $A'$; this is the same as a reduction mod $B$ of the cycles of $X \mod B'$;
- an injection $\eta: B' \to B$ or mapping into themselves of the groups of the cycles of $B'$.

The duals are

- an injection $\pi^*: A^* \to A'^*$,
- a projection $\eta^*: B^* \to B'^*$,

with their obvious interpretations in terms of the cocycles.

If we are dealing with subcomplexes of a given complex the same operations are to be understood in the sense of (III, 23). This common terminology will enable us to consider together the groups of webs of complexes and nets.
(22.2) Let $\mathcal{A} = \{A_\lambda\}, \mathcal{B} = \{B_\lambda\}$ be complementary webs of subcomplexes of a fixed complex $K$ or of subsets of a given net $X$. We suppose $\mathcal{A}$ open and $\mathcal{B}$ closed. A fixed coefficient group $G$ is chosen and will not be indicated in the notations for the groups of cycles. The groups $\mathcal{Z}(B_\lambda), \cdots, \mathcal{Z}(A_\lambda), \cdots$ are groups of absolute cycles and cocycles, while $\mathcal{Z}(A_\lambda), \cdots$ are groups of cycles mod $B_\lambda$, and $\mathcal{Z}(B_\lambda), \cdots$ are groups of cocycles mod $A_\lambda$. We will write:

$$\pi^\lambda_\mu = \text{a projection} \quad \psi^\lambda_\mu = \text{an injection} \quad A_\lambda \rightarrow A_\mu \text{ or } B_\lambda \rightarrow B_\mu.$$

(22.3) Consider first the direct web $\mathcal{A}(\lambda > \mu \leftrightarrow A_\lambda \supset A_\mu)$. We have then (22.1) a projection $\pi^\lambda_\mu : A_\lambda \rightarrow A_\mu$ and its dual operation an injection $\pi^{\mu *}_\lambda : A^{\mu *}_\lambda \rightarrow A^{\mu *}_\lambda$. Suppose for the present the group of the cycles of the $A_\lambda$ topologized and those of the cocycles discrete. Referring to (13.9) the system of the cycles and cocycles of the $A_\lambda$ under consideration constitute two dual categories with the following properties analogous to $N_i, N^*_i$ of (2.1):

HN1. If $\lambda > \mu$ there exists a unique simultaneous homomorphism $\pi^\lambda_\mu : \mathcal{Z}(A_\lambda) \rightarrow \mathcal{Z}(A_\mu)$, and $\pi^\lambda_\mu \mathcal{Z}(A_\lambda) \subset \mathcal{Z}(A_\mu)$.

HN2. $\lambda > \mu > \nu \rightarrow \pi^\lambda_\nu = \pi^\lambda_\mu \pi^{\nu *}_\mu$.

HN*12 the same as HN12 with $\pi^{\mu *}_\lambda$ instead of $\pi^\lambda_\mu$, cocycles in place of cycles and $>$ replaced by $<$.

HN3. The relation (6.1) for permanence of the index holds.

It may happen also that in addition we have:

HN4. The dual categories under consideration possess intersections with the properties of (8) and in particular the permanence relations (8.1, 8.2) hold.

(22.4) Definition. A system of dual categories with the properties HN123 and HN*12 will be called an $H$-net (H abridged for homology). When in addition HN4 holds, the $H$-net is said to be with intersection. If the homomorphisms proceed the other way around there is obtained an $H$-conet.

Under certain circumstances the dual categories admit only of weak duality (III, 31: only discrete groups and only field duality). We will then say that we have a weak $H$-net or $H$-conet as the case may be.

We observe now that HN123, HN*12 are the only properties required in developing the theory of cycles, cocycles, homology groups, duality theorems in (3, 4, 6). Moreover the only supplementary property required for intersections is HN4. We have therefore:

(22.5) There may be defined cycles, cocycles and the groups $\mathcal{Z}, \mathcal{Z}', \mathcal{D}$ for $H$-nets or $H$-conets in the same way as for nets and conets and they have the properties (3.4, $\cdots$, 3.7, 4.2, 4.5, 6.1, $\cdots$, 6.5). Furthermore in the $H$-net ($H$-conet) the resulting cycles and cocycles [cocycles and cycles] form dual categories and they have intersections when HN4 holds. In the weak $H$-net ($H$-conet) the same statement holds but with weak dual categories.
(22.5a) SUPPLEMENTARY REMARK. Since projections are unique, the true analogy is with spectra, and so there may be introduced the same projective groups C, Z, F, H as in (§4). We will not stop, however, to compare them with the groups of the H-nets or cones.

(22.6) Returning to the direct web # of (22.3) the groups of cycles and cocycles of the A, with the projections ηΛ and injections ηΛ' make up an H-net which we designate by {A, ηΛ}. The resulting limit-groups are called groups of the direct web # over G, written H(A, G), · · · . In particular if ηΛ is the simultaneous homomorphism in the groups H(A) induced by ηΛ, then {H(A); ηΛ} is an inverse system whose limit-group H(A, G) is the pth homology group of the direct web # over G. Similarly for the cohomology group H*(A, G) (G discrete).

(22.7) Suppose more particularly that # is a web of infinite open subcomplexes of K and that the AΛ are the groups of finite cycles of K mod B. Then these groups must be taken discrete and {AΛ; ηΛ} is a weak H-net. Our notation is not well designed to separate all the numerous possibilities, but the particular case under consideration will generally be clear from the context.

(22.8) If # is taken inverse the situation is essentially the same except that the H-net is replaced by an H-conet, with injections for the cycles, projections for the cocycles, and groups H*(#, G), · · · . Finally # presents the same two possibilities with cycles and cocycles, projections and injections interchanged. The following table summarizes the situation.

(22.9)

<table>
<thead>
<tr>
<th>Web</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. # direct</td>
<td>{AΛ; ηΛ}</td>
<td>H-net</td>
</tr>
<tr>
<td>II. # inverse</td>
<td>{AΛ; ηΛ}</td>
<td>Weak H-conet</td>
</tr>
<tr>
<td>III. # direct</td>
<td>{BΛ; ηΛ}</td>
<td>Weak H-conet</td>
</tr>
<tr>
<td>IV. # inverse</td>
<td>{BΛ; ηΛ}</td>
<td>H-net</td>
</tr>
</tbody>
</table>

(22.10) If a web # is cofinal [coinitial] in a web # then they have the same direct [inverse] groups (11, 13.3, 14.5).

(22.11) The Betti numbers of the groups of the table over a given field depend solely upon the characteristic of the field.

Since this is true for nets or complexes the proof of (7.2) applies here also.

(22.12) Duality. The table describes in substance eight dual categories corresponding to each of the H-nets there occurring.

EXAMPLES (22.13) Let K be a locally finite complex and # the open ideal element consisting of the finite open subcomplexes of K including Θ. Taking # direct we have Type 1a of the table and the resulting H-net is merely the spectrum of (20) corresponding to the infinite cycles with topologized groups and finite cocycles with discrete groups.
(22.15) We have already defined (21.1) a partial web \( \mathfrak{W} \) of \( \mathfrak{A} \). If \( \mathfrak{A} \) is any \( H \)-net associated with \( \mathfrak{A} \), then the analogue \( \mathfrak{W}_1 \) for \( \mathfrak{A}_1 \) is a partial \( H \)-net of \( \mathfrak{A} \) in the sense of (9), and it is readily shown that the arguments of (9) carry over to \( \mathfrak{W}, \mathfrak{W}_1 \).

23. Under certain conditions groups of a web of subnets may be replaced by those of a single net. The resulting properties have interesting topological applications (VII, 14, 15; VIII, 13.4).

(23.1) Let \( X = \{ \lambda; x_{\lambda}^x \} \) be our customary net and \( \mathfrak{B} \) a closed web of subnets of \( X \). It will be more in keeping with the prevailing notations of the chapter to write \( \mathfrak{B} = \{ X_{\alpha} \}, X_{\alpha} = \{ X_{\alpha\beta}; x_{\alpha}^x \} \). We denote by \( \mathfrak{A} \) the complement of \( \mathfrak{B} \).

Let \( Y_\lambda = \bigcap_{\alpha} X_{\alpha\beta} \). Since \( \lambda > \mu \rightarrow \pi_{\alpha}^x X_{\alpha\beta} \subset X_{\mu\beta} \), we also have \( \pi_{\alpha}^x Y_\lambda \subset Y_\mu \).

Therefore \( Y = \{ Y_\lambda; \pi_{\alpha}^x \} \) is a closed subnet of \( X \).

The two basic properties which we wish to prove are:

(23.2) The homology theory of the web \( \mathfrak{B} \) taken inverse is the same as that of the closed subnet \( Y \).

(23.3) The homology theory of the web \( \mathfrak{A} \) taken direct is the same as that of the dual categories of the cycles of \( X \) mod \( Y \) and cocycles of the open subnet \( X \setminus Y \).

As usual we denote cycles and cocycles by \( \gamma, \delta \) and their classes by \( \Gamma, \Delta \) affected with the same indices. The \( H \)-nets or comets defined by the inverse and direct webs will be denoted by \( \mathfrak{A}_i, \mathfrak{A}_d, \ldots \).

**Proof of (23.2).** It is clear that \( \mathfrak{B} = \mathfrak{B} \cup Y \) is likewise a closed web. Since \( \mathfrak{B} \) has \( Y \) as initial element the theory of \( Y \) and the theory of \( \mathfrak{B}, \mathfrak{A} \) are the same. Therefore (23.2) reduces to proving:

(23.4) \( \mathfrak{B} \) and \( \mathfrak{A} \) have the same homology theory.

Every cycle of \( \mathfrak{B} \) is of the form \( \gamma' = \gamma'' \cup \delta'' \), where \( \gamma'' \) is a cycle of \( \mathfrak{B} \) and \( \delta'' \) a cycle of \( Y \). The cycle \( \gamma'' \) is merely the set of coordinates of \( \gamma'' \) in the elements of \( \mathfrak{B} \), and \( \delta'' \) is its coordinate in \( Y \). It is clear that \( \gamma'' \rightarrow \gamma'' \) defines a simultaneous homomorphism \( \tau \) of the groups of cycles of \( \mathfrak{B} \) into those of \( \mathfrak{A} \). In its turn \( \tau \) induces a simultaneous homomorphism \( \tilde{\tau} \) in the homology groups (the same as \( \tau \) of (9) for the \( H \)-net \( \mathfrak{B} \) and its partial net \( \mathfrak{B} \)). As already observed we must show that:

(23.5) \( \tilde{\tau} \) is an isomorphism.

Since \( X_\lambda \) is finite it has at most a finite number of subcomplexes \( X_{\alpha\beta} \) and so there is a smallest \( X_{a_\lambda} \). Given any \( X_{\alpha\beta} \) there is an \( X_{\beta \lambda} \subset X_{\alpha\beta}, X_{a_\lambda} \). Hence \( X_{a_\lambda} = X_{a_\lambda} \subset X_{\alpha\lambda} \), which implies \( X_{a_\lambda} = \bigcap X_{a_\lambda} = Y_\lambda \).

We now come to the cycles. Since we have the last case in the table (22.9)
the basic operations in $\mathcal{B}_i$ are injections $\eta^\alpha_\beta$, $\alpha > \beta$. In $\mathcal{B}_i$, there are in addition the injections $\eta^\alpha_\beta: Y \to X_\alpha$.

Now a cycle of $X_\alpha$ is a collection $\gamma_\alpha^\alpha = \{\gamma_\alpha^\lambda\}$, where $\gamma_\alpha^\lambda$ is a cycle of $X_\lambda$ and $\lambda > \mu \to \pi^\lambda_\mu \gamma_\alpha^\lambda \sim \gamma_\alpha^\mu$ in $X_\alpha$. Hence a cycle of $\mathcal{B}_i$ is a collection $\gamma^\alpha = \{\gamma_\alpha^\lambda\}$, where $\lambda > \beta \to \eta^\beta_\alpha \gamma_\alpha^\beta \sim \gamma_\beta^\alpha$ in $X_\beta$. From the definition of homologous cycles in a net (here $X_\beta$) there comes: $\gamma_\alpha^\beta \sim \gamma_\beta^\alpha$ in $X_\alpha$. In particular if $\beta = \alpha_0$ then $X_{\alpha_0} = Y_1$. Therefore $Y_1$ contains a cycle $\delta^\alpha = \gamma_\alpha^\alpha$ such that if $\alpha, \beta$ are replaced by $\alpha_0, \alpha$ then

$$\delta^\alpha \sim \gamma_\alpha^\alpha \text{ in } X_{\alpha_0}.$$ (23.6)

Let $\lambda > \mu$ and denote by $\alpha_0$ the analogue of $\alpha_0$ for $\mu$. Since $\gamma_\alpha^\alpha$ is a cycle of the closed subnet $X_{\alpha_1}$, (23.6) yields:

$$\pi^\alpha_\beta \delta^\alpha \sim \pi^\lambda_\mu \gamma_\alpha^\lambda \sim \gamma_\alpha^\mu = \delta^\alpha \text{ in } X_{\alpha_1} = Y_\mu.$$ Therefore $\delta^\alpha = \{\delta^\alpha\}$ is a cycle of $Y$. By (23.6) $\delta^\alpha \sim \gamma^\alpha$ in $X_\alpha$, which may also be written $\eta_\alpha \delta^\alpha \sim \gamma^\alpha$ in $X_\alpha$. Hence $\tau^\alpha = \gamma^\alpha \cup \delta^\alpha$ is a cycle of $\mathcal{B}_i$, such that $\tau^\alpha \gamma^\alpha = \gamma^\alpha$. Or explicitly:

$$23.7 \text{ Corresponding to every cycle } \gamma^\alpha = \{\gamma_\alpha^\lambda\} \text{ of } \mathcal{B} \text{ inverse there is a cycle } \delta^\alpha \text{ of } Y \text{ such that } \tau^\alpha = \gamma^\alpha \cup \delta^\alpha \text{ is a cycle of } \mathcal{B} \text{ inverse, or equivalently such that } \gamma^\alpha \sim \delta^\alpha \text{ in } X_\alpha.$$ Since $\tau^\alpha \gamma^\alpha = \gamma^\alpha$, the mapping $\tau$, hence also $\tau$, is onto.

Suppose that $\gamma^\alpha \sim 0$ in $\mathcal{B}_i \leftrightarrow \gamma^\alpha \sim 0$ in $\mathcal{B}_i \leftrightarrow \gamma^\alpha_0 \sim 0$ in $X_\alpha \leftrightarrow \gamma^\alpha_0 \sim 0$ in $X_{\alpha_0} \leftrightarrow \delta^\alpha \sim 0$ in $Y \leftrightarrow \delta^\alpha \sim 0$ in $Y \leftrightarrow \gamma^\alpha \sim 0$ in $\mathcal{B}_i$. Therefore $\tau$ is univalent.

Let $U_{\alpha_0}$ be open in $\mathcal{B}^*(X_{\alpha_0}, G)$ and set

$$V_{\alpha_0} = \{\Gamma^\alpha \mid P^\alpha_{\alpha_0} \in U_{\alpha_0}\},$$

$$\{V_{\alpha_0}\} \text{ and } \{V_{\alpha_0}\} \text{ are bases for } \mathcal{B}^*(\mathcal{B}_i, G), \mathcal{B}^*_\beta(\mathcal{B}_i, G), \text{ and since } V_{\alpha_0} = \tau V_{\alpha_0}, \tau \text{ is open. This completes the proof of (23.5).}$$

The representatives of a cocycle $\gamma^\alpha$ of $\mathcal{B}_i$ are likewise those of a cocycle $\gamma^\alpha$ of $\mathcal{B}_i$, and $\gamma^\alpha \to \gamma^\alpha$ defines a simultaneous homomorphism $\tau^\alpha$ of the groups of the cocycles of $\mathcal{B}_i$ into those of $\mathcal{B}_i$. In its turn $\tau^\alpha$ induces a simultaneous homomorphism $\tau^\alpha$ of the cohomology groups (the analogue of $\tau^\alpha$ of (9) for the two $\mathcal{H}$-nets).

(23.8) $\tau^\alpha$ is an isomorphism.

Since the cohomology groups are discrete and over a discrete group $H$ or a field $J$, and (9.4) holds for $\mathcal{B}_i$, $\mathcal{B}_i$, the same character-group argument as for (IV, 10.10) with $\mathbb{Z}, \mathcal{B}$ replaced by $H$ and its character-group or both by $J$, will yield (23.8).

Referring now to (9), property (23.4) will follow from the remaining relations of permanence. As already observed this proves (23.2).
Proof of (23.3). It is essentially obtained by "dualizing" the proof of (23.2).
If $\mathfrak{A} = \mathfrak{A} \cup (X - Y)$, then $X - Y$ is cofinal in $\mathfrak{A}$, and so the dual categories in (23.3) have the same homology theory as $\mathfrak{A}$. This reduces (23.3) to:

(23.9) $\mathfrak{A}$ and $\mathfrak{A}$ have the same homology theory.

Let $\theta, \theta^*, \bar{\theta}, \bar{\theta}^*$ be the analogues of $\tau, \cdots, \tau^*$.

(23.10) $\bar{\theta}$ is an isomorphism.

Since we have the first case of the table the basic operations in $\mathfrak{A}$ are projections $\omega^\alpha_\beta$, where $\alpha > \beta \iff (X - X_\alpha) \supset (X - X_\beta) \iff X_\alpha \subset X_\beta$. In $\mathfrak{A}$ there are in addition the projections $\omega_\alpha : X \to (X - X_\alpha)$, or reduction mod $X_\alpha$ of the cycles of $X$.

Now a cycle of $X$ mod $X_\alpha$ is a collection $\gamma^\alpha_\beta = \{\gamma^\alpha_\beta\}$, where $\gamma^\alpha_\beta$ is a cycle of $X_\lambda$ mod $X_\alpha$, and $\lambda > \mu \iff \pi^\lambda_\gamma X_\alpha \sim \gamma^\alpha_\beta$ in $X_\mu$ mod $X_\alpha$. Hence a cycle of $\mathfrak{A}$ is a collection $\gamma^\alpha = \{\gamma^\alpha_\beta\}$ where $\alpha > \beta \iff \omega^\alpha_\beta \gamma^\alpha_\beta \sim \gamma^\alpha_\beta$ in $X_\beta$ mod $X_\alpha$ for all $\alpha > \beta$. In particular choosing $\alpha = \alpha_0$ and replacing $\beta$ by $\alpha$, we have a cycle $\delta^\alpha = \gamma^\alpha_{\alpha_0}$ of $X_\lambda$ mod $Y_\lambda = (X - X_{\alpha_0})$ such that:

(23.11) $\delta^\alpha \sim \gamma^\alpha_{\alpha_0}$ in $X_\lambda$ mod $X_{\alpha_0}$.

Let now $\lambda > \mu$ and $\alpha_0, \alpha$ as before. Since $\gamma^\alpha_{\alpha_1}$ is a cycle of $X$ mod $X_{\alpha_1}$, by combining with (23.10) we find:

$$\pi^\lambda_\gamma \delta^\alpha \sim \pi^\lambda_\gamma \gamma^\alpha_{\alpha_1} \sim \gamma^\alpha_{\alpha_1} \sim \delta^\alpha \text{ in } X_\lambda \text{ mod } X_{\alpha_1} = Y_\mu.$$

Therefore $\gamma^\alpha = \{\delta^\alpha\}$ is a cycle of $X$ mod $Y$. By (23.11) again $\delta^\alpha \sim \gamma^\alpha_\beta$ in $X$ mod $X_\alpha$. Therefore $\gamma^\alpha = \gamma^\alpha_\beta \cup \delta^\alpha$ is a cycle of $\mathfrak{A}$ such that $\theta^* \gamma^\alpha = \gamma^\alpha$. Or explicitly:

(23.12) Corresponding to every cycle $\gamma^\alpha = \{\gamma^\alpha_\beta\}$ of $\mathfrak{A}$ direct there is a cycle $\delta^\alpha$ of $X$ mod $Y$ such that $\gamma^\alpha \cup \delta^\alpha$ is a cycle of $\mathfrak{A}$ direct, or equivalently such that $\gamma^\alpha \sim \delta^\alpha$ in $X$ mod $X_\alpha$.

Thus $\theta$, hence also $\bar{\theta}$, is onto.

If $\gamma^\alpha = \gamma^\alpha_\beta \cup \delta^\alpha$ is a given cycle of $\mathfrak{A}$, we have: $\theta^* \gamma^\alpha \sim 0$ in $\mathfrak{A} \iff \gamma^\alpha \sim 0$ in $\mathfrak{A} \iff \gamma^\alpha_{\alpha_0} \sim 0$ in $X_\alpha$ mod $X_{\alpha_0}$ or $X_\lambda$ mod $X_{\alpha_0} \iff \delta^\alpha \sim 0$ in $X_\lambda$ mod $Y_\lambda \iff \delta^\alpha \sim 0$ in $Y$ mod $Y_\lambda$. Thus $\bar{\theta}$ is univalent.

The proof that $\bar{\theta}$ is open is the same as for $\tau$ and (23.10) follows.

From this point on, the conclusion of the proof of (23.9) is the same as for (23.4) and so (23.3) follows.

§7. METRIC COMPLEXES

24. Frequently complexes consist of elements represented by subsets of a metric space. The metric relationship thus arising may be utilized to advantage to introduce new significant webs and related homology groups. The following definition covers all the interesting types thus arising.

(24.1) Definitions. A metric complex is a closure-finite complex $X = \{x\}$ such that there exists a real-valued function of the element $x$, called the diameter of $x$, written $\text{diam } x$, and subjected to the sole condition: $x' < x \iff \text{diam } x' \leq \text{diam } x$. 
If \( Y \) is a subcomplex of \( X \) we may now define mesh \( \text{mesh} \ Y = \sup \{ \text{diam} \ x \mid x \in Y \} \), and hence for a chain \( C \): mesh \( C = \text{mesh} \mid C \mid \).

(24.2) In the applications metric complexes will usually occur as follows: With each \( x \) there will be associated a bounded subset \( |x| \) of a certain metric space \( \Re \) such that \( x' < x \rightarrow |x'| \subset |x| \), and then \( \text{diam} \ x = \text{diam} |x| \) will turn \( X \) into a metric complex. If \( Y \) is a subcomplex of \( X \) and \( C \) a chain of \( X \), we write \( |Y| = \bigcup \{ |x| \mid x \in Y \} \) and \( \| C \| = \bigcup \{ \| C \| \} \).

(24.3) A subcomplex \( Y \) of \( X \) is made metric in the obvious way by assigning to its elements the same diameters as in \( X \).

The subcomplex \( Y \) is said to be essential if there is an \( \epsilon > 0 \) such that \( \text{diam} \ x < \epsilon \rightarrow x \in Y \).

(24.4) Examples. Geometric and Euclidean complexes (VIII, §1), Vietoris complexes (VII, §5), singular complexes (VIII, 24) are important types of metric complexes.

25. \( V \)-cycles.

(25.1) An obvious web related to the metric complex \( X \) is readily defined. Given any \( \epsilon > 0 \), set \( B_\epsilon = \{ x \mid \text{diam} \ x < \epsilon \} \). In view of (24.1) \( B_\epsilon \) is a closed subcomplex of \( X \). If \( \epsilon' \leq \epsilon \) then

\[
B_\epsilon \cup B_{\epsilon'} = B_\epsilon, \quad B_\epsilon \cap B_{\epsilon'} = B_{\epsilon'},
\]

and so \( \mathfrak{B} = \{ B_\epsilon \} \) is a closed web of complexes. The only interesting ordering is evidently \( \{ \epsilon; < \} \). Furthermore since \( X \) and hence the \( B_\epsilon \), are merely closure-finite, only finite (absolute) cycles are admissible for them. Therefore the appropriate web homology theory is that of the inverse closed web and IVb of the table (22.9). The cycles \( \cdots \) are known as \( V \)-cycles, \( \cdots \). The “\( V \)” is abridged for “Vietoris,” as the prototype of this homology theory is the classical Vietoris theory for compacta (Vietoris [a]; VII, §5). Since we are dealing here with a weak \( H \)-net only discrete coefficient groups are admissible. There are two different approaches to the \( V \)-cycles, each with its advantages, and so both will be given here. The resulting homology groups are of course the same.

(25.2) First definition. Let \( \{ \epsilon_n \} \) be a positive sequence tending to 0. Since it is cofinal in \( \{ \epsilon \} \), it may replace it in the definition of the groups under consideration. A \( V \)-cycle \( \gamma \) over a discrete \( G \) will then be a collection \( \gamma = \{ \gamma_n \} \), where \( \gamma_n \) is a finite cycle of \( B_{\epsilon_n} \) and \( n \geq m \rightarrow \gamma_n \sim \gamma_m \) in \( B_{\epsilon_m} \). The cycle \( \gamma \sim 0 \) whenever \( \gamma_n \sim 0 \) in \( B_{\epsilon_n} \) for every \( n \). The operations on the cycles are defined in the natural way, and we have the usual groups of the cycles and bounding cycles written \( \mathfrak{N}_r(X, G), \mathfrak{\Gamma}_r(X, G) \). Their topology is defined as follows: The cycles \( \gamma \) with a given coordinate \( \gamma_n \) form an open set \( U_n \) in \( \mathfrak{N}_r(X, G) \) and \( \{ U_n \} \) is a base for the group. \( \mathfrak{\Gamma}_r(X, G) \) receives the relative topology. Since \( \sim 0 \) \( \Leftrightarrow \) “bounding” for the finite cycles in a complex we show as for (2.8) that \( \mathfrak{\Gamma}_r(X, G) \) is closed in
\(\mathcal{B}_p^\epsilon(X, G)\) and so we define \(\mathcal{S}_p^\epsilon(X, G) = \mathcal{B}_p^\epsilon(X, G) / \mathcal{F}_p^\epsilon(X, G)\). From the general theory it is known that \(\mathcal{S}_p^\epsilon(X, G)\) remains isomorphic with itself if \(\{\epsilon_n\}\) is replaced by any other similar sequence.

(25.3) Second definition. Homologies between finite chains in \(B\), are conveniently denoted by \(\sim\), and known as \(\epsilon\) homologies. Thus \(C^p \sim C'^p = FC^{p+1}\) where all the chains are finite and of mesh less than \(\epsilon\).

A p-dimensional V-cycle over a discrete group \(G\) is a countable collection of finite cycles, \(\gamma^p = \{\gamma^p_z\}\) such that

(a) mesh \(\gamma^p_z \to 0\);
(b) \(\gamma^p_{z+1} \sim_x \gamma^p_z\), where \(\{\epsilon_n\} \to 0\). The cycle \(\gamma^p\) bounds, or is a bounding cycle, written \(\gamma^p \sim 0\), whenever
(c) \(\gamma^p \sim_x 0\), where \(\{\epsilon_n\} \to 0\).

The group of the V-cycles \(\mathcal{B}_p^\epsilon(X, G)\) is obtained by defining \(\{\gamma^p_z\} \pm \{\gamma'^p_z\} = \{\gamma^p_z \pm \gamma'^p_z\}, 0 = \{\gamma^p_z \mid \gamma^p_z = 0\}\), and is taken discrete. The bounding V-cycles form a subgroup \(\mathcal{B}_p^\epsilon(X, G)\). The homology group is defined as \(\mathcal{S}_p^\epsilon(X, G) = \mathcal{B}_p^\epsilon(X, G) / \mathcal{F}_p^\epsilon(X, G)\).

Equivalent formulations, frequently useful, are: (a) as before and (b), (c) replaced by:

(b') for every \(\epsilon\) there is an \(m\) such that \(n, n' \geq m \rightarrow \gamma^p_z \sim_x \gamma^p_z\);
(c') for every \(\epsilon\) there is an \(m\) such that \(n \geq m \rightarrow \gamma^p_z \sim_x 0\);

or else also by:

(b'') there is a sequence of finite chains \(\{C^p_{n+1}\}\) such that
\[FC^{p+1}_n = \gamma^p_{n+1} - \gamma^p_n, \text{ mesh } C^{p+1}_n \to 0\];
(c'') there is a sequence of finite chains \(\{C'^{p+1}_n\}\) such that
\[FC'^{p+1}_n = \gamma^p_n, \text{ mesh } C'^{p+1}_n \to 0\].

Under the definition just given we have:

(25.4) If \(\gamma^p = \{\gamma^p_z\}\) is a V-cycle and \(\gamma'^p = \{\gamma'^p_z\}\) is a set of finite cycles such that \(\gamma^p \sim_x \gamma'^p\), where \(\{\epsilon_n\} \to 0\), then \(\gamma'^p\) is also a V-cycle and \(\gamma'^p \sim \gamma^p\).

From (25.4) follows readily (still under the second definition):

(25.5) If \(\gamma^p = \{\gamma^p_z\}\) is a V-cycle then \(\gamma'^p = \{\gamma'^p_z\}\) is likewise a V-cycle and \(\gamma'^p \sim \gamma^p\).

26. (26.1) We shall now compare the two definitions. For convenience let the cycles, classes and homology groups under the first definition be denoted by \(\gamma, \Gamma, \mathcal{S}\) and those under the second by \(\gamma, \Gamma, \mathcal{S}\). It is not difficult to prove the following properties: (a) all \(\gamma^p \in \Gamma^p\) are also elements of a fixed \(\Gamma^p\) and \(\Gamma^p \to \Gamma_\theta^p\) defines a homomorphism in the algebraic sense \(\tau: \mathcal{S}_\theta^p \to \mathcal{S}_\theta^p\); (b) each \(\gamma^p = \{\gamma^p_z\} \in \Gamma^p\) has a subsequence \(\{\gamma^p_z\}\) which is a \(\gamma^p\) in a fixed \(\Gamma^p\) and \(\Gamma^p \to \Gamma_\theta^p\) defines a homomorphism in the algebraic sense \(\tau: \mathcal{S}_\theta^p \to \mathcal{S}_\theta^p\); (c) \(\theta \tau = 1, \tau \theta = 1\). Hence \(\tau\) is an isomorphism in the algebraic sense. Thus the two homology groups differ only in their topology. We shall refer to \(\mathcal{S}_\tau^p\) as the homology group of the V-cycles, and to \(\mathcal{S}_\tau^p\) as the homology group of the V-cycles with topology. For practically all purposes \(\mathcal{S}_\tau^p\) is quite sufficient.
(26.2) **V-cycles around Y.** Let us suppose that $X$ is metrized as indicated in (24.2) so that we have spheroids $\mathcal{S}(|Y|, \epsilon)$ and let $Y$ be a closed subcomplex of $X$. Let the notations be those of (25.3b'c''), except that we impose the following supplementary requirement: given any $\epsilon > 0$, then all but a finite number of $\gamma_n^\ast, \gamma_n^{\ast+1}$ are in $\mathcal{S}(\{Y\}, \epsilon)$. This may also be expressed as: \{$\gamma_n^\ast$, \{$\gamma_n^{\ast+1}$ \} $\rightarrow Y$. Otherwise everything is as before. The new V-cycles are said to be around $Y$. The web interpretation is the same as before except that this time $B_\ast = \{x | \text{diam } x < \epsilon, x \subset \mathcal{S}(\{Y\}, \epsilon)\}$.

(26.3) **V-cocycles.** They must be determined in terms of the general theory of H-nets. A V-cocycle is thus defined by a cocycle in some $B_\ast \text{ mod } (X - B_\ast)$ (where $B_\ast$ is as in 25.1), i.e., it is a cochain $\gamma_\ast$ such that $F\gamma_\ast$ has no elements of diameter less than $\epsilon$. That is to say $\gamma_\ast$ is any cochain such that \[\inf \{\text{diam } x^{\ast+1}, \text{diam } x^{\ast+1} \in F\gamma_\ast\} > 0.\] The relation $\gamma_\ast \sim 0$ means that $\gamma_\ast = F\gamma_\ast + D_\ast$, where $D_\ast$ is in some $X - B_\ast$, or again that \[\inf \{\text{diam } x^{\ast}, \text{diam } x^{\ast} \in D_\ast\} > 0.\] The groups $\mathcal{G}_\ast, \mathcal{B}_\ast, \mathcal{B}_\ast$ are discrete and the cohomology group $\mathcal{G}_\ast(X, G) = \mathcal{B}_\ast/\mathcal{B}_\ast$. Cocycles may also be introduced for the other types considered but we will not discuss them here.

27. **(27.1) DEFINITION.** Let $X = \{x\}, X_1 = \{x_1\}$ be metric complexes and $t$ a set-transformation $X \rightarrow X_1$. Then $t$ is said to be metric if $\text{diam } t x \rightarrow 0$ uniformly with $x$. An isomorphism $t:X \rightarrow X_1$ is said to be metric if both $t$ and $t^{-1}$ are metric. If such a $t$ exists we shall say that $X$ and $X_1$ are metrically isomorphic. Similarly a chain-mapping $r:X \rightarrow X_1$ is said to be metric if $\text{diam } \|rx\| \rightarrow 0$ uniformly with $x$.

From the definition of metric isomorphism there follows readily:

(27.2) A metric isomorphism $r:X \rightarrow X_1$ induces an isomorphism of the $\mathcal{G}_\ast, \mathcal{B}_\ast, \mathcal{B}_\ast$ groups of $X$ with the corresponding groups of $X_1$.

(27.3) **APPLICATION.** Let $X$ be made a metric complex in two ways with two distinct functions $\text{diam } x, \text{diam } x_1$. Then if each approaches 0 uniformly whenever the other approaches 0, the two metrics define the same groups $\mathcal{G}_\ast, \mathcal{B}_\ast, \mathcal{B}_\ast$.

(27.4) A metric chain-mapping $r:X \rightarrow X_1$ induces homomorphisms of the groups $\mathcal{G}_\ast, \mathcal{B}_\ast, \mathcal{B}_\ast$ of $X$ into the same for $X_1$.

By (IV, 9) $r$ induces homomorphisms of the groups of finite chains $\mathcal{C}_\ast, \mathcal{B}_\ast, \mathcal{B}_\ast$ of $X$ into the same for $X_1$. Hence if $|C_n|$ are finite chains such that mesh $C_n \rightarrow 0$, then the chains $\{rC_n\}$ have the same property. Coupling this with (25.3) there is but a step to (27.4).

(27.5) Let $r_1, r_2$ be metric chain-mappings $X \rightarrow X_1$ which are chain-homotopic, the associated $\mathcal{D}$ operator being of the f-type (every $\mathcal{D} x$ finite). If mesh $\mathcal{D} x \rightarrow 0$ uniformly with $x$ then $r_1, r_2$ are said to be metrically chain-homotopic. When $r_1 = 1$ then $r_2$ is said to be a metric chain-deformation.

(27.6) If $X_1$ is a chain-retract [chain-deformation retract] of $X$ under a chain-retraction [chain-deformation retraction] which is metric, then $X_1$ is naturally called a metric chain-retract [metric chain-deformation retract] of $X$. 

The general properties of chain-homotopy, chain-deformation, chain-retraction, derived in (IV, 14–16) hold here also and with unimportant modifications in the proofs. We note in particular the following:

(27.7) Metric chain-homotopy is an equivalence (IV, 15.1).

(27.8) Metric chain-homotopic mappings $\tau_1, \tau_2 : X \rightarrow X_1$ induce the same homomorphisms in the $V$-homology groups. A metric chain-deformation does not alter the homology groups of the $V$-cycles (IV, 15.2).

This means in particular that if $\gamma^p$ is a $V$-cycle of $X$ then $\tau_1 \gamma^p \sim \tau_2 \gamma^p$. Also that if $\tau_2$ is a chain-deformation, then $\tau_1 \gamma^p \sim \gamma^p$ in $X_1$.

(27.9) If $X$ is a metric chain-deformation retract of $X_1$, then their $V$-homology groups are the same (IV, 15.12).