CHAPTER VIII
TOPOLOGY OF POLYHEDRA AND RELATED QUESTIONS

After some further preliminary properties the results of (VII) will first be
applied to the homology theories associated with polyhedra. From the topo-
logical standpoint a polyhedron may as well be replaced by a simplicial partition.
Unless otherwise stated therefore all polyhedral complexes under consideration
will be simplicial, i.e., they will be Euclidean complexes. In addition to the
general type we shall also discuss geometric manifolds and their special inter-
section properties. Closely related topics taken up are: continuous and singular
complexes, topological complexes, a rapid survey of differentiable manifolds
(Whitney's results and the related work of Cairns and Whitehead). The chapter
contains also a treatment of coincidences and fixed points for finite polyhedra
and for a general class of spaces which have been named “quasi-complexes.”

General references: Alexandroff-Hopf [A-H], Cairns [a, b], Hodge [H]; Lef-
schetz [L, L1, b, g], Reidemeister [R, R1], Seifert-Threlfall [S-T]; Veblen [V],
Whitehead [b], Whitney [a, b].

§1. GEOMETRIC COMPLEMENTS

1. (1.1) Notations. If $K = \{\sigma\}$ is an Euclidean complex we shall write
$\sigma(x) =$ the simplex of $K$ containing the point $x$;
$'\sigma =$ the centroid of $\sigma$;
$K^{(n)} =$ the $n$th barycentric derived of $K$.

These are meant to be "typical" designations. Thus if $L = \{\xi\}$ then $'\xi$ is
the centroid of $\xi$, etc.

(1.2) As on previous occasions (III, 6.12, 6.14) $|K|$ is a subset of an Euclidean
space $E^n$ or of the Hilbert parallelootope $P^n$. We consider here also $E^n$ as a
linear variety of a certain real vector space $\mathcal{B}$ and $P^n$ as a convex subset of
such a variety. Thus the points of $|K|$ are vectors of $\mathcal{B}$ and they will be dealt
with accordingly.

(1.3) Definition. Mesh $K = \sup \{\text{diam } \sigma \mid \sigma \in K\}$.

(1.4) In (III, 6.12) the antecedent of $K$ has been defined as the simplicial
complex $\mathcal{R} \cong K$, the isomorphism being the identity on the vertices of $K$. We
now extend the term antecedent to cover any simplicial complex $\mathcal{R}_1 \cong \mathcal{R}$. We
also say that $K$ is an Euclidean realization of $\mathcal{R}_1$.

2. (2.1) If $\sigma = \sigma^' \sigma^'' = a_0 \ldots a_p$, $p > 0$, is an Euclidean simplex and $x \in \sigma$,
there passes through $x$ a unique segment $\overline{x'x''}$, where $x' \in \sigma^'$ and $x'' \in \sigma^''$. (We
assume of course $\sigma', \sigma'' \neq \sigma$.)
We may assume \( \sigma' = a_0 \cdots a_q \), \( q < p \). If \( x \) has the barycentric coordinates \( \{ x^0, \ldots, x^p \} \) the following numbers are uniquely defined:

\[
\begin{align*}
  t' &= \sum x^i, \quad i \leq q; \\
  t'' &= \sum x'^{i+1}; \\
  x''^i &= \frac{x^i}{t'}, \quad i \leq q; \quad x'^{i+1} = 0; \\
  x'' &= 0, \quad i \leq q; \quad x'^{i+1} = \frac{x^{i+1}}{t''}.
\end{align*}
\]

Moreover \( \{ x''^i \}, \{ x'^{i+1} \} \) are the barycentric coordinates of two points \( x' \in \sigma', x'' \in \sigma'' \) such that \( x = t'x' + t''x'' \). Hence \( x \in x'x'' \).

Suppose that there is a second segment \( y'y'' \) containing \( x \), where \( x' \neq y' \in \sigma' \), \( x'' \neq y'' \in \sigma'' \). As a consequence \( x = t'x' + t''x'' = s'y' + s'y'' \). Expressing \( x', x'', y', y'' \) in terms of the vertices of \( \sigma \) we find a relation

\[
u a_0 + \cdots + u a_q + \cdots = 0 \]

where \( x' = y' \) implies that at least one of \( u_0, \ldots, u_q \neq 0 \). Since the vertices are independent this is excluded, and so \( x'x'' \) is unique.

(2.2) Let \( \sigma^p = a_0 \cdots a_p \) be in a space \( \mathcal{R} \) which is either Euclidean or the Hilbert parallelootope \( \mathcal{P}^p \). Then \( d(x, y), x \in \mathcal{R}, y \in \sigma^p \), does not exceed the maximum distance \( \rho \) from \( x \) to the set of vertices.

For every \( a_i \in \mathcal{S}(x, \rho) \), and since the sphere and \( \sigma^p \) are both convex, we have \( \sigma^p \subseteq \mathcal{S}(x, \rho) \), proving (2.2).

(2.3) The diameter of \( \sigma \) is the length of its longest edge.

By (2.2) \( \text{diam} \; \sigma \) is the distance from a point of \( \sigma \) to a vertex and again by (2.2) this is at most equal to a certain \( d(a_i, a_j) \).

(2.4) Mesh \((\text{Cl} \; \sigma^p)' \leq p/(p + 1) \; \text{diam} \; \sigma^p \).

Any one-simplex of \((\text{Cl} \; \sigma^p)'\) is of the form \( \sigma' \cap \sigma'' \), \( \sigma' < \sigma < \sigma'' \), and we have to show that its length \( \lambda \) does not exceed the value in question. Let \( \sigma' \) be the face of \( \sigma \) opposite \( \sigma'' \). The segment \( \sigma' \cap \sigma'' \) is the longest segment of \( \sigma^p \) carrying \( \sigma' \cap \sigma'' \) and if \( \mu \) is its length then by an elementary calculation:

\[
\begin{align*}
\frac{r - q}{r + 1} &\leq \frac{r}{r + 1} \leq \frac{p}{p + 1}.
\end{align*}
\]

From this follows

\[
\lambda \leq \frac{p}{p + 1} \mu \leq \frac{p}{p + 1} \; \text{diam} \; \sigma^p.
\]

(2.5) Let \( \{ y_n \} \) be coordinates for the space of \( \sigma^p \). Then on \( \sigma^p \) the barycentric coordinates may be expressed as linear functions of a set \( \{ y_n, \ldots, y_m \} \). Hence they are continuous in \( y_n, \ldots, y_m \) on \( \sigma^p \).

If \( \{ a_{ni} \} \) are the coordinates of \( a_i \) and \( \{ y_n \} \) those of \( x \in \sigma^p \), the system in the unknowns \( x^i \):

\[
y_n = x'^i a_{ni}, \quad \sum x^i = 1,
\]

is compatible and has a unique solution. As is well known this implies (2.5).
3. A few of the simpler properties of an Euclidean complex \( K = |\sigma| \) are:

(a) If \( K \) is finite \(| K | \) is a compactum;
(b) if \( K \) is locally finite, \(| K | \) is a local compactum;
(c) if \( L \) is a closed subcomplex, \(| L | \) is a closed set and hence \(| K - L | \) is an open set.
(d) if \(| K_\iota | \) are the components of the Euclidean complex \( K \) (not necessarily finite) then \(| | K_\iota | | \) are those of the polyhedron \(| K | \).

If \( K \) is finite then \(| K | \) is the union of a finite set of compacta, the \( \bar{\sigma} \), and hence it is a compactum. It is a consequence of (III, 6.12c) that if \( x \in \sigma \) then \( d(x, | K - St \sigma |) > 0 \). Hence \( x \in \text{Int} | St \sigma | \). If \( \sigma' > \sigma \) then \( St \sigma' \subseteq St \sigma \), and so \( x \in \sigma' \rightarrow x \in \text{Int} | St \sigma' | \subseteq \text{Int} | St \sigma | \). Therefore \( \text{Int} | St \sigma | = | St \sigma | \), and so \(| St \sigma | \) is open. It follows that \(| K - L | \) is open and hence \(| L | \) is closed. Since \( St \sigma \) is finite \(| St \sigma | \) is a compactum. Thus \( x \in \sigma \) has a neighborhood whose closure is a compactum, and so \(| K | \) is a local compactum.

Let \( x, y \in | K_\iota | \) and \( A, A' \) vertices of \( \sigma(x), \sigma(y) \). Since \( K_\iota \) is connected there is a sequence, \( \sigma(x) = \sigma_0, \sigma_1, \ldots, \sigma_r = \sigma(y) \) of simplexes of \( K_\iota \) such that any two consecutive are incident. Hence \(| \sigma_i | \) is a finite collection of connected sets of which any two consecutive ones intersect. Their union is a connected subset of \(| K_\iota | \) containing \( x \) and \( y \). Thus the two points are in the same component of \(| K | \).

Suppose now \( x \in | K_\iota | \), \( y \notin | K_\iota | \). Then by (c) \(| K_\iota | \) and \(| K - K_\iota | \) are disjoint closed sets whose union is \(| K | \), containing, respectively, \( x \) and \( y \) and so the two points are in distinct components of \(| K | \). Thus \( x, y \) are in the same component of \(| K | \) when and only when they are in the same set \(| K_\iota | \) and this implies (d).

(3.2) We have seen (1.2) that \(| K | \) is in a space \( R \) which is a \( G^n \) or \( P^* \).

We specify that \( \text{diam} \sigma \), in the sense of (VI, 24.1) is to be its diameter as a subset of \( R \) and it is clear that this turns \( K \) into a metric complex (VI, 24.2).

(3.3) **Definition.** Let \( |\sigma_1, \sigma_2, \ldots| \) be the simplexes of \( K \) ranged in some order. Then \( K \) is said to be regular if \( \text{diam} \sigma_n \to 0 \). If \( A \) is a closed subset of \( R \) (the space of 3.2) then \( K \) is said to be regular relatively to \( A \), whenever \( \sup |\text{diam} \sigma_n, d(A, \sigma_n)| \to 0 \). The two types of regularity are evidently independent of the order in which the \( |\sigma_n| \) have been ranged.

(3.4) Every countable locally finite simplicial complex \( R \) has a regular Euclidean realization \( K \) in \( P^n \). If in addition \( n = \text{dim} \ R \) is finite then \( R \) has a regular realization in any paralleotope \( P^m, m \geq 2n + 1 \).

Let \( |x| \) be the simplexes of \( R \) and \(|A_1, A_2, \ldots| \) its vertices. If \( |z_1, z_2, \ldots| \), \( 0 \leq z_i \leq 1/i \), are coordinates for \( P^* \) let \( a_i \) be the point: \( z_h = 0, h \neq i, z_i = 1/i \). If \( \sigma^* = A_{i_0} \ldots A_{i_p} \in R \) then \( a_{i_0}, \ldots, a_{i_p} \) are the vertices of an Euclidean simplex \( \sigma^* \subseteq P^m \) and \( K = |\sigma^*| \) behaves as required.
Suppose now \( n = \dim K \) finite. Take first in \( P^n \) a sequence \( \{ b_i \} \) tending to a limit \( b \). Choose now points \( \{ a_i \} \), one at a time, in \( P^n \) as follows. First we require that \( d(a_i, b_i) < 2^{-i} \). Then suppose \( a_1, \ldots, a_k \) so chosen that no subset of \( 2n + 2 \) is in an \( \mathcal{E}^n \) of the \( \mathcal{E}^n \) of \( P^n \). The spaces \( \mathcal{E}^n \) determined by all the subsets consisting of \( 2n + 1 \) of the \( a_1, \ldots, a_k \), are finite in number and \( a_{k+1} \) is chosen exterior to all these. Proceeding as before but all a finite number of the constructed \( \sigma \) are contained in any preassigned \( \mathcal{E}(b, \epsilon) \); we will therefore have once more an Euclidean realization in \( P^n \) which is clearly regular.

4. In many applications one requires subdivisions of arbitrarily small mesh. This may sometimes be accomplished by means of derivation. The possibilities are discussed below.

(4.1) When \( K \) is either (a) finite-dimensional and with bounded mesh (hence in particular when \( K \) is finite), or else (b) regular, then mesh \( K^{(p)} \rightarrow 0 \).

Consider first (a). If \( \dim K = n \) and mesh \( K = t \) then by (2.3, 2.4) we have mesh \( K^{(p)} < (n/(n + 1))^p t \rightarrow 0 \).

Consider now (b). Let \( \{ \sigma_i \} \) be the simplexes of \( K \) ranged in some order. Given any \( \epsilon \) we may select \( r \) so high that \( \text{diam} \sigma_i < \epsilon \). The union of the closures of the \( \sigma_i \), \( j \leq r \), is a finite closed subcomplex \( K_r \), and so by the result just proved we may choose \( p \) so high that mesh \( K_r^{(p)} < \epsilon \). Hence mesh \( K^{(p)} < \epsilon \).

(4.2) Every Euclidean complex \( K = \{ \sigma \} \) has a simplicial partition \( K_1 \) of arbitrarily small mesh.

If the situation is as in (4.1), and in particular if \( K \) is finite, we may merely choose for \( K_1 \) a suitable \( K^{(p)} \). In the general case let \( \mathcal{C}^p \) be the space of \( \sigma_i^p \) and let it be referred to the coordinates \( x_1, \ldots, x_p \). Take a fixed \( \epsilon > 0 \) and choose \( \eta > 0 \) such that the partition of \( \mathcal{C}^p \) by the subspaces \( x_i/\eta = 0, \pm 1, \ldots \) is of mesh less than \( \epsilon \). These subspaces decompose \( \text{Cl} \sigma_i^p \) into the elements of a polyhedron whose mesh is less then \( \epsilon \). Since \( K \) is locally finite, any \( \sigma \) occurs thus in at most a finite number of \( \text{Cl} \sigma_i \), and so it is decomposed into a finite set of convex polyhedral cells. Their totality is a partition \( \Pi \) of \( K \) whose mesh is less than \( \epsilon \) and its derived \( \Pi' \) is a \( K_1 \) whose mesh is also less than \( \epsilon \).

(4.3) Definition. An Euclidean complex \( K \) whose antecedent is a circuit, a manifold, \( \ldots \) will be called a geometric circuit, geometric manifold, \( \ldots \).

5. (5.1) Let \( K = \{ \sigma \} \) be a finite Euclidean complex and \( A \) a closed proper subset of \( |K| \). Then \( |K| - A \) may be covered with an Euclidean complex \( L = \{ \xi \} \) regular relative to \( A \) and such that each \( \xi \) is contained in \( \sigma \).

Let \( \{ m_1, m_2, \ldots \} \) be a monotone increasing sequence of integers to be specified presently. Denote by \( P_\ast \) the open subcomplex of the barycentric derived \( K^{(m_\ast)} \) consisting of the simplexes whose closures meet \( A \) and set \( P_0 = K \). Define \( Q_\ast = \text{Cl} P_\ast ; R_\ast \) as the set of simplexes of \( K^{(m_\ast+1)} \) in \( |Q_\ast| - |Q_{\ast+1}| \). We select, as we may, \( \{ m_\ast \} \) such that \( m_{\ast+1} = \text{mesh} K^{(m_{\ast+1})} < (1/3) d(A, |K| - |P_\ast|) \), with \( \gamma < (1/3) \sup d(A, x), x \in K \).
Let \( \Pi \) be the union of the simplexes of all the \( R_n \) and take any simplex \( \sigma_n \) of \( R_n \). Owing to the condition on the meshes, \( \sigma_n \) cannot meet both \( |\text{Cl } R_{n-1}| \) and \( |\text{Cl } R_{n+1}| \). Moreover if it meets \( |\text{Cl } R_{n+1}| \) then all its points, and hence also its faces are each in a simplex of \( \text{Cl } R_n \). It follows that if \( \sigma' < \sigma_n \) then \( \sigma' \) is in \( |R_n| \) or \( |R_{n+1}| \). In the former case it is a simplex of \( R_n \), in the latter it is a union of simplexes of \( R_{n+1} \). Thus \( \sigma_n - \sigma_n \) is the union of a finite set of faces of \( \Pi \). Since \( \sigma_n \) can only meet consecutive sets \( |\text{Cl } R_n| \), and the latter are finite, \( \sigma_n \) is the face of at most a finite number of simplexes of \( \Pi \). Hence \( \Pi \) is locally finite. Since the \( R_n \) are finite and disjoint \( \Pi \) is countable. Since the simplexes of \( \Pi \) are disjoint and clearly \( \sup \{ \text{diam } \sigma_n, d(A, \sigma_n) \} \to 0 \), \( \Pi \) is a polyhedron, and its derived \( L = \Pi' \) answers the question.

**Remark.** The theorem may readily be extended to infinite complexes. The chief modification required in the proof will then be replacing the derived by suitably chosen partitions of \( K \).

(5.2) Let \( K \) be a finite Euclidean complex contained in a parallelootope \( P^n \). Then \( P^n \) may be covered with a finite Euclidean complex which has a simplicial partition of \( K \) as a subcomplex.

The different faces of \( P^n \) (in an obvious sense) make up a polyhedral complex \( \Omega \) covering \( P^n \) (i.e., such that \( |\Omega| = P^n \)). Any given \( \sigma_i^p \) of \( K \) is a subset of an \( \mathbb{C}_{1}^{p} \) of the space \( \mathbb{C}_{n}^{p} \) of \( P^n \) and \( \mathbb{C}_{1}^{p} \) is the intersection of \( n - p \) subspaces \( \{ \mathbb{C}_{1}^{p} \} \) \( h = 1, 2, \ldots, n - p \) of \( \mathbb{C}_{n}^{p} \). The total set \( \{ \mathbb{C}_{1}^{p} \} \) for all \( i, h \), causes a partition \( \Pi^* \) of \( \Omega \) whose derived is related to \( K \) in the asserted way.

6. **Barycentric mappings.**

(6.1) Let \( K = \{ \sigma \}, L = \{ \tau \} \) be Euclidean complexes with respective vertices \( \{a_i\}, \{b_i\} \). Suppose that there exists a simplicial set-transformation \( t:K \to L \) and let \( \tau \) be the induced simplicial chain-mapping. Let the \( b_i \) be so labelled (with possible repetitions) that \( ta_i = b_i \). We introduce a point-set-transformation \( T:|K| \to |L| \) defined as follows: If \( x = x'a_i \), then \( y = T x = x'b_i \). This implies in particular that: (a) if \( b_i = \cdots = b_j \) while \( b_h \neq b_i \) for \( h \neq i \), \( \cdots \), \( j \), then the barycentric coordinate of \( y \) as to \( b_i \) is \( x'^1 + \cdots + x'^i \); (b) \( x \in \sigma \rightarrow Tx \in T \sigma \). We prove:

(6.2) \( T \) is a mapping \( |K| \to |L| \), said to be barycentric.

Let \( \{u_i\}, \{v_i\} \) be coordinates of reference for the spaces of \( K, L \), and let \( y_0 = Tx_0 \). Since \( K \) is locally finite, by (2.5) there is a finite set of the \( \{u_i\}, \) say \( \{u_1, \cdots, u_r\} \) such that on \( |\text{St } \sigma(x_0)| \) the barycentric coordinates are continuous functions of \( \{u_1, \cdots, u_r\} \) and hence of \( x \). It follows that \( T \) is continuous on \( |\text{St } \sigma(x_0)| \), and since \( |\text{St } \sigma| \) is an open covering it is an elementary manner to prove \( T \) continuous on \( |K| \).

Evidently \( Ta_i = ta_i \), so that \( T \) induces \( t \) and hence also \( \tau \). Hereafter we drop all mention of \( t \), and will call \( \tau \) the chain-mapping induced by \( T \).

(6.3) As an application suppose that \( K \cong L \). This means that they have a common antecedent \( \Phi \). Let the notations be so chosen that \( a_i, b_i \) correspond under the isomorphism, or which is the same that they are the images of the
same vertex of $\mathfrak{R}$. Then $t$ is one-one, and so is $T$. Moreover under the circumstances $T^{-1}$ corresponds to $t^{-1}$ like $T$ to $t$. Therefore $T'$ is topological. In fact it takes on $\varphi^t$ the values of a nonsingular affine transformation of the space $\mathbb{G}^n$ of $\sigma$. Thus:

(6.4) If the Euclidean complexes $K$, $L$ are isomorphic then $|K|$, $|L|$ are topologically equivalent. Furthermore there exists a topological mapping $|K| \to |L|$ which is barycentric.

As a consequence:

(6.5) All the Euclidean realizations $\{K\}$ of a given countable locally finite simplicial complex yield topologically equivalent polyhedra $\{|K|\}$, which may be mapped topologically into one another as indicated in (6.4).

7. Normal subcomplexes. A closed subcomplex $L$ of $K$ is said to be normal in $K$ whenever if a simplex of $K$ has all its vertices in $L$ then it is a simplex of $L$.

(7.1) The derived $L'$ of $L$ is normal in $K$.

In the notations of (IV, 25) if $\zeta = ' \sigma_i, \ldots, \sigma_j, \sigma_i < \ldots < \sigma_j$, is a simplex of $K$ with its vertices $' \sigma_i, \ldots, \sigma_j$ in $L'$, then $\sigma_i, \ldots, \sigma_j \in L$, and so $\zeta \in L'$.

(7.2) If $L$ is normal in $K$ so is $M = K - St L$.

If $\sigma \in K - M$ exists with all its vertices in $M$ then $\sigma \in St L - L$, and so $\sigma$ must have vertices in $L$. Since $L \cap M = \emptyset$ this is a contradiction proving (7.2).

(7.3) Under the same conditions as in (7.2) there passes through every point $x \in |St L - L|$ a unique segment $x'x''$ with $x' \in |L|$, $x'' \in |M|$. Moreover the transformation $\rho: |St L| \to |L|$ such that $\rho x = x'$, $\rho |L| = 1$, is a deformation-retraction.

We have $\sigma(x) = \sigma'\sigma''$, $\sigma' \in L$, $\sigma'' \in M$, and so the existence of the segment is a consequence of (2.1). It follows readily from the expression of $x'$ that it is continuous in the barycentric coordinates of $x$ and hence continuous in $x$ itself, (2.5). Hence $x \to x'$ for $x \in |St L - L|$ and $x = x'$ for $x \in |L|$, defines a retraction $\rho: |St L| \to |L|$. By (I, 47.4) $\rho$ is a deformation.

(7.4) Application. Under the same conditions as in (7.3) the homology groups of the compact cycles of $|M|$ are isomorphic with the corresponding groups of $|K - L|$.

In view of (7.2) we may interchange $L$, $M$ in (7.3) and so there is a deformation retraction $\rho_1: |St M| = |K - L| \to |M|$. Hence (7.4) is a consequence of (3.1) and (VII, 7.5).

(7.5) Let $|\Pi|$ be a finite Euclidean polyhedron in $\mathbb{G}^n$. Then $|\Pi|$ has a neighborhood $U$ in $\mathbb{G}^n$ for which it is a deformation retract.

Let $P^n$ be a parallelootope in $\mathbb{G}^n$ containing $\Pi$ in such a way that $d(\Pi, \mathbb{G}^n - P^n) = 2\alpha > 0$. By (4.1, 5.2, 7.2) a suitable simplicial partition $L$ of $\Pi$ is a normal subcomplex of an Euclidean complex $K$ of mesh less than $\alpha$ covering $P^n$. As a consequence $U = |St L|$ (star in $K$) is in $P^n$ and is a neighborhood of $|\Pi|$ in $\mathbb{G}^n$. The existence of $\rho$ is then a consequence of (7.3).

The retraction here considered is a special case of so-called “neighborhood retraction” in the sense of Borsuk. In point of fact it may be proved with Borsuk that $|\Pi|$ is a so-called “absolute neighborhood retract,” i.e., whenever
topologically imbedded in a compactum it is a retract of a suitable neighborhood of the compactum. For details regarding these questions see Lefschetz [I, III, IV].

8. Since the derived of a polyhedral complex is an Euclidean complex, the problem of the topological classification of polyhedra is equivalent to the same problem for simplicial polyhedra. We have shown, on the other hand (6.4), that if two Euclidean complexes \( K, K_1 \) are isomorphic then \( |K|, |K_1| \) are topologically equivalent. Conversely, supposing \( |K|, |K_1| \) topologically equivalent, what can be said regarding the isomorphism of \( K \) with \( K_1 \)? Since \( |K^{(m)}|, |K_1^{(n)}| \) are likewise topologically equivalent, \( K \cong K_1 \) would be no more reasonable than \( K^{(m)} \cong K_1^{(n)} \) for some \( m, n \). Or instead of the derived we may equally well compare any two partitions. Now if \( K, K_1 \) have isomorphic partitions \( \Pi, \Pi_1 \) they also have the isomorphic simplicial partitions \( \Pi', \Pi'_1 \). Thus we only need to consider simplicial partitions. Let \( K, K_1 \) be defined as partition-equivalent whenever they have isomorphic simplicial partitions. This relation is manifestly a true relation of equivalence. Evidently partition-equivalence implies topological equivalence of the polyhedra. The converse, one of Poincaré's well known unsolved problems, may be explicitly formulated as:

**Problem A.** Does the topological equivalence of the polyhedra \( |K|, |K_1| \) imply the partition-equivalence of the complexes \( K, K_1 \)?

As a special case if \( |K| \) is a closed \( n \)-cell we have the likewise unsolved

**Problem B.** If the polyhedron \( |K| \) is a closed \( n \)-cell, is \( K \) partition-equivalent to a closed Euclidean \( n \)-simplex?

For the dimension one the solutions of A, B are elementary. For the dimension two, solutions may be obtained but they lean heavily upon the Jordan-Schoenflies theorem regarding the subdivision of a two-sphere by a simply closed curve. For higher dimensions only partial extensions of this theorem are known (see Wilder [a, b]), and this is one source of difficulty. Another lies of course in our ignorance regarding the Poincaré group (see (23a)).

In connection with these problems it may be recalled that M. H. A. Newman [a] has taken as point of departure in his investigations on Euclidean complexes, their classification with respect to partition-equivalence.

§2. HOMOLOGY THEORY

9. Since we are unable to identify topological and partition-equivalence classes of complexes it is natural to investigate the topological invariants of the partition-equivalence classes. From (IV, 28.1) and (V, 16.2) we already infer that various homology and cohomology groups, class intersections and rings are partition-invariant. Our next object is to prove that they are also topologically invariant.

(9.1) Two types of cycles, \( \cdots \) related to \( K \) will occur simultaneously in the sequel: (a) those of the space \( |K| \), referred to as geometrical; (b) those of the complex \( K \) itself, referred to as combinatorial. Often the context will indicate the type: thus compact cycles are necessarily geometrical, while finite cycles are combinatorial.
(9.2) Notations. They remain those of (1.1) except that: (a) the simplexes of \( K^{(a)} \) are written \( \sigma_{a_1}, \sigma_{a_1}^p \); (b) if \( a_{n_1}, \gamma_{n_1}, \cdots \) are elements of any sort related to \( K^{(a)} \) the analogues for \( K \) will be written \( a_i, \gamma_i, \cdots \).

(9.3) In connection with certain coverings there will occur on more than one occasion instead of their nerves, suitable Euclidean representations of the nerves. Usually they will be \( K \) or one of its derived. For convenience the term "nerve" will also be applied to these realizations. The meaning will be clear enough from the context to cause no confusion.

(9.4) Method of proof. It will always consist in identifying the combinatorial elements with similar elements in a suitable net or web topologically related to \( |K| \).

10. Consider first a finite Euclidean complex and its combinatorial homology theory: homology and cohomology groups and class intersections (V, \S 2). Consider the finite open covering of \( |K| \) by the stars of the vertices: \( \mathcal{B} = \{\text{St } a_i\} \). From \( \text{St } a_i \cdot \cdots \cdot \text{St } a_j = \text{St } a_i \cdot \cdots \cdot a_j \), follows that \( \text{St } a_i \cdot \cdots \cdot \text{St } a_j = 0 \iff a_i \cdot \cdots \cdot a_j \in K \). Therefore nerve \( \mathcal{B} = K \) (9.3). Similarly if \( \mathcal{B}_a \) is the finite open covering by the stars of the vertices of \( K^{(a)} \) then nerve \( \mathcal{B}_a = K^{(a)} \).

Since by (4.1) mesh \( \mathcal{B}_a \rightarrow 0 \), \( |\mathcal{B}_a| \) is cofinal in the family of all the finite open coverings, and so it may serve to define the geometrical groups and intersections in \( K \).

Since \( \text{St } \sigma_n \) in \( K^{(n+1)} \) is contained in \( \sigma_n \) it is also contained in the star of any vertex of \( \sigma_n \). Therefore a mapping of \( \sigma_n \) into a vertex of \( \sigma_n \) defines a projection \( \pi_n^{n+1} : K^{(n+1)} \rightarrow K^{(n)} \), and \( \Sigma(K) = [K^{(n)}, \pi_n^{n+1}] \) is a simplicial spectrum whose net homology and cohomology groups and intersection theory are those of \( |K| \). Now \( \pi_n^{n+1} \) is merely the operation \( \tau \) of (IV, 23) for \( K^{(n)} \) (a reciprocal of chain-derivation in \( K^{(n)} \), IV, 26.2e). Therefore by (IV, 24.2), \( \pi_n^{n+1} \) induces an isomorphism on the homology groups, and similarly for its dual \( \pi_n^{n+1} \) and the cohomology groups. It follows that the net groups of \( \Sigma(K) \) are those of any \( K^{(n)} \), and hence those of \( K \) itself (II, 13.4b, 14.6). Likewise also for the intersections. Therefore we have:

\[ (\sigma_n, \sigma_n) = \sigma_n, \quad \delta(n, \sigma_n) = \delta(n, \sigma_n), \quad KI(\Gamma^p, \Gamma_q) = KI(\Gamma^p, \Gamma_q). \]
An interesting complement is:

(10.3) The properties of (III, 20), relating connectedness and homology, continue to hold for a finite Euclidean complex if the term “component” refers to the components of the polyhedron \(|\mathcal{K}|\). In particular the number of components of \(|\mathcal{K}|\) is \(\mathbb{E}^\mathcal{E}(\mathcal{K})\) (3.1d).

11. Suppose now that we are dealing with an arbitrary Euclidean \(\mathcal{K}\). Referring to (VI, §§5, 6) there are various associated combinatorial homology theories. That is to say, in each case there are two dual categories, with intersections since \(\mathcal{K}\) is simplicial.

(11.1) Definition. The combinatorial theory of two specific dual categories \(\mathcal{A}, \mathcal{B}\) of cycles and cocycles of \(\mathcal{K}\) is said to be invariant whenever it is the same as the theory of two dual categories \(\mathcal{A}', \mathcal{B}'\) which have topological character for \(|\mathcal{K}|\). Thus the combinatorial theory of the cycles and cocycles of a finite \(\mathcal{K}\) is invariant, since it has been shown to be the same as the theory of the geometrical cycles or cocycles of \(\mathcal{K}\).

It is implicit in the definition that when the theory of \(\mathcal{A}, \mathcal{B}\) is invariant, their homology and cohomology groups, class intersections, cohomology rings, Betti numbers, are all topological invariants of \(|\mathcal{K}|\). As an application we may state the following theorem, which is proved essentially like (10.1):

(11.2) Let \(\mathcal{K}, \mathcal{K}_1\) be finite Euclidean complexes with the closed subcomplexes \(L, L_1\). Let \(T\) be a topological mapping \(|\mathcal{K}| \to |\mathcal{K}_1|\) such that \(T|L| = |L_1|\).

Then the combinatorial theory of the cycles mod \(L\) and cocycles of \(\mathcal{K} - L\) is invariant under \(T\).

12. Complementary Remarks. It will be very convenient to associate with the cycles of \(\mathcal{K}\) certain specific cycles of the spectrum \(\Sigma(\mathcal{K})\).

(12.1) Let \(\delta\) denote chain-derivation both in \(\mathcal{K}\) and in all its derived. As applied then to \(\mathcal{K}^{(n)}\) it has \(\pi\delta\pi^{-1}\) of (10) for a reciprocal. By (IV, 26.8, 23.1),

\[
\pi^{n+1}\delta = 1, \quad \delta\pi^{n+1} \sim 1.
\]

(12.2)

We will compare more particularly \(\mathcal{A} = \mathbb{S}(\mathcal{K}^{(n)}, \mathcal{G})\) with \(\mathcal{B} = \delta \mathbb{S}(\mathcal{K}^{(n)}, \mathcal{G})\subset \mathbb{S}(\mathcal{K}^{(n+1)}, \mathcal{G})\). If we set \(\hat{\pi} = \pi^{n+1}\ |B\) then we still have \(\hat{\pi}\delta = 1\). Moreover since \(\hat{\pi}\delta\hat{\pi} = \delta(\hat{\pi}\delta) = \delta\), as operations between the two groups \(\mathcal{A}, \mathcal{B}\) we have in addition \(\hat{\pi}\delta = 1\). Therefore \(\delta\) is an isomorphism of \(\mathcal{A}\) with \(\mathcal{B}\). Hence:

(12.3) \(\delta^m\) is a simultaneous isomorphism \(\mathbb{S}(\mathcal{K}^{(n)}, \mathcal{G}) \to \delta^m \mathbb{S}(\mathcal{K}^{(n)}, \mathcal{G})\).

(12.4) Take now any cycle \(\gamma^p\) of \(\Sigma(\mathcal{K})\). We have \(\pi^{n+1}\gamma^p \sim \gamma^p\), and hence by (12.2): \(\gamma^p_{n+1} \sim \delta\gamma^p \sim \delta^m\gamma^p\). Therefore \(\gamma^p = \{\delta^m\gamma^p\} \sim \{\gamma^p\}\). Thus the geometric class \(\Gamma^p\) of \(\gamma^p\) contains a representative among the cycles of \(\Sigma(\mathcal{K})\) of the form \(\gamma^p = \{\delta^m\gamma^p\}\). We will say that \(\gamma^p\) is a geometric cycle adherent to \(\gamma^p\) in \(\mathcal{K}\). It is also convenient to refer to \(\Gamma^p, \Gamma^p\) as adherent to one another.

It is clear that \(\gamma^p \to \gamma^p\) defines an isomorphism in the algebraic sense \(\theta\) of \(\mathcal{B}(\mathcal{K}, \mathcal{G})\) with a subgroup \(\mathcal{B}(\Sigma, \mathcal{G})\). If \(U_n\) is any open set of \(\mathcal{B}(\mathcal{K}^{(n)}, \mathcal{G})\)
then $V_n = \{ \gamma^p | \delta^m \gamma^p \in U_n \}$ is open in $\mathcal{B}^p$ and $\{V_n\}$ is a base for $\mathcal{B}^p$. By (12.3), and since $\delta$ maps cycles into cycles, there is an open set $U$ in $\mathcal{B}^p(K, G)$ such that $\delta^p \gamma^p \in U_n \Rightarrow \gamma^p \in U$. Hence $V_n = \{ \gamma^p | \gamma^p \in U \}$. Thus if $U$ is any open set of $\mathcal{B}^p(K, G)$ and $V = \{ \gamma^p | \gamma^p \in U \}$, then $\{V\}$ is a base for $\mathcal{B}^p$. Therefore $\delta$ maps into another the elements of two bases $\{U\}$, $\{V\}$ for $\mathcal{B}^p(K, G)$ and $\mathcal{B}^p$ and so it is an isomorphism.

Let $\mathcal{B}^p = \mathcal{B}^p(\Sigma, G)$ a $\mathcal{B}^p = \mathcal{B}^p(\Sigma, G)$ the group of the bounding cycles of form $\gamma^p$. Since $\mathcal{B}^p(\Sigma, G)$ is closed in $\mathcal{B}^p(K, G)$, $\mathcal{B}^p$ is closed in $\mathcal{B}^p$. If $\gamma^p \sim 0$ then $\gamma^p \sim 0$, and so $\delta^{-1} \mathcal{B}^p \subset \mathcal{B}^p(K, G)$. On the other hand ($\gamma^p = \delta^m \gamma^p \Rightarrow \delta^m \gamma^p = \delta^m \delta^{-1}$) $\gamma^p \sim 0$. Hence $\delta \mathcal{B}^p(K, G) \subset \mathcal{B}^p$, and therefore $\delta \mathcal{B}^p(K, G) \subset \mathcal{B}^p$, since the latter is closed in $\mathcal{B}^p$. Therefore $\delta \mathcal{B}^p(K, G) = \mathcal{B}^p$, and so $\delta$ maps the cycles $\gamma^p$ which are $\sim 0$ into the $\gamma^p \sim 0$. It follows that $\delta$ induces an isomorphism $\delta: \mathcal{B}^p(K, G) \rightarrow \mathcal{B}^p(\Sigma, G)$ and this isomorphism is readily recognized to be the one in (10.1).

To sum up, the preceding analysis yields:

(12.5) The relation of adherence between the combinatorial cycles $\gamma^p$ of $K$ and the geometrical cycles of the spectrum $\Sigma(K)$ of the special form $\gamma^p = \{\delta^m \gamma^p\}$ is an isomorphism of the groups $\mathcal{B}^p, \mathcal{B}^p(\Sigma, G)$ with the corresponding groups of the $\gamma^p$. Similarly adherence between the combinatorial and geometrical classes is an isomorphism between the corresponding homology groups.

(12.6) Let now $L$ be a closed subcomplex of $K$. Referring to (IV, 24.3c), $\delta$ is likewise chain-derivation in $L$ and its derived. Moreover $\pi_n^{n+1} | L^{n+1}$ is a reciprocal of $\delta$ as chain-derivation in $L^{n+1}$. It follows that $\pi_n^n L^{n+1} = L^n$, a result which may also be deduced from (IV, 26.2bc). The projection $\pi_n^m, m > n$, of $\Sigma(K)$ is given by $\pi_m^n = \pi_m^{n+1} \cdots \pi_m^{n+1}$, and so $\pi_m^n L^n = L^n$. Referring now to (VI, 12) if $\pi_m^n = \pi_m^n \mod L^n$ and $\pi_m^n = \pi_m^n | L^{n+1}$, then $\Sigma_0 = \{K^n - L^n\}, \pi_m^n, \Sigma_1 = \{L^n; \pi_m^n\}$ are spectra such that $(\Sigma_0, \Sigma_1)$ is a section of $\Sigma(K)$. If we replace, in everything that precedes, $\Sigma(K)$ by $\Sigma_0, K by K - L$, and the cycles of $K$ by those of $K$ mod $L$, we will extend (12.5) automatically to adherence for the cycles mod $L$. We point out explicitly that the operation $\delta$ remains the same. For by reference to (IV, 26.2b) we verify without difficulty that the chain-mapping $(K^n - L^n) \rightarrow (K^{n+1} - L^{n+1})$ induced by $\delta$ is merely $\delta \{K^n - L^n\}$. Thus the adherent cycles will still be $\gamma^p$ and $\gamma^p = \{\delta^m \gamma^p\}$.

(12.7) Let $L_1$ be a second closed subcomplex of $K$ containing $L$ and let $\omega, \pi$ be, respectively, the topological and combinatorial projections $|K - L| \rightarrow |K - L_b|$ and $(K - L) \rightarrow (K - L_1)$. If $\gamma^p = \{\delta^m \gamma^p\}$ is a cycle mod $L$, then $\omega \gamma^p$ is obtained by reducing mod $L_1^{(n)}$ the coordinate $\delta^m \gamma^p$, and this yields merely $\delta^m (\pi \gamma^p)$. Therefore $\omega \gamma^p = \{\delta^m (\pi \gamma^p)\}$. Or $\pi \gamma^p$ and $\omega \gamma^p$ are still adherent. Thus the associated projections $\pi, \omega$ preserve adherence. This result will be useful later.

(12.8) Let $K, L$ be finite. Then a barycentric mapping $T: |K| \rightarrow |L|$ sends adherent elements into adherent elements; in other words $T$ preserves adherence.

For convenience we also designate by $T$ the induced operations on the com-
binatorial and geometrical cycles and classes. If \( \{a_i\}, \{b_i\} \) are the vertices of \( K, L \) evidently \( T \mid \text{St } a_i \mid \not\geq \mid \text{St } b_i \mid \). If \( \gamma = \gamma, \delta \gamma, \cdots \) we find by reference to (VII, 5.12) that \( T' \gamma \) is a cycle of \( L \) whose coordinate relative to \( L \) as nerve of \( \mid \text{St } b_i \mid \) is precisely \( T \gamma \). Since the relations of adherence in \( L \) are determined by the coordinates in \( L \), it follows that the classes of \( T' \gamma \), \( T' \gamma \) are adherent. A similar argument is valid for the cocycles except that the homomorphisms are \( \tau^* \) of (VII, 5.11) for the topological cocycles, and \( \theta^* \), the dual of the chain-mapping induced by \( T \), for the combinatorial cocycles. Similarly for the intersections.

13. We will now consider a few invariance theorems for infinite complexes.

(13.1) Let \( K \) be a general Euclidean complex. Then the combinatorial theory of the dual categories of the infinite cycles [cocycles] and finite cocycles [cycles] are topologically invariant.

Corresponding to \( K \) there may be introduced the direct web of sets \( \mathfrak{A} = \{A_x\} \) of the open sets with compact closures of (VII, 13.2). Since \( \mid \text{St } \alpha(x) \mid \mid \text{St } \alpha \mid \) is a covering of the compact set \( A \), there is a finite subcovering \( \mid \text{St } \alpha \mid \) and the union of its simplexes is a finite subcomplex \( K_\alpha \) of \( K \) such that \( K_\alpha \supset A \). Hence if \( K_\alpha \) are the finite subcomplexes, \( \mid K_\alpha \mid \) is cofinal in \( \mathfrak{A} \), and so both have the same homology theory. It is, however, an elementary consequence of (12.7) that the direct webs \( \{K_{\alpha}\} \) and \( \{K_\alpha\} \) have the same homology theory. Hence \( K_\alpha \) has the same as \( \mathfrak{A} \), and so the homology theory of the first, like that of the second, has topological character. Hence the theory of the dual categories of the infinite cycles and finite cocycles, which is that of \( K_\alpha \) (VI, 20) has topological character. For infinite cocycles and finite cycles the treatment is the same except that the comparison is with the direct web \( \mathfrak{S} \) of the compact subsets of \( K \) (VII, 13.1).

We have proved incidentally the following result:

(13.2) The homology [cohomology] groups of the compact cycles [cocycles] of \( K \) are isomorphic with the corresponding combinatorial groups of the finite cycles [cocycles] of \( K \).

An argument essentially similar to that of (13.1) yields the following extension of (12.2):

(13.3) The situation being as in (12.2) save that the complexes need not be finite, the combinatorial theory of the infinite cycles of \( K \) mod \( L \) and of the finite cocycles of \( K - L \) is invariant under \( T \).

We may in fact sharpen the preceding result to:

(13.4) Property (13.3) still holds if \( T \) is merely a mapping \( K \to K_1 \) such that: (a) \( T \) maps \( K - L \) topologically onto \( K_1 - L_1 \); (b) \( T \mid L \subset L_1 \).

Suppose first \( K, K_1 \) finite. The dual categories under consideration are then those of the cycles of \( K \) mod \( L \) and of the cocycles of \( K - L \), and the same in \( K_1 \). Consider the direct open web of sets \( \mathfrak{A} = \{A_x\} \) whose elements are the open subsets of \( K - L \). Let \( \Phi, \cdots \) and the other notations of (VII, 4.3) be applied to \( K \). If we form the direct open web \( \Phi_0(\mathfrak{A}) = \{\Phi_0(A_x)\} \) then by (VI, 23.3) its homology theory is the same as that of the cycles of \( \Phi \).
mod Φ₂(| L |) and cocycles of Φ₂(| K − L |), i.e., the same as that of the geometric cycles of K mod L and cocycles of K − L. Hence, by (11.2) the homology theory of the direct web Φ₂(Φ) is then the same as the combinatorial theory of the cycles of K mod L and cocycles of K − L. Since Φ₂(Φ) direct has topological character relative to | K − L |, it is clear that its theory is invariant under T. Therefore the combinatorial theory of the cycles of K mod L and cocycles of K − L is likewise invariant under T.

Passing to the general case the proof is the same as that of (13.1), the Aₐ being now merely subsets of | K − L |.

A result of somewhat different character required later is:

(13.5) Let the conditions be the same as in (13.4) except that: (a) L, L₁ are normal in K, K₁; (b) T is merely a topological mapping of | K − L | onto | K₁ − L₁ |. Then T induces an isomorphism of the combinatorial homology groups of the finite cycles of M = K − St L with the corresponding groups of M₁ = K − St L₁.

For by (13.2) the groups in question are those of the compact cycles of | M |, | M₁ |, and by (7.4) likewise those of the compact cycles of | K − L |, | K₁ − L₁ |.

(13.6) Property (10.3), relating connectedness and homology, holds for any Euclidean complex (10.3; III, 20, 40.7).

14. Groups at the points. Since a polyhedron is metric the only groups requiring consideration are those through the points (VII, 15.7). We prove

(14.1) The homology groups of a polyhedron | K |, K = { σ }, through a point x are the same as the corresponding combinatorial groups of St σ(x), and in particular the same for all points of σ.

Let σ' be any simplex of St σ(x). If σ' is replaced by the join x(Ωσ')ₐ in K, there is obtained a special case of the simplicial partition of (IV, 29.5). Let S be the resulting partition operation. If L = SK, let St₄ x denote the star of the vertex x in L. Since S(St σ(x)) = St₄ x, by (IV, 24.3b), the groups of St σ(x) are those of St₄ x. To prove (14.1) it will thus be sufficient to prove that the groups through x are the same as the corresponding groups of St₄ x. Hence in the last analysis we merely have to show that:

(14.2) The groups through a vertex a of K are the same as the corresponding groups of St a.

Let this time Stₙ a denote the star of a in K(ⁿ). Since Stₙ a is a subcomplex of the nth derived of the finite complex Cl St a, by (4.1) diam Stₙ a → 0. As a consequence Φ = { | Stₙ a | } is coinitial in the inverse web of sets Φ (VII, 14.1) whose homology groups are those of the cycles through a. Thus Φ may serve to determine the groups of the cycles through a. We have then a conet and the operations of interest to us are the projections πⁿ⁺₁:Stₙ a → Stₙ⁺₁ a. To prove (14.2), and hence (14.1), it is sufficient (II, 14.6) to show that πⁿ⁺₁ induces an isomorphism $S^n(Stₙ a, G) → S^{n⁺¹}(Stₙ⁺¹ a, G)$ (geometrical groups). Since this is proved in the same way for all n we merely need to establish:

(14.3) $π₁$ induces an isomorphism of the geometrical homology groups of St a with the corresponding groups of Stₙ a.
Set $\text{St} a = \{a, u \_a\}, \text{St}_1 a = \{a, u \_1 a\}, B = \{\sigma, b\}, B_1 = \{\sigma_1\}, C = K - \text{St} a,$  
$C_1 = K' - \text{St}_1 a$. Let also $\pi$ be the reduction mod $C_1$ of the chains of $K'$.  
By (12.7) if $\gamma' = \{\gamma\_p, \delta \gamma\_p, \cdots\}$, where $\gamma$ is a cycle of $K'$ mod $C'$, then $\pi_1^{\\delta, b} \gamma = \{\pi_1 \gamma, \delta(\pi_1 \gamma), \cdots\}$. Since $\gamma \rightarrow \gamma'$ and $\pi_1 \gamma \rightarrow \pi_1^{\\delta, b} \gamma'$ induce isomorphisms of the corresponding homology groups (12.1, 12.2) the proof of (14.3) reduces to  
(14.4) $\pi \delta$ induces an isomorphism of the combinatorial homology groups of $K$ 
mod $C$, or groups of $\text{St} a$, with the corresponding groups of $\text{St}_1 a$.  

By definition the simplexes of $B_1$ are all the simplexes of the form $'(a \_1 \sigma) \cdots \'(a \_i \sigma)$, such that $a \_1 \sigma < \cdots < a \_i \sigma$, or equivalently such that $\sigma_1 < \cdots < \sigma_i$.  
To the simplex just written there corresponds thus the simplex $'(\sigma_1, \cdots, \sigma_i)$ of $B'$. It follows that $'(\sigma_1, \cdots, \sigma_i)$ defines an isomorphism $\theta : B' \rightarrow B_1$. Hence $d = \theta \delta \theta$ is a chain-mapping $B \rightarrow B_1$. We prove  
(14.5)  
$$\pi \delta a \sigma = a \delta a \sigma.$$  

We notice first that from the definition of $\delta$ (IV, 26.2b) and since $\theta$ is an isomorphism there comes:  
(14.6)  
$$d a \sigma = '(a a \sigma); \quad d a \sigma = '\!(a a \sigma)\!\!d F a \sigma; \quad p > 0.$$  

Since (14.5) is immediate for $p = 0$, we use induction on $p$. We have:  
$$\delta(a a \sigma) = '\!(a a \sigma)\!\!d F(a a \sigma) = '\!(a a \sigma)\!\!(d a \sigma - \delta(a F a \sigma)),$$  
and therefore  
$$\pi \delta(a a \sigma) = - '\!(a a \sigma)\!\!\pi \delta(a F a \sigma) = - '\!(a a \sigma)\!\!d(F a \sigma)$$  
$$= a(a a \sigma)d(F a \sigma) = a a \sigma,$$
which is (14.5).  

If we lower all dimensions in $\text{St} a$ and $\text{St}_1 a$ one unit, they become isomorphic with the augmented complexes $B_2, B_1$. If we identify each with their isomorphs, $a \delta$ merely goes over into $d$. Since both $\theta, \delta$ induce isomorphisms of the homology groups this holds also for $d$. It follows that $a \delta$ induces isomorphisms of the homology groups of $\text{St} a$ with the corresponding groups of $\text{St}_1 a$. Hence this holds also for $\pi \delta$ (14.5). This proves (14.4) and hence also (14.1).  

15. Invariance of certain dimensional numbers. The classical results to be proved below will establish the identity of certain "combinatorial" dimensional numbers with corresponding dimensions in the sense of (I, 15.1). Since the latter are topologically invariant so will be the former. In each case the topological invariance was first proved by L. E. J. Brouwer (around 1910) and the identification with topological dimensions was made later by Menger and Urysohn.  

Let the dimension in the sense of (I, 15.1) be called temporarily topological. Thus an Euclidean space $E^n$, an $n$-cell $E^n$, a parallelop/signup $P^n$ have a combinatorial dimension namely $n$, and in addition a topological dimension. Similarly
an Euclidean complex $K$, or an open subcomplex $K - L$, have a combinatorial
dimension, namely as complexes, in the sense of (III, 1.1) and in addition there
are the topological dimensions of $|K|$, $|K - L|$, which we call temporarily
the topological dimension of $K$, $K - L$.

(15.1) **Theorem.** The combinatorial dimension of an open or closed Euclidean
complex $K$ is equal to its topological dimension and hence it is a topological
invariant. As a consequence the topological dimension of an $n$-cell or $n$-parallelotope
is precisely $n$.

(15.2) **Theorem.** A region $\Omega^n$ of an $\mathbb{C}^n$ cannot be mapped topologically on
an $\Omega^m$, $m < n$.

(15.3) **Theorem.** No $\Omega^n$ can be represented in one-one bicontinuous manner
by less than $n$ parameters.

(15.4) **Theorem.** The combinatorial and topological dimensions of an open
or closed Euclidean complex and hence of an $n$-cell, an Euclidean space, a parallelope
are the same.

(15.5) **Corollary.** The (topological) dimension of the Hilbert parallelope
$P^n$ is infinity.

Let $K$ be an $n$-complex, $x$ a point on a $\sigma^n \in K$, $R^p(x)$ the $p$th rational Betti
number for the cycles through $x$. Here St $\sigma(x) = \sigma^n$, hence the groups of
St $\sigma(x)$ are those of a single element. Thus $|K|$ is $n$-cyclic at $x$, and so $R^n(x) = 1$.
On the other hand for every $x \in |K|$, dim St $\sigma(x) \leq n$, and so $R^p(x) = 0$ for
$p > n$. Thus $n$ is the largest index for which some $R^p(x) \neq 0$. Since the $R^p(x)$
are topologically invariant so is $n$. This proves (15.1).

Let now $x \in \Omega^n$. There is a $\sigma^n$ between $x$ and $\Omega^n$ and so the groups at $x$ in
$\Omega^n$ are the same as for $\sigma^n$. Therefore as above $\Omega^n$ is $n$-cyclic at all points.
Since this property is topologically invariant $\Omega^n$ cannot be $m$-cyclic at all points
and so (15.2) holds. As for (15.3) it is a direct corollary of (15.2).

Before proceeding, we recall the following results which we borrow from
Menger's work: *Dimensionstheorie* (Springer, 1928) (see also Hurewicz-Wallman
[H–W]):

(15.6) Let $\mathcal{R}$ be a separable metric space. Then:

(a) $\dim \mathcal{R}$ as defined above is the same as the Menger-Urysohn dimension
(Menger, p. 157; [H–W, 66]; Menger's closed sets are readily replaced by open
sets);

(b) $A \subset \mathcal{R} \rightarrow \dim A \leq \dim \mathcal{R}$; (Menger, p. 81; [H–W, 26]);

(c) If $U$ is a neighborhood of $x \in \mathcal{R}$ then the dimension of $U$ at $x$ is the same as
the dimension of $\mathcal{R}$ at the point (immediate consequence of the Menger-Urysohn
definition of the dimension).

**Proof of** (15.4, 15.5). Suppose first $K$ Euclidean and finite. The stars
of the vertices of $K^{(p)}$ make up a finite open covering $U_p$ of $|K|$ with $K^{(p)}$ for
nerve. Since \(|W|\) is cofinal in the family of all the finite open coverings of \(|K|\) and \(\dim K^{(p)} \leq n\), necessarily \(\dim |K| \leq n\). Since \(|K|\) is \(n\)-cyclic at the points of the \(\sigma^2\), \(\dim |K| \geq n\), and so \(\dim |K| = n\).

Suppose now \(K\) infinite. Since \(K\) is locally finite every point \(x\) has a neighborhood whose closure is a finite polyhedron of dimension less than or equal to \(n\). Hence by (15.6bc) and the result just proved the dimension at \(x\) is at most \(n\); so \(\dim |K| \leq n\), then as above \(\dim |K| = n\).

Let now \(K - L\) be an open Euclidean \(n\)-complex. We may suppose \(K = \text{Cl} (K - L)\), and so \(\dim |K| \leq n\), hence by (15.6b): \(\dim |K - L| \leq n\), and again as before \(\dim |K - L| = n\).

Since \(\sigma^n, P^n, \mathcal{Q}^n\) may be covered with an Euclidean \(n\)-complex their dimension is \(n\). Since \(P^n\) contains a \(P^n\) for every \(n\) its dimension exceeds every \(n\). Thus (15.4), (15.5) are proved.

16. (16.1) *Let \(K - L, K_1 - L_1\) be open Euclidean complexes and \(T\) a mapping \(|K| \rightarrow |K_1|\) such that \(T\) is a topological mapping of \(|K - L|\) onto \(|K_1 - L_1|\) and that \(T|L| \subseteq |L_1|\). If \(K\) is an \(n\)-circuit mod \(L\) then \(K_1\) is an \(n\)-circuit mod \(L_1\), and if the first is orientable or simple so is the second.*

We designate as before by \(\gamma, \gamma'\) adherent combinatorial and geometrical cycles in the complexes. We also designate by \(\sigma, \sigma_1\) the simplexes of \(K - L, K_1 - L_1\).

We recall (II, 24) that \(K - L\) is an open \(n\)-complex such that: (a) \(\gamma^n = \sum \sigma^n_1\) is an \(n\)-cycle mod \((L, 2)\); (b) no proper closed subcomplex of \(K - L\) contains such a cycle. By (15.1) then \(\dim (K_1 - L_1) = n\) also, and by (13.4), \(K_1\) contains an \(n\)-cycle mod \((L_1, 2)\), \(\gamma_1^n = T' \gamma^n\), where \(\gamma_1^n = \sum g_i \sigma^n_1, g_i = 0, 1\).

Suppose that \(\gamma_1^n\) is in a proper closed subcomplex \(M_1\) of \(K_1 - L_1\). This will certainly be the case if in the expression of \(\gamma_1^n\) there is missing a simplex \(\sigma^n_1\) of \(K_1 - L_1\), since \(\gamma_1^n\) will then be in \(M_1 = K_1 - L_1 - \sigma^n_1\) which is a proper closed subcomplex of \(K_1 - L_1\). Be it as it may, if \(M_1\) exists as stated, there is a simplex \(\sigma_1 \epsilon K_1 - L_1\) such that \(\text{St} \sigma_1 \cap M_1 = \emptyset\). Consequently \(\gamma^n\) will be in a set which does not contain a certain open set \(U\) of \(|K - L|\). It follows that the coordinate \(\gamma_p^n\) of \(\gamma^n\) in a certain derived \((K^{(p)} - L^{(p)})\) will not contain any element in some star, and hence will lack some \(n\)-simplex of \((K^{(p)} - L^{(p)})\). On the other hand if \(\delta\) denotes chain-derivation, \(\delta \gamma^n\) contains all the \(n\)-simplexes of \((K^{(p)} - L^{(p)})\) and so \(\delta \gamma^n = \gamma_p^n\), and hence \(\delta \gamma^n \neq \gamma_p^n\) since \(n = \dim (K^{(p)} - L^{(p)})\). Since derivation does not alter the homology groups, \(K - L\) must contain a cycle mod \((L, 2), \gamma^n\), different from \(\gamma^n\). In view of (a) this can only be if \(\gamma^n = \sum \sigma^n_1\), where the \(\sigma^n_1\) do not include all the \(\sigma^n_1\). Therefore \(\gamma^n\) fulfills condition (a) in \(K_1 - L_1\). Moreover no such cycle may be in a proper closed subcomplex of \(K_1 - L_1\); so the latter satisfies (b) also. Therefore it is an \(n\)-circuit.

By (III, 24.2) \(K - L\) is orientable when and only when \(K\) contains a combinatorial integral \(n\)-cycle mod \(L\), and hence by (13.4) when and only when \(K\) contains a geometrical integral \(n\)-cycle mod \(L, \gamma^n\). When this takes place \(K_1\) contains the geometrical integral \(n\)-cycle mod \(L_1, T' \gamma^n\), and so \(K_1 - L_1\) is also orientable.
For the invariance of the simple circuit we require the

(16.2) **Definition.** A point \( x \) of the \( n \)-complex \( K - L \) is said to be regular whenever \( |K - L| \) is \( n \)-cyclic at \( x \); otherwise \( x \) is said to be singular.

By (14.1) if \( \sigma \) has a regular point every point of \( \sigma \) is regular. Therefore the aggregate of the singular points is a union of simplexes, and its closure \( S \) is a closed subcomplex of \( K - L \). We call \( S \) the singular locus of \( K - L \). Since every point of a \( \sigma \) is regular, necessarily \( \dim S \leq n - 1 \).

Now a n.a.s.c. for the \( n \)-circuit to be simple is that every \( \St \sigma^{n-1} \) be \( n \)-cyclic, i.e., that \( \dim S \leq n - 2 \). Since \( S \) and hence \( \dim S \) are invariant under \( T \), if \( K - L \) is a simple circuit so is its transform \( K_1 - L_1 \).

§3. GEOMETRIC MANIFOLDS

17. For convenience we revert in this section to the notations for combinatorial manifolds (V, §4). The complexes are thus designated again by \( X, \ldots \) and their elements by \( x, \ldots \). However, \( X, \ldots \) are now Euclidean complexes or their subcomplexes, and so the elements are Euclidean simplexes.

Let then \( Y \) be an Euclidean complex, with a closed subcomplex \( Z \) and set \( X = Y - Z \). We first prove:

(17.1) **Theorem.** Let \( X_1, Y_1, Z_1 \) be analogous to \( X, Y, Z \), and let \( T \) be a mapping \( |Y| \rightarrow |Y_1| \) such that \( T \) is a topological transformation \( |X| \rightarrow |X_1| \) and that \( T \) maps \( |Z| \subset |Z_1| \). Then: (a) if \( X \) is an \( M^n \), or an orientable \( M^n \) so is \( X_1 \); (b) if \( T \) maps topologically \( |\Cl X| \rightarrow |\Cl X_1| \), and \( X \) is an \( M^n \) with regular boundary so is \( X_1 \).

For simplicial complexes the manifold conditions are (V, 29, 34):

(17.2) every \( \St x \) is \( n \)-cyclic;

(17.3) under suitable orientations of the elements \( \sum x_i^n \) is an \( n \)-cycle mod \( Z \).

Condition (17.2) makes \( X \) an \( M^n \) which may or may not be orientable, while (17.3) makes it orientable. The supplementary conditions for a regular boundary are:

(17.4) when \( x \in \mathcal{B}X \) then \( \St x \) is \( n \)-cyclic mod \( Z \);

(17.5) \( \mathcal{B}X \) is an absolute \( M^{n-1} \).

Now owing to (14.1) condition (17.2) is equivalent to:

(17.6) \(|X| \) is \( n \)-cyclic at every point (every point is regular).

Since (17.6) is topological so is (17.2). Hence if \( X \) is an \( M^n \) so is \( X_1 \). Suppose now that \( X \) is an orientable \( M^n \), that is to say, that it satisfies (17.2, 17.3).

By (V, 29.3, 29.9) condition (17.3) merely asserts in the presence of (17.2) that (17.7) every component of \( X \) is an orientable \( n \)-circuit, and \((17.2), (17.3)\) \( \leftrightarrow \) \((17.6), (17.7)\).
Since the second pair of conditions is topological (14.1, 16.1), so is the first pair. Therefore if $X$ is an orientable $M^n$ so is $X_1$.

Suppose finally $X$ to have a regular boundary $B = \mathcal{B}X$, with $T$ topological on $|\text{Cl} \ X|$. Then necessarily $T | B| = |B_1| = |\mathcal{B}X_1|$. By what has just been proved (17.5) is fulfilled by $X_1$. As for (17.4) owing to (14.1), it makes an assertion regarding the groups at the points of $B$ relative to $|\text{Cl} \ X|$ which has obvious topological character. Therefore (17.4) holds for $X_1$ also, and hence it is an $M^n$ with regular boundary.

18. **Duality theorems.** We first consider a noteworthy complement to the duality theorems for relative manifolds. Since derivation does not alter the manifold properties we may replace $Y$ by $Y'$ and hence assume the subcomplex $Z$ normal in $Y$. Since $Y, Z, X_1 = Y - \text{St} Z$, and $Y', Z', X'$ form each a triple such as $K, L, M$ of (13.5) the homology groups of the compact cycles of $|X|$ are isomorphic with the corresponding groups of the finite cycles of $X_1$ or of $X'$. Now the former are the groups of the finite absolute cycles of $X$ (i.e., the cycles $\gamma^\delta$ such that the complex $|\gamma^\delta|$ is finite closed simplicial) with respect to bounding in a finite closed simplicial complex $\subseteq X$. Hence the homology groups of the finite cycles of $X'$ and those of the finite cycles of $X$ with respect to bounding in a finite closed simplicial complex $\subseteq X$ are isomorphic. Therefore:

(18.1) **In the duality theorems for relative manifolds** (V, 33.2cd) the homology groups of the finite cycles of $X'$ may be replaced either by those of the compact cycles of $X$, or else also whenever $Z$ is normal in $Y$, by those of the finite absolute cycles of $X$ with respect to bounding in a finite closed simplicial subcomplex of $X$.

19. We shall now consider an extension of the duality theorems in a new direction. We suppose $Y$ itself to be a finite absolute orientable $M^n$ and take a closed subset $Z$ of $|Y|$. By (5.1) $|Y| - Z$ may be covered with an Euclidean complex regular relative to $Z$.

(19.1) **The covering complex** $X$ of $|Y| - Z$ is an absolute orientable $M^n$.

It is to be shown that (17.6, 17.7) hold. Since $Y$ is an $M^n$, (17.6) holds in each point of $X$ relative to $Y$, hence also relative to $X$, since $|X|$ is open in $|Y|$. Consider now any component $X_1$ of $X$. It is a consequence of (17.6) that $X_1$ is a simple $n$-circuit. Therefore we merely have to prove $X_1$ orientable. Now orientability for $X_1$, comes down to the following: given $x_1^n, x_{2r+1} \in X_1$ is it possible to orient all the elements so that if $x_1^n x_2^{n-1} x_3 \cdots x_{2r+1}$ is any finite sequence of elements joining $x_1^n, x_{2r+1}$, in which any two consecutive are incident, then each $x_2^{n-1}$ is oppositely related (with incidence numbers $+1$ and $-1$) to $x_{2r+1}$ and $x_{2r+1}$. Denote this property by $(\alpha)$. It is a consequence of the construction of the complex $X$ that any sequence $x_1, \cdots, x_{2r+1}$ such as above consists of elements of a simplicial partition $Y_1$ of $Y$. Therefore if $(\alpha)$ fails in $X_1$, it fails also in $Y_1$. However, since $Y$ is an orientable $M^n$ so is $Y_1$. Consequently $(\alpha)$ holds in $Y_1$, hence also in $X_1$. Therefore $X$ fulfills (17.7), and so it is an orientable $M^n$.

(19.2) **The conditions remaining the same, the geometrical $p$th cohomology groups of** $X$ are isomorphic with the corresponding $(n - p)$th homology groups for the
compact cycles. Hence (V, 32.3; VII, 4.7) the cycles of $Y \mod Z$ and the compact
cycles of $X$ are quasi-dual categories.

Consider as in (VII, 13.2) the direct web of sets $\mathfrak{A} = \{A_\lambda\}$ whose elements
are the open subsets of $|X|$ with compact closures. Since the closure of $X$ is
in $Y$, $|\overline{X}|$ is compact and hence it is an $A_\lambda$, say $|X| = A_{\lambda_0}$. Since every
$\lambda < \lambda_0$, the groups of $\mathfrak{A}$ are those of $|X|$ itself. In particular, the geometrical
cohomology groups of $X$ are the same as those of its compact cocycles. By
(13.2) the latter are also the combinatorial groups of the finite cocycles of $X$.
Therefore the geometrical cohomology groups of $X$ are isomorphic with the
corresponding combinatorial groups of the finite cocycles of $X$. Since $X$ is an
absolute orientable $M^\ast$ its combinatorial $p$th cohomology groups of finite
cocycles are isomorphic with the corresponding $(n - p)$th combinatorial homology
groups of the finite cycles (V, 32.1) and this proves (19.2).

(19.3) Suppose now $Y$ to be $(p - 1, p)$-acyclic and let $\tau$ be the above isomorphism
of the groups of the compact cycles of $|X|$ with those of the finite
cocycles of $X$. Then if $\Gamma^{n-p}$ is a homology class of the former and $\Delta^{n-1}$ a
homology class of cycles of $Z$ we define their class linking coefficient as
$Lk(\Delta^{n-1}, \Gamma^{n-p}) = Lk(\Delta^{n-1}, \tau \Gamma^{n-p})$.

From (19.2) and (VII, 9.1; VI, 15.8; III, 39.3) we obtain the extension of Alex-
ander’s initial sphere duality theorem. Stated for convenience of the $n$-sphere,
it holds in fact for any finite absolute orientable $(0, n)$-cyclic geometric $M^\ast$.

(19.4) **Theorem.** Let $X$ be a topological $n$-sphere, $Z$ a closed subset of $X$, $\Gamma^\ast$ the
basic $n$-class of $X \mod Z$, $\Gamma^0$ the class of a point of $Z$. Then:

(a) for $n = 1$, or $n > 1$ and $1 \leq p < n$, the groups $\mathfrak{S}^{p-1}(Z, G)$ and
$\mathfrak{S}^{n-p}(X - Z, H)$, for the absolute cycles of $Z$ and compact cycles of $X - Z$, are
dually paired with the class linking coefficient as the multiplication;

(b) for $n > 1$ one must replace $\mathfrak{S}^0(Z, G)$, $\mathfrak{S}^0(X - Z, H)$ by $\mathfrak{S}^0(Z, G)/G \Gamma^0$,
$\mathfrak{S}^n(X - Z, H)/H \Gamma^0$ (Alexander [a], Pontrjagin [c]).

From (19.4) we deduce the duality relation for the Betti numbers mod $\pi$:

(19.5) \[ R^{n-p}(Z, \pi) = R^{n-p}(X - Z, \pi) + \delta^p - \delta^p. \]

(19.6) **Application.** The Jordan-Brouwer theorem. Let $Z$ be a topo-
logical $(n - 1)$-sphere contained in the topological $n$-sphere, $X$, $n > 1$. Then
$X - Z$ consists of two connected regions (open sets) whose common boundary is $Z$.

We reproduce essentially Alexander’s proof deduced from (19.5). Whatever
the closed set $Z$ we may cover $X - Z$ with a polyhedron $|K|$ such that $K$ is
regular relative to $Z$. The number of components of $K$ and $|K|$ are the same,
and so they are $R^p(K) = R^p(X - Z)$.

Suppose now $Z$ to be an $(n - 1)$-sphere. By (19.5) we have then
$R^p(X - Z) = R^{n-p}(Z) + 1 = 2$. Let $U_1, U_2$ be the two components. Clearly
$\mathcal{B}U_i \subset Z$. Suppose $\mathcal{B}U_1 \neq Z$ and let $a \in Z - \mathcal{B}U_1$. We can find an $(n - 1)$-cell $E^{n-1}$ between $a$ and $Z - \mathcal{B}U_1$ such that $Z - E^{n-1}$ is a closed $(n - 1)$-cell $E_{1}^{n-1}$. By the above $X - \mathcal{B}E_{1}^{n-1}$ has $\mathcal{Z}(X - \mathcal{B}E_{1}^{n-1}) = R^{n-1}(\mathcal{B}E_{1}^{n-1}) + 1 = 1$ component. Since $\mathcal{B}U_1 \cap E^{n-1} = \emptyset$, one component must be $U_1$ and $a$ must be in another component. Thus we have a contradiction proving (19.6).

For further information regarding the preceding questions, and notably the converse of (19.6), see Wilder [b].

Coupling now (19.2) with (VII, 22.3) we have Alexandroff's generalization of the so-called Phragmén-Brouwer theorem:

(19.7) Let $Y$ be a finite absolute orientable geometric $(p-1, n-p)$-acyclic $M^*$. Given two closed sets $Z_1, Z_2$ in $|Y|$ and a normal couple $(G, H)$ suppose that $\mathcal{S}^{p-1}(Z_1 \cup Z_2, G) = 0$. If $\gamma^{n-p}$ is a compact cycle of $|Y|$ with a carrier which meets neither $Z_1$ nor $Z_2$ and $\gamma^{n-p}$ is not linked with either of the two sets (in the sense of VII, 22) then $\gamma^{n-p}$ is likewise not linked with $Z_1 \cap Z_2$ (Alexandroff [a, 178]).

(19.8) Indicatrix. Let $X = Y - Z$ be a connected orientable geometric $M^*$. Since $X$ is connected it is an $n$-circuit and so it has a basic combinatorial $n$-cycle $\gamma^n$ which we identify with the adherent geometrical cycle. For the same reason if $E^n$ is an $n$-cell such that $\mathcal{B}E^n$ is a closed $n$-cell $\subset X$ then there is a basic geometrical $n$-cycle $\delta^n$ of $E^n$, or cycle of $|Y| \mod (|X| - E^n)$. Now the projection $\pi: |X| \to E^n$, or reduction mod $(|X| - E^n)$ of the cycles of $|X|$, yields a reduced cycle $\pi \gamma^n$. Since $\delta^n$ is the basic $n$-cycle of $E^n$ we have $\pi \gamma^n = \alpha \delta^n$.

Since $\mathcal{B}E^n$ is compact we show as in the proof of (13.1) that $E^n$ is contained in a finite open subcomplex of $Y$ and hence in a finite closed subcomplex $Z$. Since $X$ is connected it is an $n$-circuit and so it has a basic combinatorial $n$-cycle $\gamma^n$ which we identify with the adherent geometrical cycle. For the same reason if $E^n$ is an $n$-cell such that $\mathcal{B}E^n$ is a closed $n$-cell $\subset X$ then there is a basic geometrical $n$-cycle $\delta^n$ of $E^n$, or cycle of $|Y| \mod (|X| - E^n)$. Now the projection $\pi: |X| \to E^n$, or reduction mod $(|X| - E^n)$ of the cycles of $|X|$, yields a reduced cycle $\pi \gamma^n$. Since $\delta^n$ is the basic $n$-cycle of $E^n$ we have $\pi \gamma^n = \alpha \delta^n$.

We conclude then that $\pi \gamma^n = \pm \delta^n$. Suppose $\delta^n$ given. Replacing $\gamma^n$ if need be by $-\gamma^n$ we may so choose it that $\pi \gamma^n = + \delta^n$. Thus an assigned pair $(E^n, \delta^n)$ may serve to select one of the two basic $n$-cycles $\pm \gamma^n$ for $M^*$. The following terms are used for obvious reasons:

$(E^n, \delta^n)$ = an oriented $n$-cell;  
$(M^*, \gamma^n)$ = an oriented manifold;

an oriented $n$-cell utilized to determine $\gamma^n$ as above, i.e., to orient $M^n$, is called an indicatrix of $M^n$.

20. Invariance of intersections.

(20.1) Let first $X$ be a connected finite absolute orientable $M^n$ and let $\gamma^n$, $\Gamma^n$ be its basic cycle and class (V, 36). Since all the coefficients at the right in (V, 36.6) are $\pm 1$, and $X$ is an orientable $n$-circuit (V, 29.9), hence cyclic in the dimension $n$, every $n$-cycle of $X$ is of the form $g\gamma^n$. Moreover $\gamma^n$ is determined to within its sign in the sense that the only other cycle having the property...
just stated is \(-\gamma\). If \(\Gamma^p\) is any other combinatorial class of \(X\), by (V, 36.12) there is a unique combinatorial class \(\Gamma'_{n-p}\) such that

\[(20.2) \quad \Gamma'_{n-p} \cdot \Gamma^n = \Gamma^p.\]

If \(\Gamma^n, \Gamma'_{n-q}\) is an analogous pair we have (V, 37.6):

\[(20.3) \quad \Gamma^n \cdot \Gamma^q = \Gamma'_{n-p} \cdot \Gamma'_{n-q} \cdot \Gamma^n.\]

Let each combinatorial class be identified with the adherent geometrical class. The combinatorial dot-intersections are then identified with the corresponding geometrical intersections (10.2). Suppose in particular that \(X_1\) is a second finite absolute \(M^n\) such that \(|X| = |X_1|\). Then \(\Gamma^n\) defines a class \(\Gamma^n_1\) for \(X_1\) such that every other integral \(n\)th homology class of \(X_1\) is a multiple of \(\Gamma^n_1\). Hence (20.1) the basic class of \(X_1\) is \(\pm \Gamma^n_1\) and by (V, 36.6) we may suppose the \(n\)-simplexes of \(X_1\) so oriented that it is \(+\Gamma^n_1\). Thus as geometrical classes the two basic classes of \(X, X_1\) will then coincide. It follows that the geometrical intersection class \(\Gamma'_{n-p} \cdot \Gamma'_{n-q} \cdot \Gamma^n\) will be the same whether determined by means of \(X\) or \(X_1\). Hence we have the following situation: Given two geometrical classes \(\Gamma^n, \Gamma^q\) if we identify them with their combinatorial images in any complex such as \(X\) there results a combinatorial intersection whose geometrical image is unique. This topological image is called the geometrical intersection of \(\Gamma^n, \Gamma^q\) and denoted by \(\Gamma^n \cdot \Gamma^q\).

\[(20.4) \quad \text{The identification of the geometrical and combinatorial classes in } M^n \text{ causes the identification of the intersections of the geometrical homology classes. Hence in particular (V, 37.7) holds also for the latter.}\]

From the topological invariance of the dot-intersections and (20.3) there comes also:

\[(20.5) \quad \text{If } T \text{ is a topological transformation } |X| \rightarrow |X_1| \text{ inducing an isomorphism } \tau \text{ of the homology groups such that } \tau \Gamma^n = \Gamma^n_1 \text{ is the basic class of } X_1 \text{ then}\]

\[\tau \Gamma^n \cdot \tau \Gamma^q = \tau(\Gamma^n \cdot \Gamma^q)\]

\[(20.6) \quad \text{Suppose now } X \text{ not connected and let its components be } \{X_i\}. \text{ Then}\]

\[\Gamma^n = \sum \Gamma^n_i, \Gamma^q = \sum \Gamma^q_i \text{ where } \Gamma^n_i, \Gamma^q_i \text{ are unique classes of } X_i, \text{ and we define}\]

\[\Gamma^n \cdot \Gamma^q = \sum (\Gamma^n_i \cdot \Gamma^q_i),\]

with the same conclusions as before.

\[(20.7) \quad \text{The modifications required for the other types of manifolds may be deduced from the above together with reference to (V, 38).}\]

\[(20.8) \quad \text{Referring also to (V, 37.9) we observe that the right side has topological character. Hence (V, 37.9) with all classes taken geometrical defines also an index } \text{KI}(\Gamma^n, \Gamma^{n-p}) \text{ which has topological character in the same sense as in (20.5). That is to say:}\]

\[(20.9) \quad \text{If } \tau \text{ is as in (20.5) then}\]

\[\text{KI}(\tau \Gamma^n, \tau \Gamma^{n-p}) = \text{KI}(\Gamma^n, \Gamma^{n-p}).\]
§4. CONTINUOUS AND SINGULAR COMPLEXES

21. Let $K = |\sigma|$ be an Euclidean complex and $l = AB$ a segment parametrized as $0 \leq u \leq 1$. The product $\mathcal{R} = l \times K$ is a polyhedral complex known as a *prism*. Let $K$ be identified with $A \times K$ and set $K_1 = B \times K$. Consider the topological mapping (translation) $T : |K| \to |K_1|$ defined by $T(0 \times x) = 1 \times x$. Evidently $\sigma \to T\sigma$, or which is the same $A \times \sigma \to B \times \sigma$ defines a chain-mapping $\tau : K \to K_1$ which is an isomorphism. By (IV, 5.6):

$$F(l \times \sigma) = B \times \sigma - A \times \sigma - l \times (F\sigma).$$

Let $C$ be any chain of $K$ and let $\partial C = l \times C$. From (22.1) follows

$$F\partial \sigma = r\sigma - \sigma - F\partial \sigma,$$

and hence

$$F\partial C = rC - C - F\partial C.$$

Therefore $\tau$ is a chain-deformation $K \to K_1$ in $\mathcal{R}$ with the deformation operator $\partial$. There is nothing surprising in this, since the whole concept of chain-homotopy has been designed to carry over the above situation.

22. It is often convenient to replace $\mathcal{R}$ by an Euclidean complex and its derived $\mathcal{R}'$ may serve for the purpose. The property to be proved presently describes an alternate method which has the double advantage of requiring no new vertices and not modifying the bases $K, K_1$.

Let $\{A_i\}$ be the vertices of $K$ arranged in some order and let $A^+_i = TA_i$.

(22.1) If $\sigma^p = A_{i_0} \cdots A_{i_p} \in K$, $i_0 < \cdots < i_p$, then $\bar{\sigma}^{p+1} = A_{i_p} \cdots A_{i_1} A_{i_0} \cdots A_{i_p}$ is contained in $\mathcal{R}$ and $\mathcal{R}_1 = \bigcup \text{Cl} \bar{\sigma}^{p+1}$ is a simplicial partition of $\mathcal{R}$.

Since $l \times \sigma$ is convex and all the vertices of $z$ are in $l \times \sigma$, we have $z \subseteq l \times \sigma \subseteq |\mathcal{R}|$. To prove that $\mathcal{R}_1$ is a partition we must show that:

(a) the simplexes of $\mathcal{R}_1$ are disjoint;
(b) every point of $|\mathcal{R}|$ is in a $\text{Cl} \bar{\sigma}$.

Let $x \in l \times \sigma^p$, $p > 0$. The segment $A_{i_p}x$ extended meets $|l \times \mathcal{R}^p|$ in a point $y$. If $x$ is common to two distinct simplexes of $\mathcal{R}_1$ in $l \times \sigma^p$ then $y$ has the same property relative to $|l \times \mathcal{R}^p|$. This reduces the proof of (a) for the simplexes of $\mathcal{R}_1$ in $l \times \sigma^p$ to the same for an $l \times \sigma^{p-1}$, and since it is trivial for $p = 0$, (a) is proved. Similarly (b) for $x$ reduces to the same for $y$, i.e., for a point in an $l \times \sigma^{p-1}$ and since (b) is also trivial for points in an $l \times \sigma^0$, it is true for all points and (22.1) follows.

Let $S$ denote the partition of (22.1). By (IV, 29.1) $S$ is a subdivision. I say that the chain-mapping $\theta$ defined as follows is the chain-subdivision associated with $S$. Namely on $K$ and $K_1 : \theta = 1$. For $\sigma^p$:

$$\theta\partial \sigma = \sum (-1)^g A_{i_0} \cdots A_{i_g} A_{i_0}^+ \cdots A_{i_g}^+.$$

At all events $\theta$ is readily verified to commute with $F$, and so it is a chain-mapping. It has also the required carrier $S$, and is the identity on the vertices as it should
be. Since $S$ does not raise dimensions this is sufficient to prove the asserted property of $\theta$ (IV, 18.1).

(22.3) Let us set now $\mathfrak{D} = \mathfrak{D}_1$. Since $\theta = 1$ on $K, K_1$ we have in $\mathfrak{R}_1$ also

$$\mathfrak{F}\mathfrak{D}_1 + \mathfrak{D}_1 \mathfrak{F} = \tau - 1.$$  

(22.4) Therefore $\tau$ is likewise a chain-deformation $K \rightarrow K_1$ in $\mathfrak{R}_1$ with associated operator $\mathfrak{D}_1$. Writing now $\mathfrak{R}, \mathfrak{D}$ for $\mathfrak{R}_1, \mathfrak{D}_1$ we may state

(22.5) With the translation $t:|K| \rightarrow |K_1|$ there may be associated a chain-deformation $\tau:K \rightarrow K_1$ in an Euclidean complex $\mathfrak{R}$ with operator $\mathfrak{D}$ such that $\mathfrak{D}_\sigma, \sigma \in K$, has all its vertices among those of $\sigma$ and $\sigma \tau$ ($\tau\sigma$ is a simplex of $K_1$).

It is hardly necessary to point out the parallel with (IV, 16.3). In point of fact it is out of (22.5) that there arose the general concept of chain-homotopy. See notably Lefschetz [e] and [L, 78].

The following homotopy properties will be required later. The notations are as in (1.1).

(22.6) If $t_1, t_2$ are mappings $A \rightarrow |K|$ such that $\sigma(t_i x), \sigma(t_2 x)$ are always incident then $t_1, t_2$ are homotopic.

For the condition of (I, 47.4) is manifestly satisfied here.

(22.7) Consider now $K$ and $K^{(n)}$. Every $\sigma_n$ is contained in a $\sigma$. Hence if every vertex of $\sigma_n$ is sent into a vertex of the simplex $\sigma$ carrying it, there is defined a simplicial chain-mapping $\tau: K^{(n)} \rightarrow K$ which is induced by a barycentric mapping $t:|K^{(n)}| \rightarrow |K|$. It may be shown as in (IV, 26.2c) for $n = 1$ that $\tau$ is a reciprocal of $\delta^n$, where $\delta$ is chain-derivation in $K$. Since $t_{\sigma_n} < \sigma_n$, by (22.6):

(22.8) There is a barycentric deformation $|K^{(n)}| \rightarrow |K|$ whose induced chain-mapping $\tau: K^{(n)} \rightarrow K$ is a reciprocal of $\delta^n$.

23. Continuous complexes.

(23.1) Let $\mathfrak{R}$ be a topological space. By a continuous complex $\mathfrak{R}$ in $\mathfrak{R}$ is meant a pair $(K, t)$ where $K$ is an Euclidean complex and $t$ a mapping $|K| \rightarrow \mathfrak{R}$. The complex $K$ is known as the antecedent of $\mathfrak{R}$. If $K = |\sigma|$ then the pairs $\sigma^* = (\sigma^*, t)$ are the continuous p-cells of $\mathfrak{R}$. By definition if the space is metric diam $\mathfrak{R} = \sup$ diam $\mathfrak{R}$. The continuous complex $\mathfrak{R} = (K^{(n)}, t)$ is the $n$th derived of $\mathfrak{R}$. More generally if $K_1$ is a simplicial partition of $K$ then $\mathfrak{R}_1 = (K_1, t)$ is called a subdivision of $\mathfrak{R}$.

(23.2) All this may be extended in an obvious way to the mappings of polyhedra and we thus obtain continuous polyhedra in $\mathfrak{R}$, their derived and subdivisions. Suppose in particular that the continuous complexes $\mathfrak{R} = (K, t)$ and $\mathfrak{R}_1 = (K_1, t_1)$ are homotopic in $\mathfrak{R}$, that is to say, that $t, t_1$ are homotopic. There exists then a continuous polyhedron $\mathfrak{P} = (l \times K, T)$ in $\mathfrak{R}$, where $l$ is the segment $0 \leq u \leq 1$, such that $T(0 \times x) = tx, T(1 \times x) = t_1 x, x \in |K|$. The elements $(l \times \sigma, T)$ of $\mathfrak{P}$ are called homotopy-cells, or deformation-cells when $l = 1$. 

(23.3) If $\mathcal{R} = (K, t)$ is a continuous complex in $\mathcal{R}$ and $t_1$ is a mapping $\mathcal{R} \to \mathcal{S}$, then $\mathcal{R}_1 = (K, t_0)$ is a continuous complex in $\mathcal{S}$ (obvious).

(23.4) Definition. Suppose the continuous complex $\mathcal{Q} = (L, t)$ in $|K|$. If $t$ is a barycentric mapping $|L| \to |K|$ we will say that $\mathcal{Q}$ is a continuous subcomplex of $K$.

(23.5) Let $\mathcal{Q} = (L, t)$ be a continuous complex in $K$ and let $\{a_i\}, \{b_j\}$ be the vertices of $K$, $L$. If the "star condition": $\{t | St b_j | \}$ refines $\{|St a_i|\}$, holds then $\mathcal{Q}$ is homotopic to a continuous subcomplex $\mathcal{Q}_1 = (L, t_1)$ of $K$, where $t_1$ is such that if $x \in |L|$ then its path is in $\sigma(x)$.

By hypothesis corresponding to $b_j$ there may be chosen a vertex of $K$, denoted by $a_i$, such that $|t | St b_j | \subset |St a_i|$. The resulting repetitions among the $a_i$ are immaterial. If $\tau = b_j \cdots b_k \in L$ then $t \tau \subset t | St b_j \cdots b_k | = t | St b_j \cdots b_n | \subset |St a_1 \cdots a_n | \neq \emptyset$. Therefore $\sigma = a_1 \cdots a_n$ is a simplex of $K$ and so $b_j \to a_j$ defines a barycentric mapping $t_1: |L| \to |K|$. Let $\mathcal{Q}_1 = (L, t_1)$. Since $t \tau \subset |St a_i\}$, (23.5) is a consequence of (22.6).

From (23.5) we deduce the following well known proposition which until recently provided the only method for proving the invariance of the homology groups of Euclidean complexes. (See [L, 86].)

(23.6) Theorem. Every finite continuous complex $\mathcal{Q}$ on the polyhedron $|K|$, $K$ finite, has a derived $\mathcal{Q}^{(n)}$ which is homotopic with a continuous subcomplex of $K$ after the manner of (23.5) (Alexander-Veblen).

For $n$ may be chosen such that mesh $\{t | St b_j | \} < \text{Lebesgue number} \{|St a_i|\}$, and then $\mathcal{Q}^{(n)}$ will satisfy the star condition.

For general Euclidean complexes we have:

(23.7) Let $K = |\sigma|$ be an Euclidean complex and $\mathcal{Q} = (L, t), L = \{\tau\}$, a continuous complex in $|K|$, such that at most a finite number of $\tau$ meet a given $\sigma$, and conversely. Then $\mathcal{Q}$ has a subdivision which is homotopic with a continuous subcomplex of $K$ after the manner of (23.5).

If $St b$ is the star of the vertex $b$ in $L$ then some finite closed subcomplex $K_b$ of $K$ is such that $(Cl St b, t) \subset |K_b|$, and so for some $n$ the star condition will be fulfilled by $(Cl St b)^{(n)}$ relative to $K_b$. Every $\tau \in L$ will be such that $t \tau$ is in at most a finite number of $|K_b|$ and so $\tau$ will undergo at most a finite number of derivations. Moreover $t_1 < \tau$ and $t \tau \subset |K_b| \Rightarrow t_1 \subset t \tau \subset |K_b|$. Hence $\tau_1$ will undergo at least as many derivations as $\tau$. Applying all the operations indicated for every $K_b$ there will result a polyhedral complex $L_1$ which is a partition of $L$ and such that every star of a vertex of $L_1$ is imaged by $t$ in a star of a vertex of $K$. Therefore the derived $L_1$ is an Euclidean complex which is a subdivision of $L$ such that the star condition holds relative to $(L_1, t)$ and $K$, and the application of (23.5) to $(L_1, t)$ yields (23.7).

The preceding results may be utilized for convenient approximations. For the sake of simplicity we will only consider finite complexes.
Let \( t \) be a mapping \( L \to K \), \( K \) and \( L \) finite. Then: (a) there exists an \( n \) such that \( t \) is homotopic to a barycentric mapping \( L^{(n)} \to K \); (b) given any \( \varepsilon > 0 \) there exist \( m, n \) such that \( t \) is \( \varepsilon \) homotopic to a barycentric mapping \( L^{(n)} \to K^{(m)} \) (Alexander [b1]).

Property (a) is merely another formulation for (23.6). Regarding (b) choose \( m \) so high that mesh \( K^{(m)} < \varepsilon \). By (23.6) we may choose \( n \) so high that \( (L^{(n)}, t) \) is homotopic with a continuous complex \( (L^{(n)}, \omega) \) after the manner of (23.5). The path of any \( x \in L \) under the homotopy is a segment on a closed simplex of \( K^{(m)} \) and hence its diameter is less than \( \varepsilon \). Since otherwise \( \omega \) conforms with (b) the latter is proved.

23a. Homotopy groups. While these important groups lie wholly outside our program, a few words concerning them will not be amiss.

23a.1 The Poincaré group. Assume \( \mathcal{R} \) arcwise connected (any two points may be joined by an arc). We first introduce the paths in \( \mathcal{R} \) and a certain law of composition between them. A path is merely a continuous one-complex \( \lambda = (l, t) \). The points \( x = t(0) \) and \( x' = t(1) \) are known as the initial and terminal points of \( \lambda \), both as the end points of \( \lambda \). We also say that \( \lambda \) joins \( x \) to \( x' \). If \( \lambda' = (l, t') \) is a second path with initial point \( x' \) and terminal point \( x'' \) then \( \lambda'' = (l, t') \) with \( t' \) defined by

\[
\begin{align*}
t''(u) &= t(2u), & 0 \leq u \leq 1/2; \\
t''(u) &= t'(2u - 1), & 1/2 \leq u \leq 1,
\end{align*}
\]

is likewise a path joining \( x \) to \( x'' \) and denoted by \( \lambda' \lambda \). The path \( (l, t^*e) \), where \( t^*(u) = t(1 - u) \) whose end points are those of \( \lambda \) interchanged is denoted by \( \lambda^{-1} \) and called the inverse of \( \lambda \).

Let \( \mathcal{M} = \{ \mu \} \) be the paths whose end points are both the fixed point \( x_0 \). If \( \mu, \mu' \) are in the set so are \( \mu \mu' \) and \( \mu^{-1} \). Hence our laws of composition and inversion may be applied unrestric tedly to \( \{ \mu \} \). Given \( \mu = (l, t) \) in \( \mathcal{M} \) we will write \( \mu \sim 1 \) if there is a homotopy of \( t \) with the mapping \( \omega : l \to x_0 \) such that in all "intermediate" positions the path remains in \( \mathcal{M} \). Given \( \mu, \mu' \) in \( \mathcal{M} \) we will write \( \mu \sim \mu' \) if \( \mu^{-1} \mu' \sim 1 \). It is easy to see that \( \sim \) is an equivalence relation and that the equivalence classes with the previously introduced laws of composition form a group. It is denoted by \( \pi_1(\mathcal{R}) \) and known as the Poincaré group (also fundamental group or group of paths) of \( \mathcal{R} \) and is independent of \( x_0 \) (to within an isomorphism).

Not only is the Poincaré group generally noncommutative, but it is not too much to say that all the significant noncommutative groups ever discussed in topology have been \( \pi_1(\mathcal{R}) \) or its derivates. It is not surprising therefore that ignorance regarding this group seems to account for the fact that many of the major problems of topology have so far eluded all attempts at solution.

23a.2 Covering manifold. To simplify matters let \( \mathcal{R} \) be a topological \( n \)-manifold \( \mathcal{M} \) (44.1). We define a new space \( \mathcal{R}^n = \{ x \} \) as follows. Let \( N = \)}
\{v\} be the paths with \(x_0\) as initial point. For \(v, v'\) in the set we will write \(v \sim v'\) if \(v, v'\) are coterminai and \(v^{-1}v' \sim 1\). This relation is readily seen to be an equivalence and the resulting equivalence classes are the points \(y\) of \(\mathcal{M}^n\). If \(\nu \in y, x\) is the terminal point of \(\nu\), and \(E\) is an \(n\)-cell containing \(x\), then the classes of the paths \(\lambda \nu\) where \(\lambda = (l, t) \) has \(x\) as initial point and \(tl \subset E\) make up a neighborhood \(\mathcal{G}\) in \(\mathcal{M}^n\) and \(\{\mathcal{G}\}\) is chosen as a base for \(\mathcal{M}^n\). It is not difficult to verify that \(\mathcal{M}^n\) is likewise a topological \(M^n\), and is topologically independent of \(x_0\). It is called the universal covering manifold of \(M^n\).

If \(\nu \in y\) and \(x\) is the terminal point of \(\nu\) then the mapping \(\rho : y \to x\) is easily verified to be "locally" topological.

(23a.3) The homotopy groups of Hurewicz. A noteworthy generalization of the Poincaré group has been given by Hurewicz [a]. Let \(\mathcal{S}\) be an arcwise connected compactum, and in addition locally contractible. (This property means that given any \(\epsilon > 0\) there is a corresponding \(\eta > 0\) such that \(\mathcal{S}(x, \eta), x \in \mathcal{S}\), is deformable to a point in \(\mathcal{S}(x, \epsilon)\).) Take now a fixed Euclidean \((n - 1)\)-sphere \(S^{n-1}\), \(n > 1\), and let \(x_0, y_0\) be fixed points of \(\mathcal{S}, S^{n-1}\). Consider all the mappings \(t : S^{n-1} \to \mathcal{S}\) such that \(ty_0 = x_0\). If \(t, t'\) are two such mappings and we set \(d(t, t') = \sup \{|d(ty, ty')| y \in S^{n-1}\}\), it is readily seen that \(d(t, t')\) metrizes \(|t|\). The resulting space has arcwise connected components and the Poincaré group of the component containing the identity mapping is known as the \(n\)th homotopy group of \(\mathcal{S}\), written \(\pi_n(\mathcal{S})\). The first homotopy group is \(\pi_1(\mathcal{S})\). We will merely recall the following basic properties of the homotopy groups, both due to Hurewicz.

(23a.4) The groups \(\pi_n(\mathcal{S}), n > 1\), are abelian.

(23a.5) If \(n > 1\) and the groups \(\pi_n(\mathcal{S}), q < n\), reduce to the identity, then \(\pi_n(\mathcal{S})\) is isomorphic with the homology group \(H_n(\mathcal{S})\) of the finite singular \(n\)-cycles (25) of \(\mathcal{S}^n\).

It is also an elementary matter to prove:

(23a.6) \(H^1(\mathcal{S})\) is isomorphic with the commutator group of the Poincaré group \(\pi_1(\mathcal{S})\).

For further details regarding the homotopy groups and their applications the reader is referred to Hurewicz [a] and also to a recent comprehensive paper by Eilenberg [UM, pp. 57-99].

24. Singular elements. The singular elements to be introduced presently are very useful wherever homology and homotopy occur together.

By a singular \(p\)-cell in a metric space \(\mathcal{S}\) is meant a pair \(E^p = (\sigma^p, t)\) where \(\sigma^p\) is an Euclidean simplex and \(t\) a mapping \(\bar{\sigma}^p \to \mathcal{S}\). The convention is made that if \(s\) is a barycentric mapping \(\bar{\sigma}^p \to \bar{\sigma}^p\) then we still have \(E^p = (\sigma^p, ts) = (s^{-1}\sigma^p, ts)\). It is a consequence of the preceding definition that if \(\sigma^q < \sigma^p\) then \(E^q = (\sigma^q, t)\) is likewise a singular \(q\)-cell; it is called a \(q\)-face of \(E^p\). We shall also write correspondingly \(E^q < E^p\). The simplex \(\sigma^p\) is said to be an antecedent of \(E^q = (\sigma^q, t)\).

We have just assigned dimensions and incidences in \(\Sigma = \{E\}\). To make it a complex there remains to assign suitable incidence numbers. First of all, we
now assume in \( E^p = (\sigma^p, t) \) that \( \sigma^p \) is oriented (with vertices taken in a definite order). We also agree to designate \((-\sigma^p, t)\) by \(-E^p\). We may now define the finite singular chains over a group \( G \) as the linear forms

\[
(24.1) \quad C^p = g^iE_i^p, \quad g^i \text{ an integer},
\]

with the restriction that

\[
g(-E^p) = (-g)E^p.
\]

If we have

\[
(24.2) \quad F\sigma^p = \eta^i\sigma_i^{p-1}, \quad E_i^{p-1} = (\sigma_i^{p-1}, t),
\]

then we define \( FE^p \), the boundary of \( E^p \), as the singular chain

\[
(24.3) \quad FE^p = \eta^iE_i^{p-1},
\]

and \( FC^p \), where \( C^p \) is (24.1), as:

\[
(24.4) \quad FC^p = g^iFE_i^p.
\]

Since \( F(-E^p) = -FE^p, FC^p \) thus defined is unique.

The incidence number \([E^p:E_i^{p-1}]\) is now defined as the coefficient of \( E_i^{p-1} \) in (24.3) (after terms are collected). To prove that \( \Sigma \) is a complex there remains to show that (III, 1, K4):

\[
(24.5) \quad \sum_{E'} [E:E'][E':E''] = 0.
\]

Since \( E \) has at most a finite number of faces, (24.5) is equivalent to \( FEFE^p = 0 \), which under our definitions is an immediate consequence of the known relation \( FF\sigma^p = 0 \). Therefore \( \Sigma \) is a closure-finite complex.

If \( E = (\sigma, t) \) is a singular cell in \( \mathcal{R} \) then \( \tau \sigma \) is a subset of \( \mathcal{R} \) which depends solely upon \( E \) but not upon its representation. We denote \( \tau \sigma \) by \( |E| \), and these sets have just the properties required to make \( \Sigma \) a metric complex as defined in (VI, 24.1). This metric complex is known as the complete singular complex of \( \mathcal{R} \). The subcomplexes, chains, cycles of \( \Sigma \) are referred to as singular complexes, chains, cycles, of \( \mathcal{R} \).

25. There are then two possible homology theories to be considered in relation to \( \Sigma \). They correspond to:

(a) the finite singular chains and cycles;

(b) the \( V \)-cycles of \( \Sigma \), referred to as \( VS \)-cycles.

The groups for the two types are to be chosen discrete throughout. In point of fact the \( VS \)-cycles are only interesting as auxiliaries and their groups are generally reducible (in the interesting cases) to those of the finite cycles.

(25.1) That the homology groups resulting for instance from (a) are not necessarily the Vietorius groups is shown by the following example. The space \( \mathcal{R} \) is a planar set which is the union of the following three sets: a segment \( \lambda_1: x = 0, 1 \leq y \leq 1 \); the arc \( \lambda_2: y = \sin(1/x), 0 < x < 1 \); an arc \( \lambda_3 \) joining
(1,0) to (0, -1) but otherwise not meeting \( \lambda_1 \cup \lambda_2 \). The Betti number for the rational Vietoris one-cycles is readily found to be \( R^1 = 1 \). On the other hand the fact that all the "closed" curves in the set are homotopic to points enables one to prove that for the finite rational singular one-cycles we have \( R^1 = 0 \). Thus the rational Betti numbers for the finite singular one-cycles and those for the Vietoris cycles are distinct.

(25.2) If \( A \) is a subset of \( \mathcal{R} \), and \( E, \mathcal{R}, C \) are a singular cell, complex or chain, then: \( E \subset A \), \( \cdots \) signifies: \( \left| E \right| \subset A \), \( \cdots \). In particular when \( A \) is closed and \( E \subset A \) then \( E' < E \Rightarrow \left| E' \right| \subset \left| E \right| \subset A \). Hence \( \Sigma_A = \left| E \right| E \subset A \) is a closed subcomplex of \( \Sigma \). The singular cycles, \( \cdots \mod A \) are those of \( \Sigma \mod \Sigma_A \).

(25.3) Groups related to a point. Since a point \( x \) is closed the \( E \)'s in \( x \) form a closed subcomplex \( \Sigma_x \). The \( VS \)-cycles around \( x \) are known as singular cycles around \( x \). At the same time one may also define singular cycles through \( x \), as follows: a singular \( n \)-cycle \( \gamma^n \) through \( x \) over a discrete \( G \) is a singular chain over \( G \) such that \( x \in \left| FC^n \right| \); the cycle \( \sim 0 \) whenever there are singular cycles \( C^n, D^n, x \in \left| D^n \right|, \) such that \( FC^n + \gamma^n = D^n \). The groups \( \mathcal{S}_n, \mathcal{S}_n, \mathcal{S}_n \) are then defined in the customary way.

26. We will now describe a certain number of results with proofs merely outlined or even omitted in the simpler cases.

(26.1) If \( \mathcal{R}, \mathcal{S} \) are metric spaces and \( T \) is a mapping \( \mathcal{R} \rightarrow \mathcal{S} \), then \( T \) induces a homomorphism of the homology groups of the finite singular cycles of \( \mathcal{R} \) into the corresponding groups of \( \mathcal{S} \).

(26.2) The homology groups of the finite singular cycles are topologically invariant.

(26.3) Let \( \mathcal{R} = (K, t) \) be a continuous complex in \( \mathcal{R} \), where \( K = \{ \sigma \} \). Then \( \mathcal{R}_1 = \{ (\sigma, t) \} \) is a singular complex said to be induced by \( \mathcal{R} \).

(26.4) Every singular complex \( \mathcal{R}_1 \) is induced by some continuous complex \( \mathcal{R} \).

If \( \mathcal{R}_1 = \{ E_i \} \) we may set \( E_i = (\sigma_i, t_i) \) where the \( \text{Cl} \sigma_i \) are disjoint. Then if \( K = U \text{Cl} \sigma_i \), and \( t: \left| K \right| \rightarrow \mathcal{R} \) is defined by \( t|_{\sigma} = t_i \), we have in \( \mathcal{R} = (K, t) \) a continuous complex inducing \( \mathcal{R}_1 \).

(26.5) By taking operations on \( \mathcal{R} \) such that if say \( \sigma_1, \cdots, \sigma \) are antecedents of \( E \in \mathcal{R}_1 \) in \( \mathcal{R} \) then the effect of the operations is independent of the permutations of \( \sigma_1, \cdots, \sigma \), there may be introduced corresponding operations in \( \mathcal{R}_1 \). Thus we may define chain-derivation \( d \) in \( \mathcal{R}_1 \), as well as other types of subdivisions, likewise singular chain-homotopy, etc.

Let \( \mathcal{R} = \{ E^n \} \) be a singular complex, \( \mathcal{R} = \{ E^n \} \) its \( n \)th derived, \( \delta \) chain-derivation in all simplicial complexes, \( d \) singular chain-derivation in \( \mathcal{R} \).

(26.6) \( d^n \) is a singular chain-deformation \( \mathcal{R} \rightarrow \mathcal{R}^n \) with operator \( \mathcal{D} \) such that \( \left| \mathcal{D} E \right| \subset \left| E^n \right| \).

Since (26.6) is obtained by repetition from (26.6), we only need to prove the latter and so suppose \( n = 1 \). Let \( l \) be the segment \( 0 \leq u \leq 1 \) and suppose \( E = (\sigma, t) \). Consider the mapping \( T: l \times (\bar{\sigma}) \rightarrow \mathcal{R} \) such that \( T(l \times x) = tx, \ x \in \bar{\sigma} \). Apply to \( l \times (\text{Cl} \sigma) \) a simplicial partition which differs only from a bary-
centric subdivision in that no new vertices are introduced in \(0 \times \text{Cl} \sigma\), and let \(\theta\) be the induced chain-subdivision. We have \(\theta(0 \times \sigma) = 0 \times \sigma\), \(\theta(1 \times \sigma) = \delta(1 \times \sigma)\). Hence the singular images of \(\theta(0 \times \sigma)\) and \(\theta(1 \times \sigma)\) under \(T\) are, respectively, \(E\) and \(dE\). We have now the relation (22.1) in \(l \times \text{Cl} \sigma\). Applying \(\theta\) to both sides, taking the singular images under \(T\), and denoting in particular by \(\mathcal{D}E\) the singular image of \(\theta(l \times \sigma)\), we obtain \(\mathcal{D}F + F\mathcal{D} = d - 1\). It is readily seen that \(\mathcal{D}E\) depends solely upon \(E\), and not upon the particular representation \((\sigma, l)\) chosen for \(E\). Thus \(d\) is a singular chain-deformation. Since \(T(l \times \sigma) = ||\mathcal{D}E|| \subset T\sigma\), (26.6) is proved.

(26.7) The notations remaining the same suppose \(\mathfrak{R}\) finite and contained in the polyhedron \(K\), \(K = \{\sigma\}\). Then

(a) for \(n\) above a certain value there is a singular chain-deformation \(\eta: \mathfrak{R}^{(n)} \to K\) with operator \(\mathcal{D}\) such that \(||E_{n1}|| \subset \sigma \to ||\mathcal{D}E_{n1}|| \subset \sigma\);

(b) there is likewise a singular chain-deformation \(\theta: \mathfrak{R} \to K\) with the property that if \(L\) is a closed subcomplex of both \(K\) and \(\mathfrak{R}\) then \(\theta \mid L = 1\);

(c) we also notice explicitly that mesh \(\mathfrak{R}^{(n)} \to 0\).

Let \((K_1, l)\), \(K_1 = \{\xi\}\), be an antecedent of \(\mathfrak{R}\). Since \(K_1\) is finite \(l \mid K_1\) = \(|\mathfrak{R}|\) is compact. The open covering of this compact set by the intersections with the stars of the vertices of \(K\) has a finite subcovering. Hence \(\mathfrak{R}\) is contained in a finite set of the stars and so in a finite closed subcomplex of \(K\). Thus we may assume \(K\) finite. Under the circumstances by (23.6) there is an \(n\) such that (23.5) may be applied to \((K_1^{(n)}, l)\). Thus we have a mapping \(T: l \times K_1^{(n)} \to K\) corresponding to the homotopy of (23.5) as applied here. If we take now a simplicial partition of \(l \times (K_1^{(n)})\) similar to the derived but with no new vertices in \(0 \times K_1^{(n)}\) or \(1 \times K_1^{(n)}\), and pass to the singular images, the mappings of the bases \(0 \times K_1^{(n)}, 1 \times K_1^{(n)}\) of \(l \times K_1^{(n)}\) induce the chain-mappings \(I, \eta\), and the image of \((l \times \iota_{n1}'; T)\) subdivided is a singular chain \(\mathcal{D}E_{n1}\). If \(K_1^{(n)} = \{\iota_{n1}\}\) and \(\iota_{n1}', \iota_{n1}''\) are two antecedents of \(E_{n1}\) we find readily that \((l \times \iota_{n1}', T) = (l \times \iota_{n1}'', T)\), where the parentheses represent the singular chains obtained from the subdivision of the cells. Thus \(\mathcal{D}E\) is shown as in the proof of (26.6) to yield a suitable \(\mathcal{D}\) making \(\eta\) a singular chain-deformation. If \(||E_{n1}|| \subset \sigma\), then by (23.5): \(T(l \times \iota_{n1}') \subset \sigma\), and since \(||\mathcal{D}E_{n1}|| = T(l \times \iota_{n1}')\), the rest of (26.7a) follows.

If we set \(\theta = \eta d^n\) then from (26.6, 26.7a) we deduce that \(\theta\) is a singular chain-deformation \(\mathfrak{R} \to K\). If \(L\) is as in (26.7b) then \(\eta \mid L^{(n)} = \tau\), a reciprocal of \(\delta^*\), and \(d^n = \delta^*\). Hence \(\theta \mid L = \tau d^n = 1\). This completes the proof of (26.7b).

(26.8) We will now apply (26.6, 26.7) to the reduction of the finite singular cycles of \(K\) to those of \(K\) itself. If \(\gamma^p\) is such a cycle its class will be written \(T^p\). If \(\sigma^p\) is a cycle of any finite Euclidean complex then we agree to identify its combinatorial class and geometrical class, with a common designation \(\Delta(\gamma^p)\).

If \(\delta\) denotes chain-derivation in \(K\) and \(\tau\) is a reciprocal of \(\delta\) then \(\Delta = \Delta\tau\), and since \(\tau \delta \approx 1\), likewise \(\Delta\tau = \Delta\tau \delta = \Delta\).
Taking then $\gamma$ in $|K|$ by (26.7b), there is a singular chain-deformation $\theta_n: |\gamma| \to K^{(n)}$.

(26.8a) $\Delta(\theta_n \gamma^p)$ depends solely upon $\Gamma^p$.

If $\mathcal{D}_n$ is the homotopy operator for $\theta_n$ then

$$F \mathcal{D}_n \gamma^p = \theta_n \gamma^p - \gamma^p \sim 0.$$ 

If $\gamma^p$ is any other cycle of $\Gamma^p$ there will be corresponding $\mathcal{D}'_n$, $\theta'_n$ and

$$F \mathcal{D}'_n \gamma^p = \theta'_n \gamma^p - \gamma^p \sim 0.$$ 

Hence $\theta'_n \gamma^p \sim \theta_n \gamma^p$ in the singular sense. Therefore there is a singular chain $C^{p+1}$ such that $F C^{p+1} = \theta'_n \gamma^p - \theta_n \gamma^p$. If we treat $C^{p+1}$ like the cycles and notice that the chain-operations involved reduce to the identity on the chains of $K^{(n)}$ (26.7b) we find that $C^{p+1}$ is chain-homotopic with a chain in $K^{(n)}$ whose boundary is the same. Hence we may assume $C^{p+1} \subset K^{(n)}$ and so $\theta'_n \gamma^p \sim \theta_n \gamma^p$ in $K^{(n)}$. This proves (26.8a).

Notice incidentally that we may assume $\gamma^p = \gamma$ and $\theta'_n$ merely any singular chain-homotopy whatever $|\gamma| \to K^{(n)}$, and the preceding argument shows that $\Delta(\theta_n \gamma^p)$ is independent of the particular chain-homotopy $\theta_n: |\gamma| \to K^{(n)}$.

(26.9) $\Delta(\theta_n \gamma^p)$ is independent of $n$.

Let this time $\tau$ denote the reciprocal of chain-derivation $\delta$ in $K^{(n)}$. Since $\tau$ is a singular chain-deformation $K^{(n+1)} \to K^{(n)}$, (22.8), $\tau \theta_{n+1}$ is a chain-homotopy $|\gamma^p| \to K^{(n)}$, and so $\Delta(\tau \theta_{n+1} \gamma^p) = \Delta(\theta_n \gamma^p)$. Since $\Delta \tau = \Delta$ we have $\Delta(\theta_{n+1} \gamma^p) = \Delta(\theta_n \gamma^p)$ which implies (26.9).

Thus $\Delta(\theta_n \gamma^p)$ depends solely upon $\Gamma^p$ and so we denote it by $\Delta(\Gamma^p)$.

(26.10) $\Gamma^p \to \Delta(\Gamma^p)$ defines an isomorphism of the corresponding groups.

At all events there is defined a homomorphism $\omega$ of the groups. If $\gamma^p$ is a cycle of $K$ then the chain-mapping identity is a singular chain-homotopy sending $\gamma^p$ into itself. Hence $\omega$ is of "onto" type. If $\gamma^p \sim 0$ in $K$ then it is also $\sim 0$ in the singular sense, and so $\omega$ is univalent. Since we are dealing with homology groups of finite cycles, which are all discrete, $\omega$ is an isomorphism.

This proves (26.10). More explicitly:

(26.11) The homology groups of the finite singular cycles of an Euclidean complex $K$ are isomorphic with the corresponding combinatorial groups. Hence in particular the latter are topologically invariant.

This result implies (10.1) and in particular the topological invariance of the Betti numbers and torsion coefficients of a finite Euclidean complex. It was essentially along these lines, i.e., by means of the singular cycles, that the invariance of the Betti numbers was first proved by Alexander [a], and that of the Betti numbers and torsion coefficients later also by Veblen [V]. (See also [L, 87; L1, X].)

An interesting complement required later is:

(26.12) Assuming for simplicity $K$ finite, the operation $\Gamma^p \to \Delta(\Gamma^p)$ has topological character with respect to a topological mapping $t$: $|K| \to |K_1|$, where $K_1$ is also finite.
Consider the continuous complex \((K, t)\) and let \(n\) be chosen in accordance with (23.6). We have then the following two associated operations:

(a) If \([a_i], [b_j]\) are the vertices of \(K^{(n)}, K_1\), and if \(K^{(n)}, K_1\) are considered as the nerves of the coverings \(\{\{\text{St } a_i\}\}, \{\{\text{St } b_j\}\}\), then the barycentric mapping \(|K^{(n)}| \rightarrow |K_1|\) of (23.5) (implicit in 23.6) induces a chain-mapping \(\theta: K^{(n)} \rightarrow K_1\), which is of the same type as the \(p^n\) of (VII, 5.12) corresponding to the mapping \(t\). Since the combinatorial classes of \(K^{(n)}, K_1\) are identified with the corresponding geometrical classes, and since \(t\) is topological, \(\theta\) induces a mapping of \(\Delta(\Gamma^n)\) into its image class \(t\Delta(\Gamma^n)\) in \(K_1\).

(b) The second operation referred to is the homotopy \(T:\ (K^{(n)}, t) \rightarrow |K_1|\) of (23.6).

Take as a representative of \(\Gamma^n\) a cycle \(r^n\) of \(K^{(n)}\). Then we verify immediately that the singular chain-deformation \((K^{(n)}, t) \rightarrow K_1\) induced by \(T\) sends \(tr^n\) into \(\theta r^n\). Thus if \(\Gamma^n\) denotes the class of \(tr^n\) then \(\Delta(\Gamma^n)\) is the class of \(\theta r^n\) in \(K_1\). Since this is the same as \(t\Delta(\Gamma^n)\), (26.12) is proved.

(26.13) Relative cycles. All the preceding results hold with minor modifications for the cycles of \(K \mod L, L\) a closed subcomplex of \(K\). Of course in (26.12) \(t\) must be such that \(t | L = | L_1 |, L_1\) a closed subcomplex of \(K_1\).

(26.14) Cycles through the points. Essentially the same arguments are valid for these cycles. As in (14) one may reduce the treatment to the case of a vertex \(a\). Care will merely have to be taken not to send into \(a\) any singular vertex different from \(a\). This yields a new proof of (14.1) and in particular implies also:

(26.15) The homology groups of the singular cycles through the points of \(|K|\) are isomorphic with the corresponding geometrical groups.

27. Intersections of singular chains in a manifold. The questions which we shall now consider are the last dealing with intersections in the present work. It is interesting to observe that they were the first investigated by the author in his initial paper on intersections [a]. This is in keeping with the development of topology since that time: the algebraic properties are now completely to the fore, and the more special topological properties are dealt with afterwards.

(27.1) Let again \(X\) be the manifold of (20). Given a finite singular cycle \(\gamma^p\) in \(|X|\) consider the associated closed set \(P = ||\gamma^p||\) and let \(U(\epsilon) = \oplus(P, \epsilon)\). By (4.1) corresponding to a given \(\epsilon > 0\) there is an \(s\) such that if \(Y_s\) is the union of the closures of the simplexes of \(X^{(s)}\) which meet \(P\) then \(|Y_s| \subseteq U(\epsilon/4)\).

If we treat now \(\gamma^p\) as in (26.8) we obtain a cycle \(\delta^p\) of \(Y_s\), such that \(\gamma^p \sim \delta^p\) in \(Y_s\), and hence also in \(\overline{U(\epsilon)}\). Moreover (26.12) the class \(\Delta^p\) of \(\delta^p\) as a cycle of \(Y_s\), and hence of \(\overline{U(\epsilon)}\) depends solely upon \(\gamma^p\). Since derivation does not alter the manifold properties we may apply to \(\delta^p\) in \(X^{(s)}\) the operation analogous to \(\xi_p\) of (V, 36). This will give rise to a cocycle \(\delta_{\alpha-p}\) such that if \(\gamma^n\) is the basic cycle of \(X\) or rather its image in \(X^{(s)}\) then, by virtue of (V, 36.13):

(27.2) \[\gamma^p \sim \delta^p \sim \delta_{\alpha-p} \cdot \gamma^n \in \overline{U(\epsilon)}\].
Finally it is a consequence of (V, 36.12) and the topological character of the intersection classes in a finite complex, that the class of $\delta'_{n-p}$, say in $U(2\varepsilon)$, also has topological character.

Thus we have assigned to $\gamma^\varepsilon$ a definite cocycle $\delta'_{n-p}$ over the same group $G$ as $\gamma^\varepsilon$, whose class in $U(2\varepsilon)$ depends solely upon $\gamma^\varepsilon$.

(27.3) Let now $G, H$ be commutatively paired to $J$, and let $\gamma^p, \gamma^q$ be finite singular cycles over $G, H$. If $Q = ||\gamma^\varepsilon||$ and $V(\xi) = \mathcal{V}(Q, \xi)$, we find $\delta'_{n-q}$, $\delta'_{n-q}$ carried by $U(2\varepsilon), V(2\varepsilon)$. We will denote the classes of $\gamma^p, \delta'_{n-p}, \delta'_{n-q}$, by $\Gamma^p, \Delta^p, \Lambda^q, \cdots$.

Since $\Gamma^\varepsilon$ is integral there is a class intersection $\Gamma^p \cdot \Gamma^q$ of $\Gamma^\varepsilon$ with a cohomology class $\Gamma^t$ over any group whatever. Hence $\Delta^p, \Delta^q \cdot \Gamma^t = \Gamma^{p+q-t}$ is a class of cycles over $J$. By (VII, 10.4, 10.6) it has a representative $\gamma^{p+q-t}$ in $W(2\varepsilon)$, where $W(2\varepsilon) = U(2\varepsilon) \cap V(2\varepsilon)$, whose class in $W(2\varepsilon)$ is unique. Hence if $\varepsilon' < \varepsilon$ we have

$$\gamma^{p+q-t} \sim \gamma^{p+q-t} \text{ in } W(2\varepsilon).$$

If $R = P \cap Q$ then $|W(\varepsilon)|$ is coterminal in the web of all the neighborhoods of $R$. Hence by (VII, 15.1) $|\gamma^{p+q-t}|$ converges to a cycle $\gamma^{p+q-t}$ in $R$, such that

$$\gamma^{p+q-t} \sim \gamma^{p+q-t} \text{ in } W(2\varepsilon),$$

for every $\varepsilon$, and its class $\Gamma^{p+q-t}$ (as a cycle of $R$) is unique. Any cycle $\gamma^{p+q-t} \in \Gamma^{p+q-t}$ is called an intersection cycle of the singular cycles $\gamma^p, \gamma^q$ and is denoted by $\gamma^p \circ \gamma^q$. The class $\Gamma^{p+q-t}$ is likewise denoted by $\Gamma^p \circ \Gamma^q$ and called the intersection class of the singular classes $\Gamma^p, \Gamma^q$. It is an elementary matter to prove:

(27.4) $\gamma^p \circ \gamma^q$ has all the properties of (V, 37.7).

(27.5) Kronecker index. If $q = n - p$ then $\gamma^p \circ \gamma^{n-q}$ is a zero-dimensional cycle, and so it has an index. We naturally define

$$\text{KI}(\gamma^p, \gamma^{n-q}) = \text{KI}(\gamma^p \circ \gamma^{n-q}).$$

(27.6) If $\gamma^p, \gamma^q$ are disjoint (i.e., $P \cap Q = \emptyset$) then $\gamma^p \circ \gamma^q \sim 0$.

(27.7) Suppose now that $C^p, C^q$ are finite singular chains over $G, H$ in $X$ and let

$$P = ||C^p||, \quad Q = ||C^q||, \quad R = P \cap Q,$$
$$P_1 = ||FC^p||, \quad Q_1 = ||FC^q||.$$

We will say that $C^p, C^q$ are in general position if

$$P \cap Q_1 \cup P_1 \cap Q = \emptyset.$$

Using (VII, 10.7) it is now a simple matter to extend the preceding results and obtain an intersection cycle $C^p \circ C^q$ in $R$ and its intersection class. Moreover if $q = n - p$ we will have an index

$$\text{KI}(C^p, C^{n-q}) = \text{KI}(C^p \circ C^{n-q}).$$
(27.8) The intersections of singular chains and cycles in a finite geometric absolute orientable $M^n = \mathbb{X}$, as well as the Kronecker index which have just been introduced, are topologically invariant with respect to any topological mapping $t: |X| \to |X_1|$, where $X_1$ is a similar $M^n$, such that $\gamma^n$ is the basic $n$-cycle of $X$ then $\gamma^n$ is the basic $n$-cycle of $X_1$.

For under the circumstances all the elements used in defining the intersections have topological character.

(27.9) Extension to any orientable $M^n$. In the first place the finiteness restriction is manifestly unimportant. However if $M^n$ and the chains are allowed to be infinite, various cases may have to be distinguished. We merely observe that if $R = P \cap Q$ is compact then so is the intersection cycle $\gamma^n \circ \gamma^i$.

Suppose now that $X = Y - Z$, i.e., that $Y$ is an orientable $M^n \mod Z$. The notations being essentially as before the set $|X|$ is covered with an Euclidean complex $X_i$ regular with respect to $Z$ after the manner of (5.1) duly extended. We show then as in (19.1) that $X_i$ is an absolute (generally infinite) orientable $M^n$ and intersections are now defined by reference to that manifold.

(27.10) Application. Let the Euclidean space $\mathbb{E}^n$ be referred to $[x_1, \ldots, x_n]$ and let $X_1$ be the one-complex arising from $-\infty < x_i < +\infty$ and its subdivision by $x_i = 0, \pm 1, \ldots$. Then $\mathbb{E}^n = |X|, X = X_1 \times \cdots \times X^n$. Let $q < p, r = p - q$ and consider the subspaces

$$
\mathbb{E}^p : x_{p+i} = 0, \quad i > 0;
\mathbb{E}^{p-r} : x_i = 0, \quad i \leq q;
$$

$$
\mathbb{E}' = \mathbb{E}^p \cap \mathbb{E}^{p-r} : x_i = 0, \quad i \leq q \text{ or } > p.
$$

We suppose them oriented by $[x_1, \ldots, x_p], [x_{p+1}, \ldots, x_n], [x_{p+1}, \ldots, x_p]$. If $A_i$ is the point $x_i = 0$ in the $i$th line and $L_i$ its basic one-cycle (geometrical identified with adherent combinatorial cycles) then the classes of the basic cycles $\mathbb{E}_0^p, \mathbb{E}_0^{p-r}, \mathbb{E}_0^r$ are precisely those of (V, 38.7). Thus $\mathbb{E}_0^p \cap \mathbb{E}_0^{p-r}$ is a cycle of $\mathbb{E}'$ in the class of $\mathbb{E}_0', \text{ or by (V, 38.7):}

(27.11)

$$
\mathbb{E}_0^p \cap \mathbb{E}_0^{p-r} = \mathbb{E}_0'.
$$

If $p + q = n$ then $\mathbb{E}_0^p \cap \mathbb{E}_0^{p-r}$ is the origin of coordinates, taken once according to (V, 38.8), and so

(27.12)

$$
KI(\mathbb{E}_0^p, \mathbb{E}_0^{p-r}) = 1.
$$

These are precisely the intersection rules of Lefschetz [a] for convex intersections.

It is to be noted throughout that the preceding results imply a reorientation of the dual $X^*$ or equivalently of the reciprocal $\mathbb{R}$ as described in (V, 38.3).

§5. COINCIDENCES AND FIXED POINTS

28. In this and the next section we propose to deal with extensions to mappings of the results of (V, 24) for coincidences and fixed elements of chain-mappings.

(28.1) If $\mathbb{R}$ is a topological space and $T$ a mapping $\mathbb{R} \to \mathbb{R}$ then a fixed point of $T$ has its obvious meaning: it is a point $x$ such that $Tx = x$. Similarly if $\mathbb{R}, \mathbb{S}$ are two topological spaces and $T_1$ and $T_2$ are mappings $\mathbb{R} \to \mathbb{S}$ and $\mathbb{S} \to \mathbb{R}$,
then a coincidence of $T_1$, $T_2$ is a pair $(x, y)$, $x \in \mathbb{R}$, $y \in \mathcal{S}$, such that $y = T_1x$, $x = T_2y$. This is the general situation, with $T_1, T_2$ running in opposite directions: from $\mathbb{R}$ to $\mathcal{S}$ and $\mathcal{S}$ to $\mathbb{R}$. However, for finite geometric manifolds we shall also consider coincidences of two mappings $T_1, T_2 : \mathbb{R} \to \mathcal{S}$, both going in the same direction, and they are defined as any pair $(x, y)$, $x \in \mathbb{R}$, $y \in \mathcal{S}$, such that $y = T_1x = T_2x$.

(28.2) All our results will apply solely to spaces which have

Property A. The space $\mathbb{R}$ is compact, all its rational Betti numbers $R^p$ are finite and all but a finite number of the $R^p$ are zero.

(28.3) Suppose that $T$ is a mapping $\mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ possesses Property A. Since the numbers $R^p$ are finite there is a finite so-called rational homology base or maximal set of rational cycles $\{\gamma_i^p\}$, $i = 1, 2, \ldots, R^p$, independent with respect to homology. The mapping $T$ induces a homomorphism on the homology groups (VII, 7.1) and hence a transformation on the elements $\{\gamma_i^p\}$ given by homologies with rational coefficients:

\begin{align*}
(28.4) \quad T\gamma_i^p & \sim \lambda_i(p)\gamma_i^p, \\
\lambda^p & = ||\lambda_i(p)||.
\end{align*}

(For simplicity we also denote by $T$ the simultaneous homomorphisms of the homology groups which $T$ induces, and similarly for the other mappings.)

Since all but a finite number of the $R^p$ are zero, the expression

\begin{align*}
(28.5) \quad \psi(T) = \sum (-1)^p \text{trace} \lambda^p
\end{align*}

has a meaning. Similarly if $\mathcal{S}$ is a second space with Property A and rational homology bases $\{\delta_i^p\}$, and if $T_1, T_2$ are mappings $\mathbb{R} \to \mathcal{S}, \mathcal{S} \to \mathbb{R}$ then

\begin{align*}
T_1\gamma_i^p & \sim \mu_i(p)\delta_i^p, \\
T_2\delta_i^p & \sim \mu_i(p)\gamma_i^p, \\
\mu_i^p & = ||\mu_i(p)||,
\end{align*}

(28.6)

and the expression

\begin{align*}
(28.7) \quad \varphi(T_1, T_2) = \psi(T_2 T_1) = \sum (-1)^p \text{trace} \mu_i^p \mu_i^p
\end{align*}

has meaning. As is well known $\mu_i^p, \mu_i^p$ may be interchanged at the right without changing the traces.

If $\{\gamma_i^p\}$ is a second maximal independent set analogous to $\{\gamma_i^p\}$ we have

\begin{align*}
(28.8) \quad \gamma'^p = a\gamma_i^p, \quad a = ||a_i^p||,
\end{align*}

where $a$ is a rational nonsingular square matrix and $\lambda^p$ will be replaced by $a\lambda^p a^{-1}$ whose trace is the same. Similarly for trace $\mu_i^p \mu_i^p$. Coupling this with (VII, 7.1) we have:

(28.9) The numbers: trace $\lambda^p$, trace $\mu_i^p \mu_i^p$, $\psi(T)$, $\varphi(T_1, T_2)$ are independent of the particular homology bases in terms of which they have been calculated. Furthermore they depend merely upon the homotopy classes of the mappings.

29. Coincidences and fixed points for finite Euclidean complexes. We will now suppose that we have finite Euclidean complexes $K, L$ and a mapping $T : |K| \to |K|$ or a pair of mappings $T_1, T_2 : |K| \to |L|, |L| \to |K|$.
It is clear that $|K|$, $|L|$ have Property A, and so (28.9) is applicable to them.
We will now prove:

(29.1) **Fixed Point Theorem.** If $\varphi(T) \neq 0$ then $T$ has a fixed point.

(29.2) **Corollary.** If $\varphi(T_1, T_2) \neq 0$ then $T_1, T_2$ have a coincidence.

We shall assume that $T$ has no fixed point and show that this leads to a contradiction, and similarly for the coincidences.

(29.3) If $T$ has no fixed point, $d(x, Tz) > 0$ for every $x \in |K|$. Since $|K|$ is a compactum $\varepsilon = \inf d(x, Tz) > 0$. Choose an $n$ such that mesh $K^{(n)} < \varepsilon/3$ (4.1). By (28.8) there is an $m > n$ such that: (a) a suitable $T'$ homotopic to $T$ maps $K^{(m)} \rightarrow K^{(n)}$ barycentrically into $K^{(n)}$; (b) $T'x \sim \sigma_\delta(Tx)$, and so $d(Tx, T'x) < \varepsilon/3$. As a consequence $\delta_\sigma$ and $\delta'_{\sigma_\delta}$ are always disjoint. Therefore if $\theta$ is the chain-mapping $K^{(m)} \rightarrow K^{(n)}$ induced by $T'$, then $\sigma_\theta \subset \delta_\delta$ implies that $\delta_\sigma_\theta$ has no element in $\delta_\sigma$.

On the other hand if $\delta$ denotes chain-derivation in $K$, then $\delta^{m-n}$ applied to $K^{(n)}$ is a chain-mapping such that $\|\delta^{m-n}\sigma_\delta\| \subset \delta_\sigma$. Therefore $\theta_\delta, \delta^{m-n}$ are chain-mappings $K^{(m)} \rightarrow K^{(n)}$, $K^{(n)} \rightarrow K^{(m)}$ without coincidences, and so $\varphi(\theta, \delta^{m-n}) = 0$, where $\varphi$ is defined as in (V, 24).

Since $T, T'$ are homotopic they induce the same homomorphisms on the homology groups. Since $\delta$ induces an isomorphism of the homology groups of $K$ with the corresponding groups of $K^{(n)}$ we may as well identify the classes $\Gamma_\theta$, $\delta^{m-n} \Gamma_\theta$ of $K, K^{(n)}$. Hence if $\Gamma_\theta^p$ is the class of $\gamma_\delta^p$ we will have from (28.4):

$$\theta_\Gamma_\theta^p = \lambda_\theta (\psi_p) \Gamma_\theta^p.$$

To $\delta^{m-n}$ there will correspond similar relations with every $\lambda_\theta = 1$. Therefore

(29.4) \[ \varphi(\theta, \delta^{m-n}) = \psi(T), \]

and so here $\varphi(\theta, \delta^{m-n}) \neq 0$, a contradiction proving (29.1).

(29.5) Passing now to (29.2), $T_1T_2$ is a mapping $|K| \rightarrow |K|$ whose fixed points are in one-one correspondence with the coincidences of $T_1, T_2$. This together with $\varphi(T_1T_2) = \varphi(T_1, T_2)$ and (29.1) yields (29.2).

(29.6) Since $\theta, \delta^{m-n}$ are chain-maps of finite complexes into one another, $\varphi(\theta, \delta^{m-n})$ is given by a relation (V, 22.4), in which at the right all the numbers are integers. Hence $\varphi(\theta, \delta^{m-n})$ is an integer, and therefore this holds also for $\varphi(T)$, consequently also by (28.7) for $\varphi(T_1, T_2)$. Or:

(29.7) For mappings of finite polyhedra into one another both $\varphi(T)$ and $\varphi(T_1, T_2)$ are integers.

(29.8) We have just been considering coincidences for mappings $T_1, T_1$ going in opposite directions (from $|K|$ to $|L|$ and $|L|$ to $|K|$). A similar result may be derived for finite absolute orientable manifolds when the mappings proceed in the same direction. For this purpose we utilize the result of (V, 40).

Let the notations be those loc. cit., except that $X, Y$ are geometric manifolds.
and let $T_1$, $T_2$ be two mappings $|X| \to |Y|$. Replacing if need be both $X$ and $Y$ by suitable derived and proceeding as in (29.3) there will be obtained barycentric mappings $T'_1$, $T'_2$ such that if the latter have no coincidences, the same holds regarding $T'_1$, $T'_2$. In fact we may then so choose $X$, $Y$ that throughout

$$\text{(Cl St } T'_1x) \cap \text{(Cl St } T'_2x) = \emptyset.$$  

As a consequence if $\tau$, $\theta$ are the chain-mappings $X \to Y$ induced by $T'_1$, $T'_2$ then no $y \in Y$ will be found in both $\tau x$ and $\theta x$. Coupling this with (V, 36.3) we find that $\tau$, $\theta$ (the latter as in V, 40) have no coincidences. On the other hand the coincidences of $\tau$, $\theta$ are in one-one correspondence with those of $\tau$, $\theta^*$, and so the latter will have none. Therefore with $\chi$, $\omega$ as in (V, 40) we must have:

$$0 = \varphi(\tau, \theta^*) = \chi(\tau, \theta) = \omega(\tau, \theta).$$

We will set again $\omega(\tau, \theta) = \omega(T_1, T_2)$ and it is an elementary matter to prove the analogues of (29.9):

$$\omega(T_1, T_2) \text{ depends merely upon the homotopy classes of } T_1, T_2 \text{ and it is an integer.}$$

We have then from (29.10):

(29.12) Let $X$, $Y$ be finite absolute orientable geometric $n$-manifolds and $T_1$, $T_2$ two mappings $|X| \to |Y|$. If $\omega(T_1, T_2) \neq 0$ then $T_1$, $T_2$ have at least one coincidence.

Thus there has been obtained a coincidence theorem for two mappings of absolute orientable geometric manifolds into one another even when the two mappings proceed in the same direction.

30. Applications. We will prove some of Brouwer’s classical degree and fixed point theorems (Brouwer [a, b, d]). We first extend the concept of Brouwer degree (V, 25) in the following way: If $K$, $L$ are both cyclic in the dimension $n$ and acyclic in the dimensions greater than $n$, and have the basic integral classes $\Gamma^m$, $\Delta^m$ then (28.4) for $p = n$ yields

$$T \Gamma^m = c \Delta^m$$

and $c$ is the degree of the mapping $T$: $|K| \to |L|$. From (29.9) follows

(30.2) The degree depends merely upon the homotopy class of $T$, and if $T$ is a mapping $|K| \to |K|$ it is even independent of the basic $n$-cycle chosen on $K$.

Let now $T$ be a mapping of an $n$-parallelotope $P^n$ (closed $n$-cell) on itself. Since $P^n$ is zero-cyclic, a point $x$ is a rational homology base for the dimension zero, and evidently $\sigma x \sim x$. Therefore $\psi(T) = 1$, and so:

(30.3) Every mapping of a finite-dimensional parallelotope (closed cell) into itself has a fixed point.

Let now $|K|$ be a topological $n$-sphere $S^n$. If we denote also by $S^n$ the basic $n$-cycle of the sphere then (28.1) for a mapping $T:S^n \to S^n$ becomes

$$\sigma \Gamma^0 = \Gamma^0, \quad \sigma S^n = c S^n,$$
where \( c \) is the degree. Therefore

\[
(30.4) \quad \psi(T) = 1 + (-1)^n c.
\]

From (30.4) follows the

\[
(30.5) \text{Theorem. The following mappings of a topological sphere } S^n \text{ into itself have a fixed point: (a) sense-preserving [reversing] mappings } S^n \rightarrow S^n \quad [S^{n+1} \rightarrow S^{n+1}]; \text{ (b) mappings of degree different from } (-1)^{n+1}.
\]

§6. QUASI-COMPLEXES AND THE FIXED POINT THEOREM

31. Quasi-complexes. The motivation for introducing this new and last class of spaces is the search for a sufficiently general type for which our basic fixed point theorem (29.1) is still valid. It has been known for some time (Lefschetz [g]) that it holds for so-called \( LC^* \) compacta and their generalization: \( HLC^* \) compact spaces. These spaces may be briefly described as follows:

(a) \( LC^* \) spaces. Let \( K = \{ \sigma \} \) be a finite Euclidean complex and \( L \) a closed subcomplex containing all the vertices of \( K \). An \( LC^* \) space is a compactum \( \mathcal{R} \) characterized by the following property: given any \( \varepsilon > 0 \) there is an \( \eta > 0 \) such that if there is a pair \( (K, L) \) and a mapping \( t_0 : |L| \rightarrow \mathcal{R} \) with mesh \( \{ t_0(L \cap \text{Cl } \sigma) \} < \eta \), then \( t_0 \) has an extension \( t \) to \( |K| \) such that the continuous complex \( \mathcal{R} = (K, t) \) is of mesh less then \( \varepsilon \). The pair \( (L_0, t_0) \) is known as a partial realization of \( K \) to mesh less than \( \eta \). It may be said that \( LC^* \) spaces have been identified (Lefschetz [d]) with the so-called absolute neighborhood retracts of Borsuk (compacta which are neighborhood retracts of every compactum in which they are topologically imbedded). They include notably all finite Euclidean complexes as well as the Hilbert parallelootope.

(b) \( HLC^* \) spaces. These are analogues of the \( LC^* \) type which may be described in outline as follows. First \( K, L \) are now simplicial. Next \( \mathcal{R} \) is merely compact and the partial realizations and extensions are chain-mappings into nerves. For a fuller description the reader is referred to a complete treatment of a class of spaces which includes \( HLC^* \) spaces in a forthcoming paper by E. G. Begle [a]. We merely state that the \( HLC^* \) class includes the \( LC^* \) class and in addition, for instance, all paralleloptopes.

The characteristic property of all these spaces, is the presence of an operation resembling indefinite chain-derivation. That is to say, if \( \Phi = \{ \Phi_\lambda ; \tau_\mu^\lambda \} \) is the net of the finite open coverings (VIII, 4) then there exist chain-mappings \( \omega^\lambda : \Phi_\mu \rightarrow \Phi_\lambda \), \( \lambda > \mu \), for a sufficiently large class of \( (\lambda, \mu) \). Since indefinite chain-derivation has been at the root of the proof of the fixed point theorem it is natural to search for our generalization in that direction. As we shall see this surmise is completely justified: quasi-complexes, the class of spaces to which it leads, satisfy the fixed point theorem. Moreover the class includes \( HLC^* \) spaces, hence also \( LC^* \) spaces (= absolute neighborhood retracts), finite Euclidean complexes and all paralleloptopes.
32. (32.1) Let then \( R \) be compact and as in (VII, 4) let \( \{ U_a \} \) be its finite open coverings and \( \Phi = \{ \Phi_a; \pi^a_\gamma \} \) the net of their nerves. We will be dealing throughout the remainder of the section with a certain cofinal family \( M \) of \( \Lambda = \{ \lambda; \succ \} \) and the elements of \( M \) will be designated by \( a, b, f, g \). The indices \( f, g \) will be used particularly to designate a certain dependence upon other indices. Corresponding to \( a, \ldots \), the coverings, nerves, \( \cdots \) are of course written \( U_a, \cdots, \Phi_a, \cdots \).

We describe two properties \( B, C \) of \( R \).

Property B. There is a family \( M \) (cofinal in \( \Lambda \)) such that for every \( a \) there is an \( f(a) \succ a \) with one or more chain-mappings \( \omega_f^a: \Phi_a \rightarrow \Phi_{f(a)} \), called antiprojections, and such that:

(a) \( \omega_f^a \pi^a_\gamma \sim 1 \);

(b) if \( g \succ f \succ a \) and \( \omega_f^a, \omega_g^f \) are antiprojections so is \( \omega_g^f \omega_f^a \);

(c) if \( \omega_f^a, \omega_g^a \) are antiprojections then \( \omega_f^a \sim \omega_g^a \).

The antiprojections \( \omega_f^a \) behave exactly like the duals of the projections, and give to the collections of the homology groups \( \{ D^\gamma(\Phi_a, G) \} \) the character of direct systems. In the terminology of (VI, 2) \( \{ \Phi_a \} \) is thus both a net and a conet. The role of (a) is to make projections and antiprojections cancel out along the homology classes. These remarks already hint at a theorem such as (33.1) below.

Property C. First of all Property B holds with \( M \) and the antiprojections \( \omega_f^a \) as in its statement. All the indices \( a, \ldots \) being understood in \( M \), we have in addition: for every \( a \) there is an index \( g \succ a \), and for every \( b \) an index \( h(a, g, b) \succ b, g \) such that \( \omega_f^a \) exists (in accordance with Property A), and that if \( \Phi_a = \{ \sigma_a \} \) then \( [\sigma_a] \cup [\omega_f^a \sigma_a] \succ U_a \), where if \( C \) is a chain of \( \Phi_a, [C'] \) denotes the union of the kernels of the simplexes of \( C \). Thus here the union of the kernels of \( \sigma_a \) and of the simplexes of \( \omega_f^a \sigma_a \) must be contained in a set of \( U_a \).

Roughly speaking, for a compactum Property C asserts that the nerve of a finite open covering \( U_a \) of sufficiently small mesh may be \( \epsilon \) chain-mapped (in an obvious sense) into the nerve of an \( U_a \) of arbitrarily small mesh.

(32.2) Definition. A quasi-complex is a compact Hausdorff space which possesses Property C and hence also Property B.

33. (33.1) If a compact space possesses Property B then its homology groups are isomorphic with subgroups of the homology groups of a certain finite simplicial complex (in fact with those of a nerve \( \Phi_a \)). Hence:

(a) the groups above a certain \( n \) vanish;

(b) the space possesses Property \( \Lambda \) of (28.2).

(33.2) Noteworthy special case. Proposition (33.1) holds for a quasi-complex.

We assume then that \( R \) is compact and satisfies Property B, with \( M, \omega \) as in the statement of Property B. Take a fixed \( a \in M \) and let \( \gamma^a = \{ \gamma_i^a \} \) be any cycle of \( R \) over \( G \). The mapping \( \gamma^a \rightarrow \gamma_i^a \) defines a simultaneous homomorphism \( \pi^a \) of the groups of cycles of \( R \) into the same for \( \Phi_a \), and this induces in turn a
simultaneous homomorphism \( \Pi_a : \mathfrak{S}^p(\mathcal{R}, G) \to \mathfrak{S}^p(\Phi_a, G) \). To prove (33.1) it is sufficient to show that \( \Pi_a \) is a simultaneous isomorphism with subgroups of \( \mathfrak{S}^p(\Phi_a, G) \). We must first prove \( \Pi_a \) univalent or equivalently:

\[
\gamma_{\xi}^{\rho} \sim 0 \implies \gamma_{\xi}^{\rho} \sim 0.
\]

(33.4) It is an immediate consequence of Property B that there is a family \([f]\) cofinal in \( \mathcal{M} \) and hence in \( \Lambda \), such that \( \omega_{f}^{\rho} \) exists for every \( f \) of the family. Now:

\[
\omega_{f}^{\rho} \sim \gamma_{\xi}^{\rho} \sim 0,
\]

and so from (a) of Property B:

\[
\omega_{f}^{\rho} \gamma_{\xi}^{\rho} \sim \gamma_{\xi}^{\rho} \sim 0.
\]

Since all the coordinates of \( \gamma_{\xi}^{\rho} \) for \([f]\) cofinal in \( \Lambda \) are \( \sim 0 \) we have \( \gamma_{\xi}^{\rho} \sim 0 \), which is (33.3).

To prove (33.1) there remains to show that

(33.5) \( \Pi_a \) is open.

For this purpose it is more convenient to pass to the homology classes \( \Gamma_{\xi}^{\rho}, \Gamma_{\xi}^{\rho}, \ldots \) of \( \gamma_{\xi}^{\rho}, \gamma_{\xi}^{\rho}, \ldots \). The homomorphisms in the homology groups induced by \( \omega_{f}^{\rho} \), \( \omega_{f}^{\rho} \) will be denoted by \( \Pi_{\xi}^{\rho} \), \( \Omega_{\xi}^{\rho} \), so that we have as a consequence of (a) of Property B:

\[
\Omega_{\xi}^{\rho} \Pi_{\xi}^{\rho} = 1.
\]

Let \( a \) and \([f]\) be related as before. If \( b \in \mathcal{M} \) and \( U_b \) is an open set of \( \mathfrak{S}^p(\Phi_a, G) \) then \( V_b = \{ \Gamma_{\xi}^{\rho} \mid \Gamma_{\xi}^{\rho} \in U_b \} \) is open in \( \mathfrak{S}^p(\mathcal{R}, G) \), and since \([f]\) is cofinal in \( \Lambda \), \( \{ V_f \} \) is a base for \( \mathfrak{S}^p(\mathcal{R}, G) \). To prove (33.5) we merely need to show that \( \Pi_a V_f \) is open. Since \( \Omega_{\xi}^{\rho} \) is continuous \( (\Omega_{\xi}^{\rho})^{-1} U_f = U_a \) is open. Now \( \Gamma_{\xi}^{\rho} \in V_f \iff (\Omega_{\xi}^{\rho} \Pi_{\xi}^{\rho})^{-1} U_f = U_a \iff \Gamma_{\xi}^{\rho} \in U_a \). Therefore \( \Pi_a V_f = U_a \cap \Pi_a \mathfrak{S}^p(\mathcal{R}, G) \) is an open set. This proves (33.5) and hence also (33.1).

34. Coincidences and fixed points for quasi-complexes.

(34.1) Theorem. The basic fixed point theorem (29.1) and its corollary (29.2) hold also for quasi-complexes.

The derivation of (29.2) is as in (29.5); so we only need to prove (29.1). The proof is by a reduction to the analogous property for chain-mappings of certain nerves and requires a careful selection of coverings.

(34.2) If \( \mathcal{A} = \{ A_a \} \) is any aggregate of sets then the star of \( A_a \), written \( \mathfrak{S} A_a \) (not to be confused with star in a complex), is the union of \( A_a \) and all the \( A_b \) which meet it. If \( \mathcal{R} \) is a topological space and \( \mathcal{U} \), \( \mathcal{B} \) open coverings then \( \mathcal{B} = \{ V_a \} \) is said to be a star-refinement of \( \mathcal{U} \) whenever \( (\mathfrak{S} V_a) > \mathcal{U} \) (every star of \( \mathcal{B} \) is contained in a set of \( \mathcal{U} \)). Star-refinements have been repeatedly utilized in questions related to the metrization problem (see notably J. Tukey [T]). However, we merely require here:

(34.3) If \( \mathcal{R} \) is normal, every finite open covering \( \mathcal{U} \) has a finite star-refinement \( \mathcal{B} \).

Let \( \mathcal{U} = \{ U_1, \ldots, U_r \} \) and let it be shrunk to \( \mathcal{U}' = \{ U_i' \} \subseteq U_i \). Then \( \mathcal{B}' = \{ U_i , \mathcal{R} - U_i' \} \) is a binary open covering and \( \mathcal{B} = \mathcal{U}' \cap \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_r \) is a finite open covering. Let \( \mathcal{B} = \{ V_a \} \) and suppose \( V_b \cap V_j \neq \emptyset \). The set \( V_b \) is contained in a set \( U_i' \) of \( \mathcal{U}' \), and \( V_j \) in one of the sets of \( \mathcal{B}_i \), i.e., \( V_i \subseteq U_i \),
or $V_j \subset R \setminus U_i'$. The second inclusion is ruled out since $V_j$ meets the subset $V_\lambda$ of $U_i'$. Therefore the first holds. Thus $V_\lambda$ and all the sets of $B$ meeting it are in $U_i$ and so $B$ satisfies (34.3).

(34.4) We will assume now that $R$ is a quasi-complex with a mapping $T: R \to R$ and set $R = T^R$. The notations are as in the statement of Property C and the various elements there considered are selected in the following way. We choose any $\eta_a$, $a \in M$, then take $\eta_a = |U_{a1}|$ as stated in Property C. Next we choose $\eta_b > |U_{a1}| \cap T^{-1}(R \cap U_{a1})$ and finally $\eta_b$ is selected as in Property C.

Suppose $\eta_b = |U_{b1}|$. We may find for each $U_{a1}$ two sets of $\eta_a$, $\eta_b$ which we denote for convenience by $U_{a1}$, $U_{a1}$, such that $U_{a1} \subset U_{a1} \cap T^{-1}(R \cap U_{a1})$. Since $\cap U_{a1} \neq \emptyset \implies \cap U_{a1} \neq \emptyset$, $U_{a1} \to U_{a1}$ defines a simplicial chain-mapping $\tau_a^b : \Phi_b \to \Phi_b$. Hence $p_{a1} = \tau_a^b \pi_a^b$ is a chain-mapping $\Phi_b \to \Phi_a$. (It is the analogue of $p_a^b$ of (VII, 5.12) for our present mapping $T$.) Thus finally $\theta = p_{a1} \Phi_b$ is a chain-mapping $\Phi_b \to \Phi_a$.

If $\gamma^p = |\gamma^p|$ is any cycle of $R$, then

$$\theta \gamma^p = p_{a1} \Phi_b \gamma^p \sim p_{a1} \Phi_b \pi_a^b \gamma^p \sim p_{a1} \gamma^p$$

Therefore (VII, 5.12):

(34.5) $\theta \gamma^p$ is the $g$ coordinate of the cycle $T \gamma^p$.

It is convenient to specialize $a$, $g$ still further in accordance with:

(34.6) $g$ may be so chosen that all the rational cycles $\pi_a \gamma^p$ are essential.

Corresponding to $a$ there exists by (VI, 3.12) an index $\lambda > a$ such that the rational cycles $\pi_a \gamma^p$ are all essential. This remains true if $\lambda$ is replaced by any $\lambda' > \lambda$. On the other hand, in Property C if $a' > a$, $a' \in M$, then any $g(a')$ is a suitable $g(a)$. Choosing then $a' > \lambda$, and $g = g(a')$, (34.5) will be satisfied.

35. Since the Betti numbers of $R$ are finite (33.1) it possesses a finite maximal independent set of rational cycles $|\gamma^p|$, $i = 1$, 2, ..., $R^p$, $\gamma^p = |\gamma^p|$. The cycles $|\gamma^p|$, $g$ fixed, are independent for $\Phi_g$. However, to have a maximal independent set for $\Phi$, it may be necessary to add a new set $|\delta^p|$.

We have seen (34.5) that $\theta \gamma^p$ is the $g$ coordinate of $T \gamma^p$. Therefore if

$$T \gamma^p \sim \lambda_i(p) \gamma^p,$$

then by the argument in the proof of (33.1) with $a$ in place of $g$ we find:

$$\theta \gamma^p \sim \lambda_i(p) \gamma^p.$$

On the other hand

$$\theta \delta^p \sim \rho_i(p) \gamma^p + \zeta_i(p) \delta^p.$$

Now we have:

$$\omega^p \delta^p \sim \omega^p \omega^p \delta^p \delta^p.$$

By (34.6) and (II, 27.12) there exists a cycle $\delta^p = |\delta^p|$ of $R$ such that $\delta^p \sim \pi_a \delta^p$, and we will have $\delta^p \sim c \gamma^p$ in $R$. Replacing, as we may, $\delta^p$ by $\delta^p - c \gamma^p$, the
situation will be unchanged except that now $\delta^p \sim 0$ and hence $\omega^p\delta^p \sim 0$. With
this new choice of the $\delta^p$, we find then from (35.4): $\omega^p\delta^p \sim 0$, and hence (35.3)
will be replaced (in view of $\theta = p^k \omega^k$) by

$$ (35.5) \quad \theta\delta^p \sim 0. $$

From this follows

$$ (35.6) \quad \psi(\theta) = \sum (-1)^p \text{tr} \lambda^p = \psi(T). $$

Since $\theta$ is a chain-mapping of a finite complex into itself, $\psi(\theta)$ is an integer
(V, 22.7), and hence the same holds for $\psi(T)$, and consequently also for
$\varphi(T_1, T_2) = \psi(T_1T_2)$. Therefore as in (29.7):

$$ (35.7) \text{For mappings of quasi-complexes into one another both } \psi(T) \text{ and } \varphi(T_1, T_2) \text{ are integers.} $$

36. We will assume now that $T$ has no fixed point and so dispose of the
situation that $\theta$ has no fixed element. In view of (35.6) this will prove (34.1).

(36.1) There is a finite open covering $\mathcal{B}$ of $\Re$ such that no star of $\mathcal{B}$ meets its
transform.

If $x_1 = Tx$ then $x \neq x_1$ and so the two points have disjoint neighborhoods $U, U_1$. Hence $U \cap T^{-1}U_1 = U'$ is a neighborhood of $x$ which does not meet $TU'$.
Since $\mathcal{B}$ is compact the covering $\mathcal{U}' = \{U'\}$ has a finite subcovering $\{U'_i\}$.
Since $\Re$ is compact Hausdorff it is normal (I, 33.6), and so $\{U'_i\}$ has a finite
star-refinement $\mathcal{B}$, and $\mathcal{B}$ satisfies (36.1).

It is clear that if $\mathcal{B}' \succ \mathcal{B}$ then $\mathcal{B}'$ still has property (36.1). Since $M$ is cofinal
in $\Lambda$, we may choose $a \in M$ such that $U_a$ of (34.4) satisfies (36.1). The situation
being then as before we prove:

(36.2) $\theta$ has no fixed elements.

Let $\Phi = \{\sigma_a\}$. Since the coverings of (34.4) have been selected in accordance
with Property C, some $U_{ai} \supset \{\sigma_a\} \cup \{\omega^p\sigma_a\}$. Hence if $\sigma_a$ is a simplex of the chain
$\omega^p\sigma_a$ we have $[\sigma_a] \subset U_{ai}$. From this follows that each vertex of $\omega^p\sigma_a$ is con-
tained in a set of $U_a$ meeting $U_{ai}$, and so $[\omega^p\sigma_a] \subset \# U_{ai}$. Hence $[\omega^p\sigma_a] =
[p^k\omega^k] \subset T \# U_{ai} \subset \Re - U_{ai}$, and so $[\theta\sigma_a] \subset \Re - U_{ai}$. Consequently
$[\theta\sigma_a] \cap [\sigma_a] = \emptyset$, which proves (36.2).

As already observed this completes the proof of the fixed point theorem (34.1).

(36.3) Application. A topological space $\Re$ is said to have the fixed point
property whenever every mapping $\Re \to \Re$ has a fixed point.

Suppose that $\Re$ is a zero-cyclic quasi-complex. Since $\Re$ is zero-cyclic it is
connected, and so we prove readily that if $x \in \Re$ then $Tx \sim x$. From this
follows $\psi(T) = 1$, and so $\Re$ has the fixed point property. Since any parallelo-
tope may be deformed into a point it falls under this category. Thus

(36.4) A zero-cyclic quasi-complex, and in particular an arbitrary parallelo-
tope, has the fixed point property.

There are noteworthy applications to analysis to which we propose to return
elsewhere [La, IV].
§7. TOPOLOGICAL COMPLEXES

37. In the applications to analysis or geometry complexes do not always occur in the convenient aspect of polyhedra or Euclidean realizations. Thus a circumstance with two subdivision points gives rise to a complex which is not polyhedral and other examples could be multiplied. Topological complexes are intended to bridge the gap. Historically they are also of interest since they were the complexes introduced by Poincaré [b] and dealt with at length by Veblen [V]. As we shall see (40.3) topologically speaking we remain within the class of Euclidean complexes and this will justify if need be our having devoted our major efforts to that class.

38. (38.1) It will be convenient to break up the description of our complexes into two parts. We will first consider an Euclidean complex $\mathcal{K} = \{\sigma\}$ with which there is to be associated a new complex $K = |E|$ with a derived $K' \cong \mathcal{K}$. Thus the passage from $\mathcal{K}$ to $K$ is the inverse of derivation. The complex $K$ is characterized by these properties:

I. $E'_i$ is a $p$-cell covered by an open subcomplex of $\mathcal{K}$ which is a join $A(S^{p-1})_a = A S^{p-1} \cup A$ of a vertex $A$ of $\mathcal{K}$ and an augmented $(p - 1)$-sphere which is a finite closed subcomplex of $\mathcal{K}$. Thus $E'_i = A_a(S^{p-1})$ is a closed $p$-cell and it is covered by a closed subcomplex of $\mathcal{K}$.

II. The cells $E$ are disjoint and their union is $|\mathcal{K}|$.

III. $E$ is the union of the cells of $\text{Cl} E$.

IV. The description of the incidence numbers is less immediate. As above let $E'_i = |A(S^{p-1})_a|$. Since $E'_i$ is a closed $p$-cell, by (16.1) it is a simple $p$-circuit mod $S^{p-1}$. For convenience let $E'_i$ also designate a basic $p$-cycle for that circuit. Since the $E'_i$ are disjoint their $q$-simplexes may be written uniquely $\{\sigma'_{ij}\}$, $\sigma'_{ij} \subset E'_i$. In particular

$$E'_i = \sum \alpha_{ij} \sigma'_i,$$

where $\alpha_{ij} = \pm 1$ since $E'_i$ is a simple circuit.

It is clear that each $\sigma'_{p-1} \subset S^{p-1}$ is the face of one and only one $\sigma'_i$, and hence:

$$FE'_i = \sum \epsilon_{ijk} \sigma'_{jk},$$

where $\epsilon_{ijk} = \pm 1$ if $E'_{ij} < E'_i$, and is equal to 0 otherwise. If we reduce $FE'_i \mod (S^{p-1} - E'_{ij})$ there arises a cycle of $E'_{ij}$ which is a multiple of its basic cycle. Thus

\begin{equation} \sum_k \epsilon_{ijk} \sigma'_{jk} = \eta_i(p - 1)E'_{j}^{-1}, \quad \text{(no summation on i, j)}, \end{equation}

and therefore

\begin{equation} FE'_i = \eta_i(p - 1)E'_{i}^{-1}. \end{equation}

We set:

$$[E'_i : E'_i^{-1}] = \eta_i(p - 1)$$

and determine all the other incidence numbers in $K$ by means of (III, 1.1, K23).
(38.4) \( K \) is a simple complex which has \( \mathcal{R} \) as a subdivision \( \cong K' \).

(a) \( K \) is a complex. Referring to (III, 1.1, 8.3) we merely have to verify that if the boundary operator in \( K \) is defined by (38.2) then \( \text{FF} = 0 \). Now we have

\[
\text{FF} E_0^i = i^i E_0^{i-2}.
\]

Since the complexes \( E_0^{i-2} \) are disjoint the \( \sigma_{ji}^{i-2} \) are distinct. The coefficient of \( \sigma_{ji}^{i-2} \) in \( E_0^{i-2} \) is a number \( \lambda \neq 0 \), and its coefficient in \( \text{FF} E_0^i \) expressed as a cycle of \( \mathcal{R} \) is \( \lambda_j^i = 0 \), since \( \text{FF} = 0 \) holds in \( \mathcal{R} \). Hence \( i^i = 0 \) and (a) follows.

(b) \( K \) is augmentable. Since \( E_0^i \) is a one-chain in a simplicial complex we have \( \text{KI}(FE_0^i) = 0 = \sum \eta_i'(0) \), and therefore if \( \gamma_0 = \sum E_0^i \) then \( F \gamma_0 = \sum \eta_i'(0) E_0^i = 0 \). Thus \( K \) is augmentable and with \( \gamma_0 \) as fundamental cocycle.

(c) (Cl \( E \)) is acyclic. For \( |\text{Cl} E| = \overline{E} \) is a closed cell, so (c) holds for the geometrical groups, hence also for the combinatorial groups (10.1).

(d) \( K \) is closure-finite (obvious).

It is now a consequence of (abcd) that \( K \) is simple (III, 47.1). If the vertex of \( \mathcal{R} \) in \( E \) is written 'E' then the simplexes of \( \mathcal{R} \) are uniquely represented as \( 'E_0, E_1, E_2, \cdots < E_n; \) and so \( \mathcal{R} \cong K' \). Since \( K \) is simple, \( \mathcal{R} \) is a subdivision (IV, 27.1) and (38.4) is proved.

\[
[E_0^p : E_0^{p-1}] = \begin{cases} 
\pm 1 & \text{if } E_0^{p-1} < E_0^p, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \sigma_{ji}^{p-1} \) is a relative \( (p-1) \)-circuit so is \( E_0^{p-1} \), and hence the coefficient \( \lambda \) of \( \sigma_{ji}^{p-1} \) in \( E_0^{p-1} \) is different from 0. Since the \( E_0^{p-1} \) are disjoint the \( \sigma_{ji}^{p-1} \) are distinct. Thus the coefficient of \( \sigma_{ji}^{p-1} \) in the chain of \( \mathcal{R} \) at the right is \( \lambda \eta_i(p - 1) \). Hence \( \lambda \eta_i(p - 1) = \varepsilon_{ij} \). From the known values of \( \varepsilon_{ij} \) we have then \( \lambda \eta_i(p - 1) = 0 = \eta_i(p - 1) \) if \( E_0^{p-1} < E_0^p \), and \( \lambda \eta_i(p - 1) = \pm 1 = \eta_i(p - 1) \) if \( E_0^{p-1} < E_0^p \), and this proves (38.5).

(38.6) The complex \( K \) corresponding to a given collection \( \{E\} \) is unique.

At all events it is unique except possibly for the incidence numbers. In their determination the only allowable modification is a change in the signs of certain basic cycles \( E_i^p \) and this is equivalent to reorienting \( K \) by means of an \( \alpha(E) \) taking the value \(-1\) in those \( E_i^p \). Under our conventions this does not modify \( K \).

39. The ground is now prepared for topological complexes. Here again the only difficulty will be caused by the incidence numbers.

(39.1) Definition. A topological complex is a countable locally finite complex \( K = \{E\} \) with the following properties:

I. The elements of \( K \) are disjoint cells and their union is a topological space, written \( |K| \) and said to be polyhedral.

II. \( E_0^p \) is a \( p \)-cell, \( E_0^p \) is a closed \( p \)-cell which is the union of the cells of \( \text{Cl} E_0^p \).

III. If \( \varphi(E) \) is the union of the cells of \( K - \text{St} E \) then \( \varphi(E) \cap E = \emptyset \).
The specification of the incidence numbers will be given presently.

The analogy of these properties with (III, 6.1abc) for polyhedral complexes is evident.

(39.2) Definition. A topological complex $K = \{E\}$ is said to be simplicial if there exists an Euclidean complex $\mathcal{R} = \{\sigma\}$ and a topological mapping $t: |\mathcal{R}| \to |K|$ such that $t\sigma$ is a cell $E$ of $K$.

(39.3) Definition. If $K = \{E\}$, $K_1 = \{E_1\}$ are topological complexes then $K_1$ is said to be a partition of $K$ whenever every $E$ is the union of a finite set of $E_1$'s and every $E_1$ is contained in some $E$.

(39.4) Returning to $K$ of (39.1) let its elements ranged in some order be denoted by $E_1, E_2, \cdots$, and select a point $A_i \in E_i$. In the Hilbert parallelo
tope $P^n$ referred to $\{u_1, u_2, \cdots\}$, $0 \leq u_i \leq 1/i$, let $B_i$ be the point $u_i = 1/i$, $u_j = 0$ for $i \neq j$. Corresponding to every set $E_i < \cdots < E_j$ introduce the Euclidean simplex $\sigma = B_i \cdots B_j \subset P^n$. Evidently $\mathcal{R} = \{\sigma\}$ is an Euclidean complex $\cong K'$. We will now define a topological mapping $t: |K| \to |\mathcal{R}|$ as follows. Let $K^q$ be the $q$-section of $K$. We choose $tA_i = B_i$ and this defines $t|K^q$. Suppose $t|K^{q-1}$ known and let $\dim E_i = p$. Take a parallelo
tope $P^p$ and let $S^{p-1}$ be its boundary sphere. Introduce a topological mapping $s: P^p \to P^n$ and suppose $sA_i = C_i$. Then $ts^{-1}$ is a topological mapping $S^{p-1} \to \mathcal{R}$ such that $ts^{-1}S^{p-1} = S^{p-1}$ is a subcomplex of $\mathcal{R}$, and that $ts^{-1}C_i = B_i$. This mapping is extended to a topological mapping $t: P^p \to \mathcal{R}$ as follows: if $R \in S^{p-1}$ then $C_iR$ is mapped barycentrically on $B_i(ts^{-1}R)$. If we choose $t|E_i = t,s$ and operate likewise for all $E^p$ we obtain $t|K^p$, then proceeding in this fashion $t$ itself.

(39.5) We will take advantage of $t$ to identify $|\bigcup E|$ with $|\mathcal{R}|$ so that points corresponding under $t$ coincide. Under the circumstances $|E|$ becomes a collection related to $\mathcal{R}$ as in (38.1). We now specify:

IV. The incidence numbers in $K$ are in accordance with (38.1, IV).

40. Several interesting conclusions may quickly be drawn from the definition.

(40.1) The topological complex $K$ is uniquely determined by the collection of cells $\{E\}$ and properties (38.1, I, II, III).

In particular $[E^q_1 : E^q_1 \cap E^q_1] = \begin{cases} \pm 1 \text{ if } E^q_1 \cap E^q_1 \subset E^q_1; \\ 0 \text{ otherwise.} \end{cases}$

For the decomposition of $\mathcal{R}$ by $\{E\}$ is independent of the construction and the resulting incidence numbers obtained in accordance with (38.1, IV) determine $K$ uniquely (38.6). That the incidence numbers behave as stated is a consequence of (38.5).

The identification of the topological complex with the complex $K$ of (38) together with (38.4) yield:
(40.2) A topological complex is simple.

(40.3) A topological complex $K$ has a simplicial partition which is a subdivision, and at the same time a realization of the derived $K'$ such that the vertex of $K'$ corresponding to $E_i^p$ is imaged into a preassigned point of the cell.

The distinction between combinatorial and geometrical groups (9.1) may of course be made for topological complexes. From (40.3) together with (IV, 24.1) and the results of (10, · · · , 13) we deduce:


Further results of this nature could be obtained but the topological identification of Euclidean and topological complexes implicit in (40.3) makes their derivation superfluous.

(40.5) A polyhedral complex is topological.

Let $\Pi = \{E\}$ be a polyhedral complex, $\Pi'$ its derived, $E$ the new vertex of $\Pi'$ in $E$, $D$ and $\delta$ set and chain-derivation in $\Pi$. If

$$FE_i^p = \eta'_i(p - 1)E_i^{p-1},$$

where the $\eta'_i(p - 1)$ are the incidence numbers in $\Pi$, then (IV, 29.6, 29.1):

(40.6) $$\delta FE_i^p = F(\delta E_i^p) = \eta'_i(p - 1)\delta E_i^{p-1}.$$  

Since $DE_i^p$ coincides with $E_i^p$, $\{DE_i^p\}$ and $\Pi'$ are related like $K$ and $\mathfrak{I}$ in (38). The cycles denoted there by $E_i^p$ are the cycles $\delta E_i^p$. The comparison of (40.6) with (38.3) shows that the incidence numbers of the polyhedral complex $\Pi$ conform with (38.1, IV). This proves (40.5).

§8. DIFFERENTIABLE COMPLEXES AND MANIFOLDS

41. The parametric representation of topological complexes and manifolds brings to the fore a wealth of questions and problems, many of which are yet unsolved, which are of great interest in both topology and differential geometry in the large. We will do little more than touch upon these questions here, and describe some of the more important recent contributions, together with certain complements regarding homology and intersections. This will necessitate, however, a number of preliminary definitions most of which are standard (see notably: Veblen-Whitehead: The foundations of differential geometry, Cambridge Tract 29, (1932); Cairns [a, b], Whitehead [b], Whitney [a, b]).

42. Systems of class $C^\alpha$.

(42.1) Let $\{u_1, \ldots, u_m\}$ be real variables and $L_i$ the real line $-\infty < u_i < +\infty$. The product $L_1 \times \cdots \times L_m$ is called the space of the parameters $u_i$, and is denoted by $U^m$. It is merely an Euclidean space with a specified coordinate system. The point $(u_1, \ldots, u_m)$ is denoted by $u$.

(42.2) If $V^n$ is the space of $[v_1, \ldots, v_n]$ then the space corresponding to $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ is called the product of $U^m, V^n$, written $U^m \times V^n$.

(42.3) A finite set of real functions $S = \{f_i(u_1, \ldots, u_m)\}$ is said to be:

(a) of class $C^\alpha$ in a region $\Omega$ of $U^m$ whenever the $f_i$ and all their partial derivatives
of order at most \( q \) are defined and continuous in \( \Omega \); (b) \textit{regular} if it is of class \( C^q \) and the Jacobian matrix \( |\frac{\partial f_i}{\partial u_j}| \) is of maximal rank at all points of \( \Omega \). We also say that \( S \) is \textit{differentiable} in \( \Omega \) if it is of class \( C^1 \) in \( \Omega \), and that it is \textit{analytical} or of class \( C^\infty \) in \( \Omega \), if the \( f_i \) are analytical in \( \Omega \) and (b) holds. Throughout the sequel one may freely interchange “differentiable” with “class \( C^1 \),” and “analytical” with “class \( C^\infty \).”

If \( A \subseteq U^n \) then \( S \) is \textit{regular} if \( S \) is \textit{regular} of class \( C^q \) in \( A \) if it is \textit{regular} of class \( C^q \) in some neighborhood of \( A \).

(42.4) Let \( V^n \) be the space of \( (v_1, \ldots, v_n) \) and let \( t \) be a transformation \( \Omega \rightarrow V^n \), \( \Omega \) a region of \( U^n \), given by relations

\[
v_i = f_i(u_1, \ldots, u_m).
\]

We will say that \( t \) is \textit{regular} of class \( C^r \) or is a \textit{regular} \( C^r \)-mapping if \( \{f_i\} \) is \textit{regular} of class \( C^r \) in \( \Omega \).

43. \textbf{Parametric cells.}

(43.1) A parametric \( n \)-cell is an \( n \)-cell \( E^n \) together with a topological mapping \( t:E^n \rightarrow U^n \). The coordinates of \( U^n \) are called the \textit{parameters} of \( E^n \). A convenient designation for the parametric cell is \( (E^n, t, U^n) \).

(43.2) Consider two parametric cells \( (E^n, t, U^n) \) and \( (E'^n, t', U'^n) \) and let \( A = E^n \cap E'^n \). We say that \( A \) is a \textit{regular} \( r \)-\( C^r \)-\textit{intersection} of the two parametric cells whenever the following holds. For each \( x \in A \) there is an \( (E', s, V) \) such that: (a) \( E' \) is a neighborhood of \( x \) in \( A \); (b) on \( t'^{-1}E' \) and \( t^{-1}E' \) the \( u_i \) and \( u'_i \) satisfy systems

\[
u_i = f_i(v_1, \ldots, v_n), \quad u'_i = \varphi_i(v_1, \ldots, v_n)
\]

such that \( \{f_i\}, \{\varphi_i\} \) are \textit{regular} of class \( C^r \) in \( t'^{-1}E', t^{-1}E' \). Two noteworthy special cases are:

(a) \( E'^n \subset E^n \). Here \( r = n \) and one may take for \( (E', s, V) \) the parametric cell \( (E'^n, t', U'^n) \) itself. Thus the \( u \) coordinates of the points of \( E'^n \) form then a system of functions of class \( C^r \) of \( u' \)-coordinates in \( t'E'^n \). We have then a \textit{regular} \( C^r \)-\textit{imbedding} of the second parametric cell in the first.

(b) \( n = m \) and \( A \) is an open set in both \( E^n \) and \( E'^n \). Then the \textit{regular} \( C^r \)-intersection becomes a \textit{regular} \( C^r \)-\textit{overlap}.

(43.3) The product of two parametric cells \( (E^n, t, U^n), (E'^n, t', U'^n) \) is by definition the parametric cell \( (E^n \times E'^n, t \times t', U^n \times U'^n) \), i.e., \( E^n \times E'^n \) parametrized by \( (u_1, \ldots, u_m, u'_1, \ldots, u'_n) \), the mapping \( t \times t' \) being defined by \( (t \times t')(x, y) = (tx, ty) \).

44. \textbf{Manifolds.}

(44.1) A \textit{topological} \( n \)-\textit{manifold} is a metric space \( \mu \) with a countable locally finite open covering consisting of \( n \)-cells.

(44.2) An \( n \)-\textit{manifold} \( M^n \) of class \( C^r \) or \( C^r \)-\textit{manifold} is a topological \( n \)-manifold \( \mu \) with a countable locally finite open covering by parametric \( n \)-cells, \( \{E_\gamma^n\} \) (supposed to exist) such that if \( E_\gamma^n \), \( E_\delta^n \) intersect then they have a regular \( C^r \)-\textit{overlap}. The collection \( \{E_\gamma^n\} \) is called the \textit{basic covering} of \( M^n \) and \( \mu \) is written \( |M^n| \).
Let \( \mu \) give rise to two \( C^- \)-manifolds \( M^n, M'^n \) by means of \( \{E^\mu_n\}, \{E'^\mu_n\} \). If the two basic coverings together may serve as a basic covering for a \( C^-n \)-manifold associated with \( \mu \) then \( M^n \) and \( M'^n \) are considered as identical \( C^- \)-manifolds. This allows for a certain latitude in the choice of the basic covering.

(44.2a) Example. \( U^n \) is a parametric \( n \)-cell and so it is an analytical \( M^n \) with itself as basic covering consisting of a single cell. On the other hand the cells defined by the inequalities \( u_i > -1/2, u_i < 1/2 \), make up a basic covering consisting of \( 2^n \) elements.

(44.3) Let \( M^n, M'^n \) be \( C^- \)-manifolds with the basic coverings \( \{E^\mu_n\}, \{E'^\mu_n\} \). Then \( \{E^\mu_n \times E'^\mu_n\} \) is a basic covering for the topological manifold \( |M^n| \times |M'^n| \) which turns it into a \( C^-{(m + n)} \)-manifold known as the product of \( M^n \), \( M'^n \) and written \( M^n \times M'^n \). Furthermore this product is independent of the particular choice of basic coverings for \( M^n, M'^n \).

Example. If \( M^n \) is a \( C^- \)-manifold and \( L \) the real line \( -\infty < u < +\infty \) turned into an analytical manifold then \( L \times M^n \) is a \( C^-{(n + 1)} \)-manifold.

(44.4) The notations remaining those of (44.3), we say that \( M^n, M'^n \) have a [regular] \( C^- \)-intersection if the intersections of the elements of the basic covering of \( M'^n \) with those of the basic covering of \( M^n \) are all [regular] \( C^- \). If \( M'^n \subset M^n \) we also say that \( M^n \) is [regularly] \( C^- \)-imbedded in \( M'^n \), or merely \( M^n \) is [regularly] \( C^- \)-imbedded in \( M'^n \). Here again the situation is independent of the basic coverings.

(44.5) Let the \( C^- \)-manifolds discussed so far be termed absolute. A relative \( C^-n \)-manifold, or \( C^-n \)-manifold with regular boundary \( M'^{n-1} \) is a region \( \Omega \) of \( |M^n| \), where \( M^n \) is an absolute \( C^- \)-manifold, whose boundary \( \partial \Omega \) is a regular \( C^-{(n - 1)} \)-manifold in \( M^n \).

(44.6) Example. Let again \( L \) be the real line of (44.3) turned into a \( C^- \)-manifold and let \( \lambda \) be the interval \( 0 < u < 1 \). If \( M^n \) is a \( C^- \)-manifold then \( \lambda \times |M^n| \) is a region of \( L \times M^n \) whose boundary \( 0 \times M^n \cup 1 \times M^n \) is a \( C^-n \)-manifold regularly \( C^- \)-imbedded in \( L \times M^n \). Thus we have a manifold with regular boundary which will be written \( \lambda \times M^n \).

(44.7) Let \( M^n, M'^n \) be as in (44.3). A mapping \( T : |M^n| \rightarrow |M'^n| \) is said to be a [regular] \( C^- \)-mapping \( M^n \rightarrow M'^n \), whenever if \( x \) is in the parametric cell \( (E^\mu, t, U^n) \) of \( M'^n \) and \( Tx \) in the parametric cell \( (E'^\mu, t', U'^n) \) then the \( U'^n \) coordinates of \( Tx \) make up a system of functions of the \( u \) coordinates of \( x \) which is [regular] of class \( C^- \) at \( tx \) in \( U'^n \). If \( |M^n| \subset |M'^n| \), the resulting imbedding is readily seen to be a [regular] \( C^- \)-imbedding in the sense of (44.4).

(44.8) Let \( \lambda \) be an interval of \( -\infty < u < \infty \) containing 0, 1. Two \( C^- \)-mappings \( T_1, T_2 \) of \( M'^n \) into \( M'^n \) are said to be [regular] \( C^- \)-homotopic if there exists a [regular] \( C^- \)-mapping \( T_0 \) of \( \lambda \times M'^n \) which agrees with \( T_1, T_2 \) on \( 0 \times M'^n, 1 \times M'^n \).

(44.9) Let \( M^n \) be a topological manifold in \( \mathbb{E}^n \). The subspace \( \mathbb{E}^n \) through \( x_0 \in M^n \) is said to be transversal to \( M^n \) at \( x_0 \), if \( \mathbb{E}^n \) and the tangent \( \mathbb{E}^n \) to \( M^n \) at \( x_0 \) intersect at no other point than \( x_0 \). The manifold is said to be in normal position in \( \mathbb{E}^n \) if for each \( x \in |M^n| \) there is a transversal \( \mathbb{E}^{n-m} \) to \( M^n \) at \( x \) with a system of coordinates which vary continuously with \( x \).
45. \textit{C'}-complexes and other structures.

(45.1) A \textit{C'}-complex is a topological complex \( K = \{ E'_i \} \) such that: (a) \( E'_i \) is a subset of a \textit{C'}-p-manifold denoted by \((E'_i)^p\); (b) if \( E' < E \) then \((E')\) is regularly \textit{C'}-imbedded in \((E)\). We add the following convention: if the \(( \ )\) are replaced by \textit{C'}-manifolds \(( \ )^p\), which are identical with them in the common parts, then the \textit{C'}-complex remains the same.

(45.2) An \textit{analytical spread} is a subset \( \Sigma \) of \( U^n \) with the property that every point \( x \in \Sigma \) has a neighborhood \( N \) in \( U^n \) such that the points of \( N \cap \Sigma \) are the solutions of a system

\[
 f_i(u_1, \ldots, u_n) = 0,
\]

where the \( f_i \) are analytical in \( N \).

(45.3) \textbf{Examples.} Euclidean and real or complex projective spaces, algebraic loci in such spaces are all analytical spreads.

(45.4) Let \( M^n \) be a \textit{C'}-manifold and \( K \) a \textit{C'}-n-complex such that: (a) every \((E'_i)^p\) is \( M^n \) itself; (b) the set \(| M^n | = | K | \). Then \( M^n \) is said to be \textit{C'}-covered by \( K \).

46. \textbf{Imbedding and covering theorems.} Only the statements are given, the reader being referred to the original papers for the proofs.

(46.1) Every differentiable \( M^n \) may be differentially and topologically mapped onto an analytical \( M^n \) in some Euclidean space \( \mathbb{E}^{2m+1} \). More precisely if \( M^n \) is of class \( \textit{C'} \) \((r \neq \omega)\) then the mapping may be chosen of the same class (Whitney \([a, b]\)).

(46.2) If \( M^n \) is a manifold of class \( \textit{C'} \) in normal position in \( \mathbb{E}^n \) then it may be indefinitely approximated by a topologically equivalent analytical \( M'^{\infty} \) in \( \mathbb{E}^n \) (Whitney \([a]\)).

(46.3) Let \( t \) be a mapping \( M^n \rightarrow M^n \) where \( t \) and the manifolds are \( \textit{C'} \), and let \( \eta \) be a continuous positive function on \( M^n \). Then:

(a) \( t \) is \( \textit{C'} \)-homotopic to a mapping \( T \) such that \( d(tx, Tx) < \eta(x), x \in M^n \);

(b) if \( n \geq 2m \) the mapping \( T \) may be chosen regular and if \( n > 2m \) it may be chosen topological (Whitney \([a]\)).

(46.4) Every differentiable manifold may be covered with a differentiable simplicial complex. More precisely, if the manifold is of class \( \textit{C'} \), \( r > 0 \), then the complex may be chosen of the same class (Cairns \([a, b]\), Whitehead \([b]\)).

(46.5) Every bounded analytical spread may be covered with an analytical complex (Brown-Koopenman \([a]\), Lefschetz-Whitehead \([a]\)).

It may be noted that (46.5) leads rapidly to the following property which is also implicit in (46.4):

(46.6) Every analytical manifold may be covered with a simplicial analytical complex.

From (46.1, 46.5) we deduce the following property implicit in (46.4):

(46.7) Every differentiable manifold is a polyhedral space.

A problem as yet unsolved for dimensions greater than 2 is:
(46.8) Are topological manifolds polyhedral spaces? That is to say, can they be covered with topological complexes?

For \( n = 1 \) the answer is affirmative and elementary and for \( n = 2 \), at least when \( M^n \) is compact, it is likewise affirmative, a result due to Radon. Nothing is known for \( n > 2 \).

47. Differentiable manifolds: Homology and intersection theory.

(47.1) If \( M^n \) is differentiable then it may be covered with a topological complex \( K \). Since the latter has a simplicial derived we may already assume \( K \) simplicial. We apply directly to \( K \) whatever properties will be required which hold for an Euclidean complex under a topological transformation of the underlying topological space (polyhedron).

(47.2) Now if \( x \in |M^n| \), \( x \) has a neighborhood which is an \( n \)-cell \( E^n \). Hence \( E^n \) is \( n \)-cyclic in \( x \) and therefore the same holds for \( |M^n| \). By (17.6) \( K \) is thus an absolute combinatorial \( n \)-manifold. Let us assume this manifold orientable. Then by (17.1) every topological simplicial complex covering \( M^n \) is an absolute orientable \( M^n \). We say then that \( M^n \) itself is orientable.

(47.3) Suppose \( M^n \) orientable and connected. It has then a basic \( n \)-cycle \( M_0^n \) which is specified by any indicatrix. Consider in particular a parametric \( n \)-cell \( (E^n, t, U^n) \). Let \( A_0 \) be the point \( z_0 = (u_{01}, \cdots, u_{0n}) \), \( x_0 \in E^n \), and let \( A_1 \in U^n \) be a point \( u_{0j} + \epsilon_0 j \) where \( \epsilon > 0 \) is chosen so small that the Euclidean simplex \( \sigma^n = A_0 \cdots A_n \subset tE^n \). Replacing if need be one of the \( u_i \) by \( -u_i \) we may assume the coordinates such that \( \sigma^n = tE^n \) is an indicatrix of \( M^n \).

(47.4) The situation remaining the same let \( M^p, M^{q-\epsilon}, p \geq q \), be orientable connected differentiable manifolds regularly and differentiably imbedded in \( M^n \) and suppose that they have a regular differentiable intersection which is an orientable connected differentiable \( M^n, r = p - q \). The related basic cycles are written \( M_0^p, M_0^{q-\epsilon} \) and \( M_0^r \). By (27.3) we will have an intersection cycle \( M_0^p \cap M_0^{q-\epsilon} \) and

\[
M_0^p \cap M_0^{q-\epsilon} = aM_0^r.
\]

We assume now explicitly that for some \( z_0 \in |M^p| \) there may be chosen a parametric cell \( (E^p, t, U^p) \) such that \( |M^p|, |M^{q-\epsilon}|, |M^r| \) intersect it in cells \( E^p, E^{q-\epsilon}, E^r \) mapped by \( t \) into cells \( E_0^p, \cdots, \) where \( E_0^n \subset U^n \) and the others are in the spaces:

\[
\mathcal{E}^p: u_i = 0, \quad i > p; \quad \mathcal{E}^{q-\epsilon}: u_i = 0, \quad i \leq q;
\]

\[
\mathcal{E} = \mathcal{E}^p \cap \mathcal{E}^{q-\epsilon}: u_i = 0, \quad i \leq q \text{ or } > p.
\]

Furthermore the situation may be so disposed that \( E_0^p, E_0^p, \cdots \) are indicatrices for \( U^p, \mathcal{E}^p, \cdots \). Let \( e^n, \cdots \) be indicatrices for \( M^n, \cdots \), where \( \epsilon_0, \cdots = \pm 1 \). If \( \mathcal{E}^n, \cdots \) are the basic cycles of \( \mathcal{E}^n, \cdots \), then (27.11) holds. If we apply the projection \( \mathcal{E}^n \rightarrow E_0^n \) (VII, 5.10) we obtain

\[
E_0^p \cap E_0^{q-\epsilon} = \alpha E_0^n.
\]
where $E^p_0, \cdots$ stand also for the basic cycles of the cells. Similarly the projection $|M^n| \rightarrow E^n$ followed by $t$ yields:

$$\epsilon_n(\epsilon_{pE^p_0})^q(\epsilon_{n-q}E^{n-q}_0) = \epsilon_nE^p_0,$$

and hence $\alpha = \epsilon_p \epsilon_{n-q} \epsilon_{n}$, or finally:

$$(47.6) \quad M^p_0 \circ M^{n-q}_0 = \epsilon_p \epsilon_{n-q} \epsilon_{n} M^n_0.$$  

Once more we recognize here the rule for determining intersections given in Lefschetz [a], and in closely related form in [L, IV].

If $q = p$ then $r = 0, \epsilon_0 = 1$, and $M^n$ is a single point. Its index is

$$(47.7) \quad \text{KI}(M^p_0, M^{n-q}_0) = \epsilon_p \epsilon_{n-q} \epsilon_{n}.$$  

(47.8) If $|M'|$ is not connected each part is determined by the above rule and the intersection is the sum of the individual cycles thus obtained. Similarly if $M^n$ is a finite point set $\{A_1, \cdots, A_i\}$ each gives rise to an index $\lambda_i$ determined by our rule and

$$(47.9) \quad \text{KI}(M^p_0, M^{n-q}_0) = \sum \lambda_i.$$  

(47.10) Noteworthy special case: the manifolds are complex algebraic varieties. Then orientations may be assigned in advance to all manifolds such that the index (47.7) is always $+1$, and hence (47.9) always greater than 0. For details see [L, 379].

(47.11) We have assumed that the intersection $M^p \cap M^{n-q}$ is an $M'$. Using Whitney's theorem (46.3) this restriction may be removed by a suitable method of approximation (Whitney [a]). This was carried out in [L, 383] for algebraic varieties. One may state more precisely this: If $M^r, M^{n-r}$ are $C^r$-imbedded ($r$ finite and greater than 0) in $M^n$, then an arbitrarily small deformation of $M^r$ may be chosen such that afterwards $M^r \cap M^{n-r}$ is an $M^{p-q}$ $C^r$-imbedded in $M^n$. If $M^n \subset \mathbb{R}^n$ then this holds also for $k = \omega$.

(47.12) We may conclude with the historical remark that the structures with which Poincaré initiated his epoch-making investigations in topology were essentially differentiable manifolds (see his paper [a]). It was only in order to straighten out difficulties with torsion which were pointed out to him by Heegaard that Poincaré in [b] invented the first complexes. In a certain sense the work of Whitney and Cairns may be said to have placed for the first time these first contributions of Poincaré on a solid basis.

§9. GROUP MANIFOLDS

48. A number of properties of the homology groups of group manifolds have been obtained by various authors. These investigations culminated recently in a highly interesting theorem due to H. Hopf [c] (see 49.1) which we propose to discuss in the present section. It is noteworthy that Hopf's methods belong chiefly to the domain of chain-multiplication. Further references notably to the related work of E. Cartan and Pontrjagin will be found in Hopf's paper.
(48.1) Conventions. Unless otherwise stated in the present section all cycles, \(\cdots\) are rational, and all manifolds are finite connected absolute orientable geometric.

(48.2) Definitions. Returning for a moment to standard group terminology, let \(G = \{g\}\) be a multiplicative group. Then \(G\) is said to be a topological group, or a group space, whenever it is assigned a Hausdorff topology making \(g_{i}^{-1}\) continuous. If \(G\) has further properties such as being a manifold, \(\cdots\), we describe it as a "group manifold," \(\cdots\). Thus a closed connected Lie group is an analytical group manifold \(M\) such that the group operations are analytical functions on \(M\).

(48.3) \(\Gamma\)-complexes. Hopf himself has shown that the properties under consideration are those of a large class of manifolds, the \(\Gamma\)-manifolds, described below. Using essentially the same methods but with cocycles replacing more or less cycles, we shall show that they are properties of a large class of complexes called by analogy \(\Gamma\)-complexes.

I. \(\mathcal{V} = \{x\}\) be a finite connected simple \(\text{complex}\) (special case: \(X\) is simplicial) and let its classes (rational unless otherwise stated) be written \(\Gamma^{\alpha}, \Gamma_{\beta}\), or alternately \(\Delta^{\alpha}, \Delta_{\beta}\). The intersections existing in \(X\) are denoted as usual by a dot-product. The rational cohomology ring will be written \(R(X)\). We say that \(X\) is a \(\Gamma\)-complex whenever:

(a) For some \(n\) the Betti number \(R^{n}(X) = 1\). Choosing a fixed rational homology class \(\Gamma^{n} \neq 0\) every other is of the form \(t\Gamma^{n}, \, t\text{ rational}\).

(b) \(\Gamma_{p} \cdot \Gamma^{n} = \Gamma^{n-p} = 0 \iff \Gamma_{p} = 0\).

(c) There exists a multiplication \(\mu\) of \(X\), \(X\) to \(X\), i.e., a chain-mapping \(X^{2} \to X\) such that if \(\Gamma^{n}\) is the class of a vertex then both \(\mu \Gamma^{n} \times 1^{n}\) and \(\mu 1^{n} \times \Gamma^{n}\) are different from 0.

Notice that

\[\mu \Gamma^{n} \times 1^{n} = c_{1}\Gamma^{n}, \quad \mu 1^{n} \times \Gamma^{n} = c_{2}\Gamma^{n},\]

and so (c) is equivalent to the relation

\[(c') \quad c_{1} \neq 0.\]

It has been observed by Hurewicz that the proof of Hopf's theorem (49.1) merely requires the following property: There exists a multiplication \(\mu\) of \(X\) to \(X\) such that \(\Gamma^{p} \to \mu (\Gamma^{p} \times \Gamma^{p})\) and \(\Gamma^{p} \to \mu (\Gamma^{p} \times \Gamma^{p})\) define isomorphisms of the rational homology groups of \(X\) with themselves. However (abc) appear to be better adapted to the applications.

(48.4) \(\Gamma\)-manifolds. A \(\Gamma\)-manifold is a connected \(M^{n}\) such that there exists a mapping \(t: (M^{n})^{2} \to M^{n}\), written also as a product \(xy (x, y \in M^{n})\), whose values are in \(M^{n}\), and with the following property: if \(x, y\) is fixed then \(xy\) defines a mapping \(t_{x}[t_{y}]: M^{n} \to M^{n}\) with a degree \(c_{1}[c_{2}]\), and we must have \(c_{1}c_{2} \neq 0\).

(48.5) A \(\Gamma\)-manifold is a \(\Gamma\)-complex.

That (48.3ab) hold is a consequence of (V, 29.9, 36.14). Regarding (48.3c),
or its equivalent (48.3c'), consider first a mapping \( t: K \rightarrow L \) where \( K, L \) are finite geometric complexes. By (23.8) there is a barycentric mapping \( t_1: K^{(s)} \rightarrow L \) homotopic to \( t \). If \( t_1 \) induces the chain-mapping \( \theta: K^{(s)} \rightarrow L \) then \( \theta \) induces the same simultaneous homomorphism \( \hat{\theta} \) in the homology groups as \( t_1 \), and hence as \( t \). If \( \delta \) denotes chain-derivation in \( K \) then \( \theta \delta \) is a chain-mapping \( K \rightarrow L \) inducing likewise \( \hat{\delta} \). The same considerations hold also if \( K, L \) are polyhedral complexes, since their derived are still simplicial. Applying them to the mapping: \( (M')^2 \rightarrow M \) induced by \( xy \), we find that there is a chain-multiplication \( \mu \) of \( M' \), \( M' \) to \( M \) such that if \( \Gamma^0 \) is the class of a vertex then, by the definition of the degrees: \( \mu \Gamma^0 \times \Gamma^0 = c_1 \Gamma^0, \mu \Gamma^0 \times \Gamma^0 = c_1 \Gamma^0 \), and so under our assumption (48.3c'), holds. This proves (48.5).

(48.6) A group manifold is a \( \Gamma \)-manifold and hence also a \( \Gamma \)-complex.

For if \( xy \) is the group operation then for \( x = 1 \) we have \( t_1 = 1 \) and hence \( c_1 = 1 \), and similarly \( c_0 = 1 \).

(48.7) Conclusion. The homology properties of \( \Gamma \)-complexes are also homology properties of \( \Gamma \)-manifolds and group manifolds.

(48.8) Returning to the \( \Gamma \)-complex \( X \) we will designate by \( \Gamma_0 \) the class of its fundamental cocycle (sum of the duals of the vertices) and we note that by (V, 4.13, 8.9):

\[
\Gamma_0 \cdot \Gamma_0 = \Gamma_0 \cdot \Gamma_0 = \Gamma_0.
\]

Thus \( \Gamma_0 \) is the unit of the ring \( R(X) \).

49. We are now ready for our main argument and prove:

(49.1) Theorem of Hopf. The rational homology group and ring of a \( \Gamma \)-complex, hence those of a \( \Gamma \)-manifold or group manifold, are isomorphic with those of a certain finite product \( S \) of odd-dimensional spheres.

The proof will rest upon the following proposition:

(49.2) There may be chosen \( l \) cohomology classes \( \{ \Delta^i \} \) in \( R(X) \) and \( \{ C^i \} \) in a suitable \( R(S) \) such that \( \dim \Delta^i = \dim C^i \), and that \( \Delta^i \rightarrow C^i \) define isomorphisms of \( R(X), R(S) \).

To complete the picture we will also prove with Hopf:

(49.3) Every finite product of odd-dimensional spheres is a \( \Gamma \)-manifold.

(49.4) Remark. Actually Hopf's treatment was given only for \( \Gamma \)-manifolds. However, by making use of the co-theory, his proof extends to \( \Gamma \)-complexes.

50. (50.1) Let \( \mu^*: X^* \rightarrow X^{*2} \) be the dual of \( \mu \). As regards both \( \mu, \mu^* \) we designate also by \( \mu, \mu^* \) the induced homomorphism in the homology and cohomology groups. If we choose bases for the rational cohomology classes then every cycle of \( X^{*2} \), i.e., cocycle of \( X^2 \), is a linear rational combination of products of elements of the bases (IV, 6.7). Therefore with \( p > 0 \):

\[
\mu^* \Gamma_p = \Gamma_0 \times \lambda \Gamma_p + (\rho \Gamma_p) \times \Gamma_0 + \sum_{\alpha \neq 0} \Gamma_\alpha \times \Gamma_\beta.
\]
According to (V, 14.1c) the operations \( \lambda : \Gamma_p \to \lambda \Gamma_p \), \( \rho : \Gamma_p \to \rho \Gamma_p \) are homomorphisms \( R(X) \to R(X) \). We first prove:

(50.3) \( \lambda \) and \( \rho \) are isomorphisms.

It will be sufficient to consider \( \lambda \). Since \( R(X) \) is a vector space, we merely need to show that \( \lambda \) is univalent. Suppose then \( \lambda \Gamma_p = 0 \). By (V, 17.6):

\[
\mu^* \Gamma_p \cdot \Gamma^0 \times \Gamma^n = (\Gamma_0 \times \lambda \Gamma_p) \cdot (\Gamma^0 \times \Gamma^n)
= (\Gamma_0 \cdot \Gamma^0) \times (\lambda \Gamma_p \cdot \Gamma^n)
= \Gamma^0 \times (\lambda \Gamma_p \cdot \Gamma^n).
\]

Applying now (V, 14.1a) and with an obvious permutation of terms in accordance with (V, 8.8a) we have if \( \lambda \Gamma^n = 0 \):

\[
\mu(\mu^* \Gamma_p \cdot \Gamma^0 \times \Gamma^n) = \Gamma_p \cdot c_i \Gamma^n = c_i(\Gamma_p \cdot \Gamma^n)
= \mu(\Gamma^0 \times (\lambda \Gamma_p \cdot \Gamma^n)) = 0.
\]

Since \( c_i \neq 0 \) by (48.3b): \( \Gamma_p = 0 \). Thus \( \lambda \) is univalent and (50.3) follows.

(50.4) Generators for \( R(X) \). If \( \{ \Gamma^i \} \) is any subset of \( R(X) \), including \( \Gamma_0 \), then the operations of the ring applied between the elements of the set give rise to a subring \( R(\Gamma^i) \) of \( R(X) \). If \( R(\Gamma^i) = R(X) \), then the \( \Gamma^i \) other than \( \Gamma_0 \) are called generators of \( R(X) \). A set of generators is said to be irreducible if no proper subset is a set of generators for the ring.

We say that a class \( \Gamma_p \), \( p > 0 \), is maximal if we cannot write \( \Gamma_p = \sum \Gamma_k \cdot \Gamma^i \), where \( 0 < q, r < p \).

An irreducible system of generators may be constructed as follows: choose for each \( p > 0 \) a maximal set of maximal \( \Gamma_p \) linearly independent modulo the non-maximal elements. The total set \( \{ \Delta^i \}, i = 1, 2, \cdots, l \), thus obtained, is clearly an irreducible set of generators consisting of actual cohomology classes, and not merely of sums of classes of mixed dimensions. We prove:

(50.5) \( \Delta^1 \cdots \Delta^l \neq 0 \).

Since every \( \Delta^i \neq 0 \) and the order of the \( \Delta^i \) is immaterial the proof reduces to:

(a) \( \Delta^1 \cdots \Delta^k \neq 0 \Rightarrow \Delta^1 \cdots \Delta^k \neq 0 \), where we also assume \( \dim \Delta^{i+1} \leq \dim \Delta^i \), and where \( \{ \Delta^1, \cdots, \Delta^l \} \) are merely any \( k \) generators of the initial set.

Let \( \Delta^i = \lambda \Delta^1 \), and denote by \( \mathfrak{A} \) the ideal of \( R(X) \) based on \( \{ \Delta^2, \cdots, \Delta^k \} \), or set of elements \( \{ \sum_{i>1} \Gamma_i \cdot \Delta^i \} \). Corresponding to \( \mathfrak{A} \) the elements \( \{ \Gamma_p \times \Delta^i \} \), \( i > 1 \), generate an ideal \( \mathfrak{A}^* \) of \( R(X) \). Now \( \mu^* \Delta^i = \Gamma_0 \times \Delta^i + \rho \Delta^i \times \Gamma_0 + \sum \Gamma_p \times \Delta^i \times \Gamma_r, 1 \leq i \leq k \), where \( r < \dim \Delta^i \). Since \( \lambda \) is an isomorphism it preserves dimensions and furthermore \( \{ \Delta^i \}, j = 1, 2, \cdots, l \), is likewise a set of generators for \( R(X) \). On dimensional grounds then \( \Gamma_p \) is a linear rational combination of intersections of the \( \Delta^j, j > 1 \), i.e., \( \Gamma_p \in \mathfrak{A}^* \) and hence \( \Gamma_p \times \Gamma_r \in \mathfrak{A}^* \). Thus we have

\[
\mu^* \Delta^i = \Gamma_0 \times \Delta^i + \rho \Delta^i \times \Gamma_0 \mod \mathfrak{A}^*.
\]
From this follows by repeated application of (V, 17.6):
\[ \mu^*(\Delta_1 \cdots \Delta^k) = \rho(\Delta_1 \cdots \Delta^k) \times \Gamma_0 \pm \rho(\Delta_1 \cdots \Delta^i) \times \Delta^i \text{ mod } \mathfrak{A}^*. \]
Hence \( \Delta^1 \cdots \Delta^k = 0 \implies \rho(\Delta_1 \cdots \Delta^i) \times \Delta^i \equiv 0 \text{ mod } \mathfrak{A}^* \). Since \( \Delta_1 \cdots \Delta^i \neq 0 \) and \( \rho \) is an isomorphism we also have \( \rho(\Delta_2 \cdots \Delta^i) \neq 0 \), and so by a transparent application of (II, 37.5): \( \Delta^i \in \mathfrak{A} \). Since \( \lambda \) is an isomorphism this implies that \( \Delta^i \) is in the ideal based on \( \{ \Delta_2, \ldots, \Delta^1 \} \), contrary to assumption. This proves (a), and hence (50.5).

(50.6) If \( p \) is even \( \Gamma_p \) is not maximal.

Let \( \mathfrak{B} \) be the ideal of \( R(X) \) generated by the elements \( \Gamma_r, r \neq 0, p \). Thus \( \mathfrak{B} \) consists of all the elements of dimension different from 0, \( p \) and of the nonmaximal \( \Gamma_p \). Denote also by \( \mathfrak{B}^* \) the ideal of \( R(X^*) \) based on the elements \( \Gamma_q \times \Gamma_r, \Gamma_q, \Gamma_r \in \mathfrak{B} \). We have:
\[ \mu^* \Gamma_p = \Gamma_0 \times \lambda \Gamma_p + \rho \Gamma_p \times \Gamma_0 \text{ mod } \mathfrak{B}^*. \]

Supposing now \( p \) even, we find by multiplying \( m \) times and recollecting (V, 8.8a, 17.6):
\[ \mu^*(\Gamma_p)^m = \rho(\Gamma_p)^m \times \Gamma_0 + m\rho(\Gamma_p)^{m-1} \times \lambda \Gamma_p \text{ mod } \mathfrak{B}^*, \]
where the powers refer to repeated dot-products. Since \( X \) is finite-dimensional there exists an \( m \) such that \( (\Gamma_p)^{m-1} \neq 0 \), \( (\Gamma_p)^m = 0 \) and we shall have:
\[ \rho(\Gamma_p)^{m-1} \times \lambda \Gamma_p = 0 \text{ mod } \mathfrak{B}^*. \]

Since \( \rho \) is an isomorphism \( \rho(\Gamma_p)^{m-1} \neq 0 \), and so by (II, 37.5): \( \lambda \Gamma_p \in \mathfrak{B} \), proving (50.6).

(50.8) Every generator \( \Delta^i \) is of odd dimension.

For \( \Delta^i \) is maximal and so by (50.6) its dimension is odd.

(50.9) \[ \Delta^i \cdot \Delta^j = -\Delta^i \cdot \Delta^j, \quad \Delta^i \cdot \Delta^i = 0 \quad (V, 8.8a). \]

51. (51.1) By (50.8) \( p_i = \dim \Delta^i \) is odd. Let \( S^{p_i} \) denote a \( p_i \)-sphere (boundary of a \( (p_i + 1) \)-simplex), and let \( A^i \) be the fundamental zero-cohomology class, and \( B^i \) a \( p_i \)-cohomology class different from 0 for \( S^{p_i} \). Referring also to (V, 8.9) or (V, 20.5) there comes
\[ A^i \cdot A^i = A^i, \quad A^i \cdot B^i = B^i \cdot A^i = B^i, \quad B^i \cdot B^i = 0, \]
(the last on dimensional grounds). Set now \( S = \mathbb{P}S^{p_i} \) and define
\[ C^i = A^i \times \cdots \times A^{i-1} \times B^i \times A^{i+1} \times \cdots \times A^i, \]
\[ C^{ij\cdots k} = D^1 \times \cdots \times D^i, \]
where \( D^h = A^h \) for \( h \neq i, j, \ldots, k \) and \( D^h = B^h \) otherwise.

By (IV, 6.7) the \( C^{ij\cdots k}, i + \cdots + k = m > 0 \), form a base for the rational group \( \mathcal{S}_m(S) \), while \( \mathcal{S}_m(S) \) consists merely of the multiples of the cohomology class \( C = \mathbb{P}A^i \) which is the \( \Gamma_0 \) of \( S \). It follows that the rational ring \( R(S) \)
is generated by the intersections of the classes $C^i$, $C^{i_1 \cdots i_k}$. It is easily seen that the $C^i$ are maximal and that $C^{i_1 \cdots i_k} = \pm C^i \cdots C^i$. Hence $\{C^i\}$ is an irreducible system of generators. It is also an elementary matter to verify that $C^i \leftrightarrow \Delta^i$ defines an isomorphism $R(S) \leftrightarrow R(X)$ and consequently also $\mathcal{F}_m(S) \leftrightarrow \mathcal{F}_m(X)$. This completes the proof of (49.2) and hence also of (49.1).

(51.3) Proof of (49.3). An outline of the argument will suffice. It breaks up into two parts:
(a) An odd-dimensional sphere $S^m$ is a $\Gamma$-manifold.
(b) The product of two $\Gamma$-manifolds is a $\Gamma$-manifold.

Clearly (a, b) together prove (49.3).

Take first (a). If $x, y \in S^m$ we denote by $xy$ the reflection of $x$ in the diameter through $y$. In the notations of (48) $l$ is topological, and so $c_r = \pm 1$. A fairly simple argument whose detail is omitted (see Hopf [c, 30]) shows then that $c_l = \pm 2$. This is sufficient to prove (a).

Take now two $\Gamma$-manifolds $M, M'$, and let $t_i, t'_i, \cdots$ analogous to $t_i, \cdots$ of (48) have their obvious meaning. If $(x, x'), (y, y') \in M \times M'$ we define multiplication in the latter by $(x, x') \times (y, y') = (xy, x'y')$. By an argument such as the one leading to (48.5) it is shown that the corresponding chain-multiplication $X^k \times Y^k \to X \times Y$ is $\mu \times \mu'$ (notation of IV, 21) and the analogues of $c_l, c_r$ for $\mu \times \mu'$ are $c_{\mu l} = 0, c_{\mu r} = 0$. Since $M \times M'$ is a connected geometric manifold (b) follows and (49.3) is proved.

52. Many interesting properties may now be deduced from (49.1). We reproduce a few, all taken from Hopf [c], where references to earlier related results, notably those due to E. Cartan, are also given. In the statements $X$ may be a $\Gamma$-complex, a $\Gamma$-manifold or a group manifold.

(52.1) The elements $\Gamma^k, \Delta^i, \Delta^{i_1 \cdots i_k}$, $(t_i < t_i)$, $\cdots$ form a base for the vector space $R(X)$. Hence $2^l = \dim R(X)$, and so $l$ depends solely upon $X$.

(52.2) No even-dimensional class $\Gamma_{2p}$ is maximal. In other words a $\Gamma_{2p}$ is a sum of intersections of classes of dimension less than $2p$.

(52.3) The Poincaré polynomial of $X$ (III, 15.3) is given by:

$$P(t, X) = (1 + t^p) \cdots (1 + t^{p_i}),$$

where the $p_i$ are all odd.

(52.4) The Euler characteristic $\chi(X) = P(-1, X) = 0$.

(52.5) The sum of the Betti numbers of $X$ is a power of two, namely $2^l$.

(52.6) $R^p(X) = R^{n-p}(X)$.

In other words, the duality theorem of Poincaré for the Betti numbers of a manifold (V, 33.1a) holds for $X$. Needless to say this is far from making $X$ a manifold. It must be remembered that $n$ is the integer occurring in (48.3ab), and need not be the dimension of $X$ which may well exceed $n$. However, if $X$ is a $\Gamma$-manifold, hence also when it is a group manifold, then $n$ is the dimension of the manifold.
(52.7) The dimensions $p_i$ of the generators satisfy the relation

$$p_1 + \cdots + p_i = n.$$  

For $S$, hence $X$, has an $n$-cycle and none of higher dimension.

(52.8) Various relations for the Betti numbers $R^j$ follow from the expression of $P(t, X)$. Thus:

$$R^1 \leq n, \quad R^2 = \binom{R^1}{2}, \quad R^* \cong \binom{R^1}{s}.$$  

53. Complements.

(53.1) The Pontryagin ring. To the multiplication $\mu$ of (48) associated with the $\Gamma$-complex $X$ there corresponds a rational homology ring $P(X, \mu)$ in the sense of (V, 1.10). This ring was introduced for the first time for group manifolds by Pontryagin, and for this reason Hopf designates it as the Pontryagin ring. When $X$ is a group manifold the ring is associative, but it need not be so otherwise. It has been proved by Samelson [a] that whenever the ring $P(X, \mu)$ is associative and $\Gamma^0$ (the class of a vertex) is a unit of this ring, then the isomorphism of $R(X)$ with $R(S)$ ($S$ is the sphere-product) may be so chosen as to yield an isomorphism of $P(X, \mu)$ with the analogue $P(S, \bar{\mu})$ for $S$, where $\bar{\mu}$ is the multiplication in $S$ which corresponds in the obvious way to $\mu$ in $X$.

(53.2) $\Gamma$-spaces. We will designate by that term a compact connected Hausdorff space $\mathcal{R}$ such that: (a) it has properties (48.3abc), where $\mu$ refers merely to a simultaneous homomorphism of the rational homology groups of $\mathcal{R}^2$ into these of $\mathcal{S}$; (b) the rational cohomology ring of $\mathcal{R}$ has a finite number of generators. It may be seen by reference to (VII, 16) that all the machinery is at hand for the extension to such spaces of the results centering around Hopf's theorem proved for $\Gamma$-complexes.

An arbitrarily large supply of examples of $\Gamma$-spaces which are not group or $\Gamma$-manifolds may be based upon the following property:

(53.3) If $K$ is a finite Euclidean $\Gamma$-complex and $\mathcal{R}$ has $|K|$ for deformation retract then $\mathcal{R}$ is a $\Gamma$-space.

For $\mathcal{R}$ has the same homology properties as $|K|$, and hence as $K$ (10.1; VII, 7.1, 7.5).

As an explicit example take an odd-dimensional sphere $S^{2p+1}$ and on the sphere a countable dense set $\{x_\alpha\}$. Extend the radius to $x_\alpha$ by a segment of length $2^{-n}$. The resulting space $\mathcal{R}$ has $S^{2p+1}$ for deformation retract and so it is a $\Gamma$-space, which is manifestly not a manifold.

§10. NOMENCLATURE OF COMPLEXES AND MANIFOLDS

54. Complexes. The complexes in the sense utilized in the present work (III, 1.1) have generally been called "abstract complexes." They may be finite or infinite and in the latter case the most important subtypes are: star-finite, closure-finite and locally finite complexes. The following special types have been introduced:
A. **Simplicial complexes** (III, 5). The simplexes have usually been designated by \( \sigma, \tau \). If \( K \) is a simplicial complex, \( L \) a closed subcomplex then \( K - L \) is known as an open simplicial complex, and by contrast \( K, L \) as closed simplicial complexes.

B. **Polyhedral complexes**. The collection \( \Pi = |E| \) of the faces of a polyhedron turned into a complex (III, 6). The polyhedron as a space is denoted by \( |\Pi| \).

C. **Euclidean complexes**. Polyhedral complexes whose elements are Euclidean simplexes in an Euclidean space or in the Hilbert parallelootope (III, 6.9).

D. **Topological complexes**: Collections \( K = |E| \) whose elements are cells with incidences and dimensions similar to those in a polyhedral complex (VIII, §7). The union of the cells is a metric space denoted by \( |K| \) and called a polyhedral space.

E. **Differentiable complexes, \( C^r \)-complexes**. Topological complexes whose cells are subjected to certain differentiability conditions (VIII, 45).

F. **Simple complexes**. Closure-finite complexes whose elements satisfy certain algebraic conditions (III, 47.1). The class includes all the types A, \( \ldots, E \).

G. **\( \Gamma \)-complexes**. Simple complexes, with certain special properties described in (48.3).

55. **Manifolds**. All the manifolds to be described are supposed to be \( n \)-dimensional.

A. **Combinatorial manifolds**. The types investigated in (V, §4). They may be finite or infinite, absolute or relative, orientable or non-orientable, simplicial or merely simple complexes.

B. **Geometric manifolds**. Euclidean realizations of the preceding simplicial types.

C. **Manifolds in the sense of Brouwer**. Euclidean complexes such that the star of each vertex is isomorphic with a set of simplexes in an Euclidean \( \mathbb{S}^n \) having a common vertex \( P \) and making up a neighborhood of \( P \) in \( \mathbb{S}^n \).

D. **Manifolds in the sense of Neuman**. Euclidean complexes such that if \( a \) is a vertex and \( St a = aB \), then \( B \) is partition-equivalent to an \( (n - 1) \)-sphere.

E. **Manifolds in the sense of Poincaré [b] and Veblen [V]**. Topological complexes such that every point has for neighborhood an \( n \)-cell.

F. **Topological manifolds**. An \( M^n \) of this type is a separable metric space with a countable locally finite open covering consisting of \( n \)-cells. (See Flexner [a, b].) Noteworthy special cases: \( C^r \)-manifolds, differentiable manifolds, analytical manifolds (44.1), \( \Gamma \)-manifolds, group manifolds (48.2, 48.3).

G. **Generalized manifolds**. Locally compact spaces discussed by Čech [b], Lefschetz [c], Wilder [a] and others and characterized by certain properties of so-called "local connectedness" or "local connectedness in the sense of homology" and also by the property: each point is \( n \)-cyclic. They have been in-
vestigated at length in a forthcoming paper by E. Begle [a] to which the reader is referred for all details.

H. *Pseudo-manifolds.* This term has been applied by Brouwer and other authors to what we have called a simple geometric $n$-circuit.

I. *Manifolds of grade $p.*$ Simplicial $n$-complexes, investigated by Čech, and which behave like an $M^*$ only as regards the two consecutive dimensions $p - 1, p.$