Lecture 5

Local Bifurcation of Vector Fields

5.1 Introduction

Consider the parameter dependant vector field

$$\dot{x} = X(x, \mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^p,$$

and suppose there is an isolated equilibrium at $$(x, \mu) = (\bar{x}, \bar{\mu}).$$ The basic questions we wish to address are:

- Is the equilibrium stable or unstable?
- How is stability affected as the parameter $$\mu$$ varies?

Key tools in addressing these questions are the center manifold theorem, normal forms, and the structure of the linearization. Set $$\xi = x - \bar{x},$$ then the linearization takes the form

$$\dot{\xi} = J_X(\bar{x}, \bar{\mu}) \xi$$  \hspace{1cm} (5.1)

If the equilibrium is hyperbolic, then Hartman’s theorem (Theorem 2.8) tells us that stability is completely determined by the eigenvalues of the Jacobian. Furthermore, stability/instability is preserved under small changes in $$\mu.$$ Indeed, we have $$X(\bar{x}, \bar{\mu}) = 0$$ and $$J_X(\bar{x}, \bar{\mu})$$ has no eigenvalues with real part zero, hence is invertible. By the implicit function theorem there is a unique $$C^\infty$$ function $$x(\mu)$$ such that $$x(\bar{\mu}) = \bar{x},$$ and

$$X(x(\mu), \mu) = 0$$

for small $$|\mu - \bar{\mu}|.$$ Continuity of the eigenvalues with respect to the parameters implies that $$J_X(x(\mu), \mu)$$ has no eigenvalues with real part zero. This is the property of persistence of hyperbolic equilibria.
5.2 The Zero Eigenvalue

5.2.1 Problem Structure & Examples

Let’s assume that $J_X(\bar{x}, \bar{\mu})$ has a single zero eigenvalue and all other eigenvalues have non-zero real part. Then the center manifold theorem applies and the dynamics near the equilibrium $(\bar{x}, \bar{\mu})$ are determined by a one-dimensional vector field on the center manifold. For simplicity we write this vector field in the form

$$\dot{y} = Y(y, \mu), \ y \in \mathbb{R}, \ \mu \in \mathbb{R}^p;$$

and assume the equilibrium is at $(0, 0)$. Then the conditions for equilibrium and a zero eigenvalue are

$$Y(0, 0) = 0$$

$$\frac{\partial Y}{\partial y}(0, 0) = 0. \quad (5.2)$$

Recall the following examples. They provide specific vector fields satisfying the above conditions. The first three actually provide normal forms associated with the three types of bifurcations.

Example 5.1 $\dot{y} = \mu - y^2$. It is easy to see that for $\mu < 0$ there are no equilibria, for $\mu = 0$ then $y = 0$ is the only equilibria and it is stable. For $\mu > 0$ there are two equilibria $y^2 = \mu; y = \sqrt{\mu}$ is stable, $y = -\sqrt{\mu}$ is unstable. Hence, the stable equilibrium at $(y, \mu) = (0, 0)$ splits into two equilibria for $\mu > 0$. This is called the saddle-node bifurcation. Notice that in the $\mu - y$ plane there is a unique curve of fixed points passing through the origin which lies entirely on one side of $\mu 0 = 0$.

Example 5.2 $\dot{y} = \mu y - y^2$. Here we have equilibria $y = 0$ and $y = \mu$. The stability is depicted in the following table.

<table>
<thead>
<tr>
<th></th>
<th>$y = 0$</th>
<th>$y = \mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu &lt; 0$</td>
<td>stable</td>
<td>unstable</td>
</tr>
<tr>
<td>$\mu = 0$</td>
<td>coalesce, unstable</td>
<td></td>
</tr>
<tr>
<td>$\mu &gt; 0$</td>
<td>unstable</td>
<td>stable</td>
</tr>
</tbody>
</table>

At the equilibrium $(y, \mu) = (0, 0)$, there is an exchange of stability. This is called the transcritical bifurcation. Geometrically, there are two curves of equilibria which intersect at the origin and lie on both sides of $\mu = 0$. Stability of the equilibrium changes along either curve on passing through $\mu = 0$. 

47
Example 5.3 \( \dot{y} = \mu y - y^3 \). The equilibria are \( y = 0 \) and \( y = \pm \sqrt{\mu} \). Here the following table is apparent.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( y = 0 )</th>
<th>( y = \sqrt{\mu} )</th>
<th>( y = -\sqrt{\mu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu &lt; 0 )</td>
<td>stable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = 0 )</td>
<td>stable</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu &gt; 0 )</td>
<td>unstable</td>
<td>stable</td>
<td>stable</td>
</tr>
</tbody>
</table>

In this case \( y = 0 \) is always an equilibrium, stable when \( \mu \leq 0 \). For \( \mu > 0 \), the equilibria becomes unstable and splits into two stable equilibria. This is called the pitchfork bifurcation. Again, two curves of equilibria intersecting at the origin. One curve \((y = 0)\) lies on both sides of \( \mu = 0 \); stability changes upon crossing. The other curve lies on one side with stability opposite of the equilibrium \( y = 0 \) on that side.

Example 5.4 \( \dot{y} = \mu - y^3 \). The equilibria are given by the curve \( y^3 = \mu \). All are stable, no bifurcation occurs. This is an example of a curve of non-hyperbolic equilibria whose orbit structure is qualitatively the same for all \( \mu \).

Here is a loosely stated definition of bifurcation.

An equilibrium point \((y, \mu) = (0,0)\) of a one-parameter family of one-dimensional vector fields is said to undergo a bifurcation at \( \mu = 0 \) if the flow for \( \mu \) near zero and \( y \) near zero is not qualitatively the same as the flow near \( y = 0 \) at \( \mu = 0 \).

Remark 5.1 The phrase ‘qualitatively the same’ can be made precise by replacing it with ‘\( C^0 \)-equivalent’. This is, at least, suitable in the one dimensional case.

Remark 5.2 >From a geometric perspective, an equilibrium point \((0,0)\) of a one-dimensional vector field is a bifurcation point if either more than one curve of equilibria passes through \((0,0)\) in the \( \mu - y \) plane or if only one curve passes through \((0,0)\), the curve lies locally entirely on one side of \( \mu = 0 \).

Remark 5.3 The last example shows that the condition of non-hyperbolic equilibrium is necessary for bifurcation in the one-dimensional case, but not sufficient.

5.2.2 Bifurcation Theorems

We will now recognize that the examples of the preceding section provide normal forms for the types of bifurcations present.
We wish to establish a result which gives sufficient conditions for a vector field

\[ \dot{x} = f(x, \mu), \]  

such that

\[ f(0, 0) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(0, 0) = 0 \]  

(5.3)
to exhibit a saddle-node, transcritical, or pitchfork bifurcation at the equilibrium point \((0, 0)\). The result is based on the geometric conditions implied in Examples 5.1, 5.2, and 5.3.

**Theorem 5.1** Consider a one-dimensional vector field (5.3).

(i) Sufficient conditions for the vector field to undergo a saddle-node bifurcation are

\[ \frac{\partial f}{\partial \mu}(0, 0) \neq 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(0, 0) = 0. \]

(ii) Sufficient conditions for the vector field to undergo a transcritical bifurcation are

\[ \frac{\partial f}{\partial \mu}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0. \]

(iii) Sufficient conditions for the vector field to undergo a pitchfork bifurcation are

\[ \frac{\partial f}{\partial \mu}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial x^2}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0, \quad \text{and} \quad \frac{\partial^3 f}{\partial x^3}(0, 0) \neq 0. \]

**Proof.** We shall prove (iii) and leave the others to the reader. The geometry of the curves of equilibria in Example 5.3 had the following characteristics: (a) Two curves of equilibria passed through \((0, 0)\) in the \(\mu - x\) plane, one given by \(x = 0\), the other \(x^2 = \mu\). (b) The curve \(x = 0\) existed on both sides of \(\mu = 0\), \(x^2 = \mu\) only on one side; (c) The equilibria on the curve \(x = 0\) had different stability on opposite sides of \(\mu = 0\); the equilibria on \(x^2 = \mu\) all had the same stability type. We must have the condition

\[ \frac{\partial f}{\partial \mu}(0, 0) = 0, \]

in order to insure two curves of equilibria passing through \((0, 0)\) (otherwise, the implicit function theorem would allow only one curve). We require \(x = 0\) to be a curve of equilibria by assuming the vector field has the form

\[ F(x, \mu) = \begin{cases} \frac{\partial f}{\partial \mu}(x, \mu), & x \neq 0, \\ \frac{\partial f}{\partial x}(0, 0), & x = 0. \end{cases} \]
Clearly we need
\[ F(0, 0) = 0, \quad \text{and } \frac{\partial F}{\partial \mu}(0, 0) \neq 0. \]

The first condition insures a second curve of equilibria passing through \((0, 0)\); the second insures there is only one. The implicit function theorem implies that for sufficiently small \(x\), there exists a unique function \(\mu(x)\) such that
\[ F(x, \mu(x)) = 0. \]

To force this curve to lie on one side of \(\mu = 0\) we must have
\[ \frac{d\mu}{dx}(0) = 0, \quad \frac{d^2\mu}{dx^2}(0) \neq 0. \]

Now we have by implicit differentiation
\[
\begin{align*}
\frac{d\mu}{dx}(0) &= -\frac{\frac{\partial F}{\partial x}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)} \\
\frac{d^2\mu}{dx^2}(0) &= -\frac{\frac{\partial^2 F}{\partial x^2}(0, 0)}{\frac{\partial F}{\partial \mu}(0, 0)}
\end{align*}
\]
or in terms of the original vector field
\[
\begin{align*}
\frac{d\mu}{dx}(0) &= \frac{\frac{\partial^2 f}{\partial x^2}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)} = 0 \\
\frac{d^2\mu}{dx^2}(0) &= -\frac{\frac{\partial^2 f}{\partial x^2}(0, 0)}{\frac{\partial^2 f}{\partial x \partial \mu}(0, 0)} \neq 0.
\end{align*}
\]

The result is clear. \( \blacksquare \)

**Remark 5.4** The conditions imposed in each of the three cases actually show that the vector field has the same orbit structure near \((x, \mu) = (0, 0)\) as one of the following three forms respectively:
\[ \dot{x} = \mu \pm x^2, \quad \dot{x} = \mu x \pm x^2, \quad \dot{x} = \mu x \pm x^3. \]

These can be viewed as the **normal forms** of the saddle-node, transcritical, and pitchfork bifurcations, respectively.

**Remark 5.5** Notice that the higher order terms in the power series expansion of \(f\) do not affect the conditions in the above theorem.
5.3 Imaginary Eigenvalues

5.3.1 Problem Structure & Examples

Returning to (5.1), we suppose \( \bar{x} = 0 \) and \( \bar{y} = 0 \). Further, \( J_x(0,0) \) has a pair of imaginary eigenvalues and all other eigenvalues have non-zero real part. In this case the center manifold is two dimensional and the dynamics at the equilibrium are determined by a vector field of the form

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\text{Re}\lambda(\mu) & -\text{Im}\lambda(\mu) \\
\text{Im}\lambda(\mu) & \text{Re}\lambda(\mu)
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
f_1(x,y,\mu) \\
f_2(x,y,\mu)
\end{bmatrix},
\]

where \( \lambda(\mu) = \alpha(\mu) + i\omega(\mu) \) are the eigenvalues with \( \alpha(0) = 0 \) and \( \omega(0) \neq 0 \).

From (4.10), the normal form is

\[
\begin{align*}
\dot{x} &= \alpha(\mu)x - \omega(\mu)y + (a(\mu)x - b(\mu)y)(x^2 + y^2) + O(5) \\
\dot{y} &= \omega(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2 + y^2) + O(5).
\end{align*}
\]

(5.4)

Passing to polar form

\[
\begin{align*}
\dot{r} &= \alpha(\mu)r + a(\mu)r^3 + O(5) \\
\dot{\theta} &= \omega(\mu) + b(\mu)r^2 + O(4).
\end{align*}
\]

One more approximation using the expansions of the coefficients about \( \mu = 0 \):

\[
\begin{align*}
\dot{r} &= \alpha'(0)\mu r + a(0)r^3 + O(\mu^2 r, \mu r^3, r^5) \\
\dot{\theta} &= \omega(0) + \omega'(0)\mu + b(0)r^2 + O(\mu^2, \mu r^2, r^4).
\end{align*}
\]

Example 5.5 We neglect the higher order terms in the above equations and consider

\[
\begin{align*}
\dot{r} &= d\mu r + ar^3 \\
\dot{\theta} &= \omega + c\mu + br^2.
\end{align*}
\]

(5.5)

NOTE: Values of \( r > 0 \) and \( \mu \) which yield \( \dot{r} = 0 \) and \( \dot{\theta} \neq 0 \) correspond to a periodic orbit. Indeed, if \( -\infty < \frac{\mu d}{a} < 0 \) then with \( \mu \) sufficiently small

\[
(r(t), \theta(t)) = \left( \sqrt{\frac{-\mu d}{a}}, \left[ \omega + \left( c - \frac{bd}{a} \mu \right) \right] t + \theta_0 \right)
\]

is a periodic orbit. In fact, since we must have \( r > 0 \), this is the only periodic solution possible. Hence for \( \mu \neq 0 \), (5.5) possesses a unique periodic orbit having
amplitude $O(\sqrt{|\mu|})$. There are four cases to consider.

Case 1: $d > 0$, $a > 0$. In this case the origin is unstable for $\mu > 0$ and asymptotically stable for $\mu < 0$. The periodic orbit is unstable for $\mu < 0$.

Case 2: $d > 0$, $a < 0$. Here the origin is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$ with asymptotically stable periodic orbit.

Case 3: $d < 0$, $a > 0$. In this case the origin is unstable for $\mu < 0$ and asymptotically stable for $\mu > 0$ with unstable periodic orbit.

Case 4: $d < 0$, $a < 0$. Here the origin is asymptotically stable for $\mu > 0$ and unstable for $\mu < 0$ with an asymptotically stable periodic orbit.

5.3.2 The Poincaré-Andropov-Hopf Bifurcation

Here we wish to capture the spirit of Example 5.5. Historically, the result was known to Poincaré, first published by Andropov (1933), later published independently by Hopf (1941). It is usually referred to as the Hopf bifurcation theorem; better as the Andropov-Hopf bifurcation theorem.

**Theorem 5.2** (Andropov-Hopf) Consider the normal form (5.4) with $\alpha(0) = 0$ and $\omega(0) \neq 0$. Let $d = \alpha'(0)$ and $a = a(0)$. If $-\infty < d\mu/a < 0$, then cases (1-4) of Example 5.5 hold. In particular, a stable limit cycle bifurcates in cases 2 and 4.

This is known as the normal form version of the Andropov-Hopf theorem. What needs to be shown is that the higher order terms in (5.4) do not affect the conclusions of the Example. Very fortunately, the normal form has structure allowing the application of the following theorem due to Poincaré.

**Theorem 5.3** (Poincaré) Let $U$ be a closed bounded region in the plane which is an invariant set for a vector field. If $U$ contains no equilibria of this field, then it contains a closed orbit.

A second version of the Andropov-Hopf theorem is expressed in terms of the functions defining the vector field (assuming the linear part is in canonical form).

**Theorem 5.4** Consider a vector field

\[
\begin{align*}
\dot{x} &= f(x, y, \mu) \\
\dot{y} &= g(x, y, \mu),
\end{align*}
\]

with $f(0, 0, \mu) = g(0, 0, \mu) = 0$ and Jacobian

\[
J(0, 0, 0) = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},
\]
with $\omega \neq 0$. If

(i) $d = (f_{xx} + g_{yy})(0,0) \neq 0$, and

(ii) $a = \left( \frac{1}{16}(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}) + \frac{1}{16}\omega (f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \right)_{(0,0,0)} \neq 0$,

then

(a) a curve of periodic solutions bifurcates from the origin for $\mu < 0$ if $ad > 0$ or $\mu > 0$ if $ad < 0$; (b) the origin is stable for $\mu > 0$ (respectively $\mu < 0$) and unstable for $\mu < 0$ (respectively $\mu > 0$) if $ad < 0$ (respectively $ad > 0$); (c) the periodic solutions are stable (respectively, unstable) if the origin is unstable (respectively, stable).

The proof\(^1\) of this result is a careful computation of the terms of the normal form in terms of the power series expansions of $f$ and $g$.

5.3.3 An Example: the Brusselator

The following equations arise in chemical kinetic models.

\begin{align*}
\dot{x} &= A - (\alpha + 1)x + x^2y \\
\dot{y} &= \alpha x - x^2y
\end{align*}

There is one equilibrium point $(x, y) = (A, \alpha/A)$. The Jacobian is given by

\[ J(x, y) = \begin{bmatrix} -(\alpha + 1) + 2xy & x^2 \\ \alpha - 2xy & -x^2 \end{bmatrix} \]

\[ J(\alpha) = J(A, \alpha/A) = \begin{bmatrix} \alpha - 1 & A^2 \\ -\alpha & -A^2 \end{bmatrix}. \]

Now $T = Tr(J) = \alpha - 1 - A^2$ and $\det J = A^2$. Consequently we will have complex eigenvalues if $T^2 - 4D = (\alpha - 1 - A^2)^2 - 4A^2 < 0$. This is valid provided $1 - A^2 < \alpha < 1 + 3A^2$. We note that there is a pair of imaginary eigenvalues, $\pm iA$, when $\alpha = 1 + A^2$.

The eigenvectors when $\alpha = 1 + A^2$ are given by

$$
\begin{bmatrix}
A \\
-A + i
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A \\
-A - i
\end{bmatrix}.
$$

Consequently, the transformation

$$
T = \begin{bmatrix}
A & 0 \\
-A & -1
\end{bmatrix}, \quad T^{-1} = -\frac{1}{A} \begin{bmatrix}
-1 & 0 \\
A & A
\end{bmatrix},
$$

will convert $J = J(1 + A^2)$ to canonical form, i.e.,

$$
T^{-1}JT = \begin{bmatrix}
0 & -A \\
A & 0
\end{bmatrix}.
$$

Translating the equilibrium point to the origin we arrive at

$$
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\alpha - 1 & A^2 \\
-A & -A^2
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
\frac{\partial}{\partial x} (x^2 + 2Ax + x^2y) \\
\frac{\partial}{\partial y} (x^2 - 2Ax - x^2y)
\end{bmatrix}.
$$

Set

$$
\begin{bmatrix}
x \\
y
\end{bmatrix} = T \begin{bmatrix}
u \\
v
\end{bmatrix} = \begin{bmatrix}
Au \\
-Au - v
\end{bmatrix},
$$

then the system becomes

$$
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} = \begin{bmatrix}
A^2 - (\alpha - 1) & -A \\
A & 0
\end{bmatrix} \begin{bmatrix}
u \\
v
\end{bmatrix} + \begin{bmatrix}
(\alpha - 2A^2)u^2 - 2Au - A^2u^3 - Au^2v \\
0
\end{bmatrix}.
$$

This is the necessary form to apply Theorem 5.4. In this case

$$
d = \frac{1}{2} > 0
$$

and

$$
a = \frac{1}{16} (f_{uuu} + f_{uvv}) + \frac{1}{16A} (f_{uv}(f_{uu} + f_{vv})) \bigg|_{(0,0,1+A^2)}
\quad = \quad -\frac{A^2}{8} - \frac{1}{4} < 0.
$$

It follows that a stable limit cycle bifurcates for $\alpha > 1 + A^2$. 

54