Energy Level Statistics of a Critical Random Matrix Ensemble

Macleans L. Ndawana† § and Vladimir E. Kravtsov‡∥

† Institut für Physik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany
‡ International Center for Theoretical Physics, P. O. Box 586, 34100 Trieste, Italy

Abstract. We study energy level statistics of a critical random matrix ensemble of power-law banded complex Hermitian matrices. We compute the level compressibility via the level-number variance and compare it with the analytical formula for the exactly solvable model of Moshe, Neuberger and Shapiro.

1. Introduction

Random matrix ensembles (RME) provide a natural framework for the statistical description of quantized energy levels of complex systems. This is probably best explained by paraphrasing Dyson [1]: in ordinary statistical mechanics a renunciation of knowledge about the system is made i.e., by assuming that all states of a large ensemble are equally probable. Therefore only the overall behavior of the system is accessible. Thus orthodox statistical mechanics is clearly inadequate for the discussion of energy level statistics of complex systems. In energy level statistics one wishes to make statements about the fine detail of the energy level structure, and such statements cannot be made in terms of an ensemble of states. Random matrix theory (RMT) turns out to be a suitable tool in making such statements about the eigen-spectrum of complex systems e.g., energy levels of highly excited nuclei [2]. The underlying assumption in RMT is that one renounces exact knowledge not of the state but of the nature of the system itself. This is the fundamental difference between ordinary statistical mechanics and energy level statistics.

There are three RME that emerge from an analysis of real eigen-spectra of random Hermitian matrices [3]: the Gaussian orthogonal ensemble (GOE) for a system with time-reversal without spin-\(\frac{1}{2}\) interactions, the Gaussian symplectic ensemble (GSE) for a system with time-reversal and spin-\(\frac{1}{2}\) interaction, and the Gaussian unitary ensemble (GUE) for a system without time-reversal. These ensembles (generally called Wigner-Dyson (WD) statistics) exhibit strong energy-level correlations leading to the phenomenon of level repulsion [4] in their respective eigen-spectra. One could add the

§ Permanent address: University of Zambia, Department of Physics, Lusaka, Zambia
∥ Landau Institute for Theoretical Physics, Kosygina str. 2, 117940 Moscow, Russia
Poisson statistics (PS) as the fourth ensemble which does not exhibit level repulsion in it’s corresponding eigen-spectra.

In general, the WD statistics describe a system of delocalized states and the PS describe a system of localized states. The transition between these two cases controlled by the strength of disorder \textit{i.e} the Anderson model \cite{5} is a quantum critical phenomenon. At a critical disorder the states acquire a special property of \textit{multifractality} \cite{6,7} which marks a qualitative difference as compared to truly delocalized states in a metal and truly localized states in an insulator. These critical states correspond to the \textit{critical level statistics} (CLS) \cite{8,9}.

There are numerous suggestions in literature \cite{6,10,11,12,13,14,15,16,17} describing CLS by ensembles of random matrices. Initially, there were two main requirements for the critical RME: the first was that the level statistics be system-size independent and secondly, the ensemble should be able to interpolate between WD and PS \cite{8}. Later on it was realized \cite{18,6} that the key property of the critical states relevant for the CLS is multifractality. The power-law banded RME suggested in \cite{11} emerged as the best candidate for the critical RME as its level statistics is known to possess these two properties and its eigenfunction statistics is multifractal. Thus it has been suggested \cite{6} to be the critical random matrix ensemble (CrRME). In the limit of large bandwidth this model turns out to be the best deformation of the WD-RME that generates correlations in the eigenstates but retains all the basic properties of WD theory, including the factorizability of higher spectral correlation functions and mapping onto the Calogero-Sutherland model of fictitious interacting fermions \cite{19}.

In the absence of an exact solution of CrRME various mappings on and approximate correspondences with exactly solvable models have been used in order to study the properties and CrRME. Here we mention the mapping onto the nonlinear sigma-model \cite{11}, and the approximate equivalence \cite{6} between the CrRME and the exactly solvable RME suggested in Refs. \cite{12,10}. Making use of the exact solution \cite{12} it was possible \cite{20} to compute the level spacing distribution function (LSDF) for this model and compare it with the exact diagonalization of the 3d Anderson model for the critical value of disorder. The coincidence of results (for certain values of the parameter \(c\) which were found from the fitting of the far tails of LSDF) was amazing \cite{20,23}: in all three Dyson symmetry classes the deviation was not larger than between the Wigner surmise and the exact WD distribution function.

In this paper we study the level number variance in the CrRMT suggested \cite{11}. The ultimate goal here is to compare the numerical simulations of this critical ensemble and the analytical results for the exactly solvable ensemble in Ref. \cite{10} (which will be referred to as MNS) for all values of the control parameter \(c\). For large \(c \gg 1\) (weak multifractality) these two ensembles are known to be equivalent \cite{6}. The first correction to the Poisson spectral statistics at small \(c \ll 1\) (strong multifractality) is also the same \cite{21,22}. We would like to check these predictions numerically for the unitary CrRME and to see how large are the deviations between these two RME for \(c \sim 1\).

The paper is organized as follows: in Section \ref{2} we give a general overview of the
level-number variance ($\Sigma_2$-statistics) and the calculation of the level compressibility, in Section 3 we give some numerical details and finally a summary and outlook of this work is given in Section 4.

2. $\Sigma_2$-Statistics and Level Compressibility

Our model is the power-law random banded matrix defined as an ensemble of random Hermitian $N \times N$ matrices belonging to GUE. The real and imaginary parts of the matrix elements are independently distributed Gaussian random variables with zero mean $\langle H_{ij} \rangle = 0$ and the variance of the matrix elements is given by:

$$\langle |H_{ij}|^2 \rangle = \left[ 1 + \frac{1}{b^2} \sin^2 \left( \frac{\pi |i-j|}{N} \right) \right]^{-1} \times \begin{cases} \frac{1}{\beta} & \text{if } i = j \\ \frac{1}{2} & \text{if } i \neq j \end{cases}$$

where $b$ is a parameter that controls the fractal dimensions of the critical eigenfunctions and $\beta$ is the symmetry parameter i.e., $\beta = 1, 2$ and 4 for the GOE, GUE and GSE respectively. This model remains critical for any arbitrary value of $b$.

From a physical point of view the GUE can be realized by putting a system in a strong external magnetic field i.e., provided the splitting of levels by the magnetic field is of the same order of magnitude as the average level spacing in the absence of the field.

In this paper we study the level compressibility $\chi(N, b)$ via the level-number variance $\Sigma_2(\langle n \rangle)$ of the above model. The level-number variance is a statistical quantity that provides a quantitative measure of the long-range ”rigidity” of the energy spectrum $[3]$:

$$\Sigma_2(\langle n \rangle) = \langle n^2 \rangle - \langle n \rangle^2$$

The behavior of $\Sigma_2$ is known (as a function of the mean number of levels $\langle n \rangle \gg 1$) in the delocalized and localized phases respectively. However, in the critical regime, the level-number variance has been conjectured to be Poisson-like $[24, 18]$;

$$\Sigma_2 \sim \begin{cases} \frac{2}{\pi^2 \beta} \ln (\langle n \rangle) & \text{delocalized,} \\ \chi \langle n \rangle & \text{critical,} \\ \langle n \rangle & \text{localized,} \end{cases}$$

where the level compressibility $\chi$ is another important parameter to characterize the localization-delocalization transition (LDT) and takes values $0 \leq \chi \leq 1$, being zero in the delocalized state and unity in the localized state. Formally, one computes $\chi$ as;

$$\chi \approx \lim_{\langle n \rangle \to \infty} \lim_{N \to \infty} \frac{d \Sigma_2(\langle n \rangle)}{d \langle n \rangle}$$

The limits in Eqn. 4 do not commute and their non-commutativity is attributed to the violation of the normalization sum rule $[18]$. It is remarkable (see $[25, 6, 18, 26]$ for details) that $\chi$ can be expressed in terms of the fractal or the correlation dimension $D_2$ and the spatial dimension $d$. In Ref. $[18]$ an explicit and very simple relationship between the critical spectral and eigenfunction statistics has been suggested:

$$\chi = \frac{1}{2} \left( 1 - \frac{D_2}{d} \right)$$
There is vast analytical and numerical evidence that this formula is correct at weak multifractality. However, its status for the case of strong multifractality (small $D_2$) raises much doubts [22]. We later see that whilst our numerical data are in agreement with Eqn.(5) in the limit $b^{-1} \to 0$ they are in disagreement [21,22] for the opposite limit. At the same time the overall agreement with the analytical result Eqn.(10) for the MNS model is unexpectedly good for all values of $b$ and the moderately large matrix sizes up to $N = 400$. We were not able to detect any systematic dependence on the size of the matrix $N$ in this range.

In the next section we turn to the numerical details of how to compute the level compressibility.

3. Results and Discussion

We performed numerical simulations of the RME described by Eqn.(11) for values of $b^{-1} \in [0.4,40]$. The matrices are not sparse and as such the use of fast and efficient algorithms i.e., the Lanczos algorithm is not possible. However, we calculated the spectra of $N \times N$ matrices for $N = 50$ with 10000 realizations to $N = 400$ with 2000 realizations by exact numerical diagonalization and hence accumulating $\sim 1 \times 10^5$ eigenvalues for each value of $N$. The results we discuss here are for the GUE only.

The average density of states as a function of energy $E$ is well described by;

$$\rho(E) = \frac{N}{\pi^2 b} \int_{0}^{+\infty} \frac{dt}{1 + (1/z) \exp \left[ (\beta/2) (E^2 + t^2) \right]}$$

where $z = \exp (\pi b) - 1$. The two limiting cases of Eqn.(6) are;

$$\rho(E) = \begin{cases} 
\frac{N}{\pi b} \sqrt{2 \pi b - E^2} & \text{Semi-circle if } \pi b \gg 1 \\
N \sqrt{\frac{\beta}{2\pi}} \exp (-\beta E^2/2) & \text{Gaussian otherwise} 
\end{cases}$$

All our calculations are done at the band center (containing $\langle n \rangle \gg 1$) where the density of states is nearly constant. The calculation of $\chi$ from $\Sigma_2$ is non-trivial as there are finite-size effects $\sim -\langle n \rangle^2 / N$ in $\Sigma_2$ originating from the long tail $R(s) \sim -1/N$ of the two-level correlation function. This is circumvented by fitting $\Sigma_2$ data with a polynomial in $\langle n \rangle$ of degree $m$:

$$\Sigma_2 (\langle n \rangle) \approx \chi_0 + \chi \langle n \rangle + \sum_{k=2}^{m} \chi_k \langle n \rangle^k$$

Then we identify the coefficient of the linear term as $\chi$ and $\chi_0$ is the constant term. Both these terms are expected to have only weak $N$-dependence. This is in contrast to that of the coefficients $\chi_k \propto 1/N^{k-1}$ with $k > 1$ that have a strong $N$-dependence which describe finite-size effects in $\Sigma_2$. By making the polynomial fit Eqn.(8) we separate the finite-size effects in $\Sigma_2$ and the weak $N$-dependence in $\chi$ and in $\chi_0$. This weak $N$-dependence in $\chi$ and in $\chi_0$ could be of fundamental interest and needs further investigation.

In Fig.4 we plot $\Sigma_2$ for $N = 400$ and $\Sigma_2 (\langle n \rangle)$ is linear as a function of $\langle n \rangle$ as is expected for a critical ensemble. The lines through the data points are in agreement.
Figure 1. The typical behavior of the level-number variance $\Sigma_2 \langle n \rangle$ for $N = 400$ and for $b^{-1} = 0.4, 1.2, 2.0, 2.8$ and 3.6. The lines through the data points correspond to $\chi_0 (b) + \langle \frac{1}{n} \rangle \langle n \rangle$. The value of $\chi_0$ is on average $N$ independent (see Fig.3). The linear behavior of $\Sigma_2$ is in perfect agreement with the theoretical prediction of [24, 18] (see Eqn.(9)). For clarity in the numerical data we skip every 3rd symbol.

with the Poisson-like behavior i.e., $\Sigma_2 \approx \chi \langle n \rangle + \chi_0$. For large $b \gg 1/2\pi$ the value of $\chi$ is in good agreement with Eqn.(5) and Eqn.(6) that follows from Eqn.(5) and the correlation dimension $D_2$ found in Refs. [13, 25] :

$$\chi = 1/4\pi b \tag{9}$$

However, our results are in disagreement with Eqn.(5) for small $b \ll 1/2\pi$ [21, 22]. The analytic formula for $\chi$ (MNS in Fig.2) that follows from [10] for all values of $b$ reads:

$$\chi = \frac{d \ln (I)}{d \ln (z)} = \frac{\text{PolyLog} \left[ -1/2, 1 - \exp (2\pi b) \right]}{\text{PolyLog} \left[ +1/2, 1 - \exp (2\pi b) \right]} \tag{10}$$

where

$$I = \int_0^\infty \frac{dp}{(1 + z^{-1} \exp (p^2))}$$

and PolyLog $[n, z]$ is an analytic continuation of $\sum_{k=1}^\infty z^k/k^n$. This function can easily be computed by use of mathematica.

The main result of this work is that the values of $\chi$ (for modestly large $N < 400$) of the critical ensemble due to [11] agree unexpectedly well with the theoretical formula due to [10], Eqn.(10). This agreement is, however, not perfect since the data points lie below Eqn.(10) in the range $3 \leq b^{-1} \leq 10$, where the function $\chi (b^{-1})$ in Eqn.(10) has a peculiar curved shape.
Figure 2. The spectral level compressibility $\chi$ as a function of $b^{-1}$ and $N$. The values of $\chi$ were calculated by a fit-polynomial of degree $m = 3$. The solid line is the analytic formula (MNS) due to [10] and the dashed line is Eqn. (9). $\chi$ is on average $N$ independent.

Figure 3. The constant term $\chi_0(N, b)$ of the fit-polynomial with degree $m = 3$ in Eqn. (8). $\chi_0$ decays to zero for values of $b^{-1} \leq 10$. The data here is the same as in Fig. 2.
4. Conclusion

We have been able to compute $\chi$ as a function of $b^{-1}$ for the GUE critical ensemble. Our results are in agreement with the theoretical formula Eqn.(10) except for the range $3 \leq b^{-1} \leq 10$ where the numerical data are below Eqn.(10).

Furthermore, $\chi_0$ and $\chi$ are almost $N$-independent which is a signature of criticality. It is important to note that our simulations exclude the strong $N$-dependence of $\chi$ but they do not exclude the weak $\ln(N)$-dependence of $\chi$ which is not trivial to realize numerically. Recently, it has been shown analytically [27] that in the CrRME the corrections to $\chi$ of order $b^2$ have a logarithmic $N$-term \( i.e \ -b^2 \ln(N) \) in the crossover between the GOE and the GUE. In both the GUE and the GOE this logarithmic factor is absent up to order $b^2$. It’s presence is not excluded in higher $b$ corrections, \( e.g \) as $-b^3 \ln(N)$. Such a logarithmic dependence is of fundamental interest. This is because the presence of $\ln(N)$-corrections would signal the incomplete criticality of the model. The logarithmic terms may drive the system to a different, stable, set of critical points where the $\chi$ may be drastically different from the MNS result. Unfortunately, the accuracy of our simulations is not enough to detect such a weak logarithmic dependence. This point needs further investigation.

We note also that the constant term $\chi_0$ carries important information about CLS since it has a finite limit as $N \to \infty$. As is seen in figure 3 the value of $\chi_0$ as a function of $b^{-1}$ decays to zero with increasing $b^{-1}$ which is (so far unnoticed) an important characteristic feature of CLS.

In summary we have been able to compare via numerical simulations of the GUE critical ensemble [11] and the theoretical formula Eqn.[10]. The two results agree but in the region $3 \leq b^{-1} \leq 10$, the simulations are below the theoretical calculations. However, we emphasize that our results are preliminary and further work is in progress.

5. Acknowledgments

MLN would like to thank R. A. Römer for encouraging discussions, funding from the Deutschen Forschungsgemeinschaft through the Sonderforschungsbereich 393 and the ICTP where this work was started.

References

Critical Random Matrix Ensemble

[27] O. M. Yevtushenko and V. E. Kravtsov (unpublished)