A simple formula
for Bose-Einstein corrections

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Abstract

In analogy with the quantum field theory of free bosons a simple integral representation is derived for recently proposed corrections describing the Bose Einstein effect. The saddle point approximation to these integrals results in a compact expression which sums effectively the original $n!$ terms with an accuracy better than 2% for $n > 7$ strongly correlated bosons.

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1 Introduction

The very nature of the Bose-Einstein correlations makes them rather difficult to include in the Monte Carlo event generators and consequently the problem has been the area of intensive studies for many years [1]-[8]. Recently Bia/łas and Krzywicki have proposed a simple recipe to effectively include Bose-Einstein correlations in the event generating Monte Carlo programs [3]. The basic step in their procedure consists of calculating a positive weight for each generated event

\[ W_n(p_1, \ldots, p_n) = \sum_{\mathcal{Q}_n} \prod_i A_{i \mathcal{Q}_n(i)} = perm(A), \]

where \( A \) is a correlation matrix which depends on particle momenta and on the choice of variables parameterizing the whole phenomenon. One commonly used form reads

\[ A_{ik} = \exp \left( -\frac{(p_i - p_k)^2}{2\sigma^2} \right). \]

The sum in Eq. (1) extends over all permutations, \( \mathcal{Q}_n \), of \( n \) elements (1,2,\ldots,n). Hence the numerical cost to calculate above weights grows like \( n! \) and this limits practical applications.

In this letter we derive a simple \( 2n \)-dimensional integral representation for a general permanent of a correlation matrix, \( perm(A) \). Our representation has two advantages. First, the integrals can be done analytically, in the saddle point approximation, providing relatively simple and accurate expression for the sum (1). Secondly, the integral representation has a straightforward probabilistic interpretation. This indicates that implementation of the correction scheme proposed in Ref. [3] might be possible stochastically, and may advance the novel generation of MC programs.

The motivation for this work came from the simple observation that even the free quantum field theory of bosons automatically incorporates symmetrization. It is therefore plausible that the simple integral representation, analogous to that for the propagation of \( n \) identical bosons, should exist for the sum (1). In the next section we prove that this is indeed the case.
2 Integral representation

Let us recall the Wick theorem for finite, say $n$, number of gaussian variables $(\phi_1, \phi_2, \ldots, \phi_n) \equiv \vec{\phi}^T$. If $\vec{\phi}$ is distributed according to

$$P(\phi) \sim \exp \left( -\frac{1}{2} \phi^T G \phi \right),$$

then moments of $\phi$’s are given by the sum over all contractions

$$<\phi_{i_1} \phi_{i_2} \ldots \phi_{i_k}> = (\phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \ldots) + (\phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \ldots) + \ldots,$$

where a single contractions reads

$$\phi_{i_1} \phi_{i_k} = (G^{-1})_{ik},$$

for a non-singular matrix $G$.

To match the combinatorics of contractions with that of simple permutations required in Eq.(1) we introduce $2n$ gaussian variables

$(\phi_1, \ldots, \phi_n), (\psi_1, \ldots, \psi_n) = (\vec{\phi}^T, \vec{\psi}^T) \equiv \vec{\Psi}^T$ distributed accordingly to

$$P(\Psi) \sim \exp \left( -\frac{1}{2} \Psi^T M \Psi \right),$$

with the $2n \times 2n$ matrix

$$M = \begin{pmatrix} 0 & A^{-1} \\ A^{-1} & 0 \end{pmatrix},$$

and $A$ as given by Eq.(2). Since, by construction, $\phi$ and $\psi$ variables are not coupled within themselves, all non-zero contractions occur between $\psi$’s and $\phi$’s only. Hence it follows from the Wick theorem (3) that

$$<\prod_{i}^{n}(\phi_i \psi_i)> = W_n(p_1, \ldots, p_n).$$

\footnote{Where evident the vector label ** will be omitted.}
since $M^{-1} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$. This is the integral representation sought of in the Introduction. However the choice of variables, although simple conceptually, is not very practical, since the form $\Psi^T M \Psi$ is not positive definite, and consequently the paths of $\prod_i^n d\psi_i d\phi_i$ integrations should lie in the complex plane of the corresponding variables. This deficiency is removed and the integrand further simplified by the following series of transformations of variables $(j,k = 1, \ldots, n)$

$$\phi_j = \frac{1}{\sqrt{2}}(u_j + iv_j), \quad u_j = O_{jk} \zeta_k, \quad \zeta_j = \sqrt{\lambda_j} x_j,$$

$$\psi_j = \frac{1}{\sqrt{2}}(u_j - iv_j), \quad v_j = O_{jk} \eta_k, \quad \eta_j = \sqrt{\lambda_j} y_j,$$

(9)

(10)

where the orthogonal matrix $O$ transforms $A$ into the diagonal form $O^T A O = \text{Diag}[\lambda_1, \ldots, \lambda_n]$. One obtains

$$W_n = \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \sum_j (x_j^2 + y_j^2) \right) \prod_{\mu} \frac{1}{2} ((x \cdot e^{(\mu)})^2 + (y \cdot e^{(\mu)})^2) \prod_i^n dx_i dy_i. \quad (11)$$

Vectors $\vec{e}^{(\mu)}$ are given by

$$e_i^{(\mu)} = O_{\mu i} \sqrt{\lambda_i}, \quad (12)$$

and all integrals are along the real axis. It turns out that all eigenvalues of the correlation matrix, Eq.(2), are nonnegative. The more general case of arbitrary eigenvalues can be readily dealt with by the appropriate Wick rotations which will be different for different coordinates.

Equation (11) represents the sum of $n!$ terms in (1) by the $2^n$-fold integral of the gaussian type. The whole dependence on the original momenta $(p_1, \ldots, p_n)$ is contained in vectors $\vec{e}^{(\mu)}$ which, in view of Eq.(12), are closely related to the eigenvectors of the correlation matrix $A$.

The integral (11) can be calculated by the Monte Carlo technique. We have found, however, that the importance sampling according, to the normal distribution, is not sufficient to produce a decent estimate. The product of $n$ factors dramatically modifies the relatively mild gaussian dependence and should be included in the weight. We have therefore solved the integral analytically in the saddle point approximation. This gives rather satisfactory results as discussed in the next Section.
3 Saddle point approximation

3.1 The action and the saddle point equations

The interplay between the gaussian fall off and the polynomial rise of the two factors in the integrand of Eq.(11), produces a maximum far from the origin. This maximum becomes more and more narrow with increasing $n$. Thus, we expect that the saddle point approximation will be better for higher $n$, just where the exact summation (1) fails in practical terms.

If one writes the integrand in Eq.(11) as

$$I(x, y) = \frac{1}{(4\pi)^n} \exp(-S(x, y)),$$

then the ”action” $S(x, y)$ reads

$$S(x, y) = \frac{1}{2}(\vec{x}^2 + \vec{y}^2) - \sum_{\mu} \ln(x^2_{\mu} + y^2_{\mu}),$$

where the greek subscript denotes the scalar product $z_{\mu} \equiv (\vec{z} \cdot \vec{e}(\mu))$, $z = x, y$. Then the location of the saddle point (or points) is determined by the following system of nonlinear equations

$$\vec{x} = 2\sum_{\mu} \vec{e}(\mu) \frac{x_{\mu}}{(x^2_{\mu} + y^2_{\mu})},$$

$$\vec{y} = 2\sum_{\mu} \vec{e}(\mu) \frac{y_{\mu}}{(x^2_{\mu} + y^2_{\mu})}.$$ 

In general many solutions may exist and there is no simple method to find all of them. The problem is partially alleviated by identifying the continuous symmetries of the integrand. This is also necessary for the proper application of the saddle point technique. Therefore we will first discuss the symmetries of the problem and identify corresponding zero mode.

3.2 Global U(1) symmetry and gauge fixing

The integrand in Eq.(11) is invariant under the ”horizontal” U(1) rotation

$$\vec{x}\left(\alpha\right) = \vec{x} \cos(\alpha) + \vec{y} \sin(\alpha),$$

$$\vec{y}\left(\alpha\right) = -\vec{x} \sin(\alpha) + \vec{y} \cos(\alpha),$$

which
with the same $\alpha$ for every coordinate $x_i, y_i$. Consequently there is no unique solution to Eqs.\((15-16)\) and one has to fix this freedom before the saddle point technique can be used. We follow the standard procedure known from the field theory. In this language the symmetry \((18)\) is the global one, and the single ”gauge fixing” condition is required. We choose the symmetric gauge

$$x^2 = \bar{y}^2,$$

and insert the following representation of unity under the integral \((11)\)

$$\Delta(x, y) \int_0^{2\pi} \delta(x(\alpha)^2 - y(\alpha)^2) d\alpha = 1. \quad (20)$$

The gauge invariant function $\Delta(x, y)$ follows from Eq.\((20)\)

$$\Delta(x, y) = \frac{1}{2} (x^2 + \bar{y}^2). \quad (21)$$

### 3.3 The formula

With the aid of Eqs.\((20)\) and \((21)\), the volume of the gauge orbit \((2\pi)\) can be readily factored out and the zero mode corresponding to the symmetry \((18)\) becomes fixed. Further calculations are straightforward and the final result reads

$$W_n = 2^{1-n} |s_0| \sqrt{\frac{\pi}{\prod_i \Lambda_i}} \exp (-S(s_0, s_0)), \quad (22)$$

where the product is taken over all, but one (equal to zero), eigenvalues of the 2n dimensional covariance matrix.

$$S^{(2)}_{jk} = \frac{\partial^2 S(x, y)}{\partial z_j \partial z_k}|_{(s_0, s_0)}, \quad (23)$$

evaluated at the saddle point \((\bar{s}_0, \bar{s}_0)\). For completeness we quote the explicit form of $S^{(2)}$ for arbitrary $x$ and $y$.

$$S^{(2)}(x, y) = \begin{pmatrix} S^{(xx)} & S^{(xy)} \\ S^{(yx)} & S^{(yy)} \end{pmatrix}, \quad (24)$$
\[ S_{ik}^{(xx)} = \delta_{ik} - 2 \sum_{\mu} e_i^{(\mu)} e_{k}^{(\mu)} \frac{y_{\mu}^2 - x_{\mu}^2}{(x_{\mu}^2 + y_{\mu}^2)^2}, \]  \hspace{1cm} (25)

\[ S_{ik}^{(xy)} = S_{ik}^{(yx)} = 4 \sum_{\mu} e_i^{(\mu)} e_{k}^{(\mu)} \frac{x_{\mu} y_{\mu}}{(x_{\mu}^2 + y_{\mu}^2)^2}, \]  \hspace{1cm} (26)

\[ S_{ik}^{(yy)} = \delta_{ik} - 2 \sum_{\mu} e_i^{(\mu)} e_{k}^{(\mu)} \frac{x_{\mu}^2 - y_{\mu}^2}{(x_{\mu}^2 + y_{\mu}^2)^2}. \]  \hspace{1cm} (27)

Condition (19) does not entirely fix the freedom (18). For each symmetric solution, \((\vec{s}, \vec{s})\) say, there exist also three rotated solutions \((-\vec{s}, \vec{s})\) \((\vec{s}, -\vec{s})\) and \((-\vec{s}, -\vec{s})\) which belong to the orbit (18) and satisfy (19). Of course all four (equal) contributions are included in the final result. Choosing different gauge may lead to the different discrete degeneracies, but the final result is not changed. More important, we have found that, except for the above degeneracy, there is no other, essentially different, solution of the saddle point equations. This seems to be a general property of our \(S(x, y)\) which was checked for a number of choices of correlated momenta \((p_1, \ldots, p_n)\).

Finally, a few comments about the numerical solution of the system (15-16). Since our experiments indicate that after a complete gauge fixing there is only a single maximum, any numerical routine should perform satisfactorily. We have used a simple iterations of Eqs.(15-16) which quickly converged to the cycle between two parallel vectors. Then, a repeated mixture of the arithmetic average between the two followed by one iteration of the map (15-16) converged rapidly to the unique solution. Condition (19) was assured by the choice of the symmetric starting point.

4 Results and further applications

Table I contains comparison between the saddle point formula, Eq.(22), and the exact one (1). As expected, the accuracy of the saddle point calculation increases with increasing \(n\) reaching the level of 2% at \(n = 8 - 9\). Consequently, both methods applied complementarily would give a satisfactory approach for all \(n\).

Results presented in Table I were obtained for the one dimensional momenta generated randomly from the interval \(0 < p_k < 2\) and with the width in Eq.(2) \(\sigma = 1/\sqrt{2}\). This corresponds to very strongly correlated particles. In real applications, few particles are clustered within a range of typically
measured $\sigma \sim 0.1\text{GeV}$. This situation corresponds to the block diagonal (after suitable permutation) form of matrix $A$ - each block describing one cluster of identical bosons. It can be readily seen that, for exactly block diagonal (reducible) $A$, the integral (11) factorizes

$$W_n = W_{n1}W_{n2} \ldots W_{n\text{cluster}}.$$  \hspace{1cm} (28)

This factorization follows even more clearly from the definition (1). On the other hand the saddle point formula applied directly to the whole matrix $A$ would fail since it does not accommodate multiple zero modes (one for each cluster) which occur in the reducible case. The remedy of this difficulty is evident. One should first perform reduction of a given event into irreducible clusters and then, depending on the size of the cluster, apply either exact or saddle point formula to each block separately. The reduction procedure will contain an essential parameter, say $\epsilon$, which can be defined as a tolerance beyond which bosons would be considered uncorrelated. This tolerance cannot be too small since then a quasi block diagonal matrix would be qualified as a single irreducible cluster. Therefore the value of epsilon is determined by the performance of the saddle point formula for weakly broken additional U(1) symmetries. Our numerical experiments indicate that the present "one loop" formula works for $\epsilon > 0.05 - 0.1$.

Further improvement in the accuracy of the saddle point result can be easily achieved by calculating higher corrections. Including for example the two additional terms of the Taylor expansion around the saddle should not increase much the computational effort but would reduce the errors considerably. More important, this should allow to decrease the tolerance parameter $\epsilon$ discussed above.

Another path for an improvement is the Monte Carlo calculation of the corrections to the saddle point formula. As mentioned Section 2, MC calculation of the average (11) was not satisfactory, because the samples were drawn from the wrong distribution (centered gaussians). However one can use the saddle point approach to locate the maximum of the integrand and then use the saddle point approximation of the integrand as the proper distribution of generated samples. In that case the observable (the ratio between exact and saddle point integrands) will be a slowly varying function of $\vec{x}$ and $\vec{y}$ and statistical estimates will be reliable.

Finally, there exists an intriguing possibility of defining a new family of event generating MC programs which would have the corrections (1) included.
on the stochastic basis. That may be possible since the integrand in our representation (11) is positive. Therefore it might be feasible to modify existing event generators by incorporating that factor.

To summarize, we have derived an exact integral representation for the Bose-Einstein weights, which until now have been computed by summing over $n!$ terms. Resulting $2n$-dimensional integral was calculated by the saddle point technique, and obtained formula approximates the exact sum to less than 2% for more than 7 correlated bosons. Various applications and improvements have been outlined.

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References

Table 1: Saddle point results and the exact summation for different number \( n \) of identical, strongly correlated, bosons. Timing (in seconds) of the Mathematica runs on a HP735 workstation are also quoted. The last column gives the result of a simple fit \( W_n = n! a^n \), where the parameter \( a^2 \) is a measure of the average value of the matrix elements of \( A \).