1) Suppose that the free energy for an Ising system is given by
\[
F = \int d^3x \left\{ \frac{\lambda}{2} \vec{\nabla} \vec{\sigma} \cdot \vec{\nabla} \vec{\sigma} + V(\vec{\sigma}) \right\}
\]
where
\[
V(\sigma) = \frac{1}{2} a(T - T_c) \sigma^2 + \frac{b}{4} \sigma^4
\]
Compute the tension (the energy per unit area) of a domain wall separating the two low temperature phases. Give a reasonable estimate for the thickness of the domain wall.

2) Suppose that the free energy for a real-valued order parameter \(\phi\) is given by
\[
F = \frac{1}{2} a(T - T_c) \phi^2 - \frac{b}{4} \phi^4 + \frac{c}{6} \phi^6
\]
where \(b\) and \(c\) are both positive. Show that there is a first order phase transition at \(T \geq T_c\). What is the latent heat required for the transition?

3) A superconductor can be thought of as a charged superfluid. For a charged particle of mass \(m\) and charge \(q\), the relation of the velocity to the momentum is
\[
\vec{v} = \frac{1}{m}(\vec{p} - q \vec{A})
\]
where \(\vec{A}\) is the gauge potential, which is related to the magnetic field by \(\vec{B} = \vec{\nabla} \times \vec{A}\). Assume that the relation of \(\vec{p}\) to an order parameter \(\Psi\) is
\[
\vec{p} = -i\hbar \frac{\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*}{2\Psi^* \Psi}
\]
a) Derive the London equation:
\[
\vec{\nabla} \times \vec{v} = -\frac{q}{m} \vec{B}
\]
b) Assume that the charged supercurrent \(\vec{j}\) is given by \(n_s q \vec{v}\) where \(n_s\) is the number density of the superfluid. Suppose that a magnetic field is applied parallel to the surface of the superconductor. Using Ampere’s law, show that inside the superconductor, the field falls off as \(B_\parallel = B_0 \exp(-x/\lambda_L)\). \(\lambda_L\) is the London penetration depth. Deep inside the superconductor, what is the supercurrent?

c) Consider a superconducting ring with thickness \(\gg \lambda_L\). Show that the magnetic flux through a closed contour that circles deep inside the ring is quantized. The flux, \(\Phi\), is given by the surface integral of the magnetic field over the area inside the contour, \(\Phi = \int \vec{B} \cdot d\vec{A}\).

d) The charges that make up the fluid are bound pairs of electrons (These are
called Cooper pairs). Assuming that each electron has charge $e$, what would be one quantum of magnetic flux?

4) The lattice gas model is a way of modeling the short range repulsion and longer range attraction between gas molecules. The volume of the gas is divided into cubes. Each cube is allowed to have up to one gas molecule inside of it (this takes into account the short range repulsion). The Hamiltonian for the system is given by

$$H = -\epsilon \sum_{\langle i,j \rangle} n_i n_j$$

where the sum is over nearest neighbors and $n_i$ is the occupancy number for cube $i$. Hence the energy is lowered if two neighboring cubes are occupied which means there is an attractive force between the molecules. Suppose there are an average $\bar{N}$ molecules spread out over $N_0$ cubes.

a) Find the mean field Hamiltonian in terms of $\bar{n} = \bar{N}/N_0$, which is the average occupancy of each cube.

b) Find the grand partition function $Z$ for the mean field Hamiltonian in terms of a chemical potential $\mu$.

c) Find the grand potential $\Omega = -\frac{1}{\beta} \ln Z$.

d) Find the pressure for the system, assuming each cube has volume $v_0$. Write down the final answer in terms of $\bar{N}$ by eliminating $\mu$ from the equations.

e) Assuming that $\bar{N}/N_0 << 1$, show that the pressure has the van der Waal’s form.